Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture – 05
Inner Product Space and its Properties:Cauchy Schwarz Inequality

Welcome to another module in this massive open online course. So, you are looking at the inner product of matrices and we have defined the inner product of two matrices.

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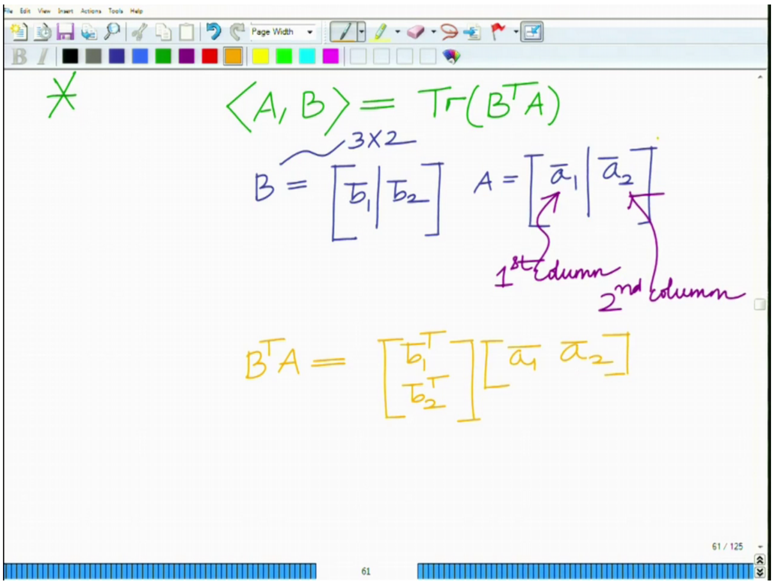

In fact, two matrices of the same size A B as trace of to B transpose A. In fact, this can be seen as if you have B for instance for our 3 cross 2 considering 3 cross 2 matrices if I can write b as two columns b 1 because remember 3 cross 2 matrix has two columns each of size 3 and the matrix A as this is just for the purpose of illustration a 1 bar first column a 2 bar. So, this is the first column this is the second column and if you write B transpose into A that will be well transpose of b remember columns becomes rows. So, this will become b 1 bar transpose second row will become b 2 bar transpose times a which is first column a 1 bar second column a 2 bar.

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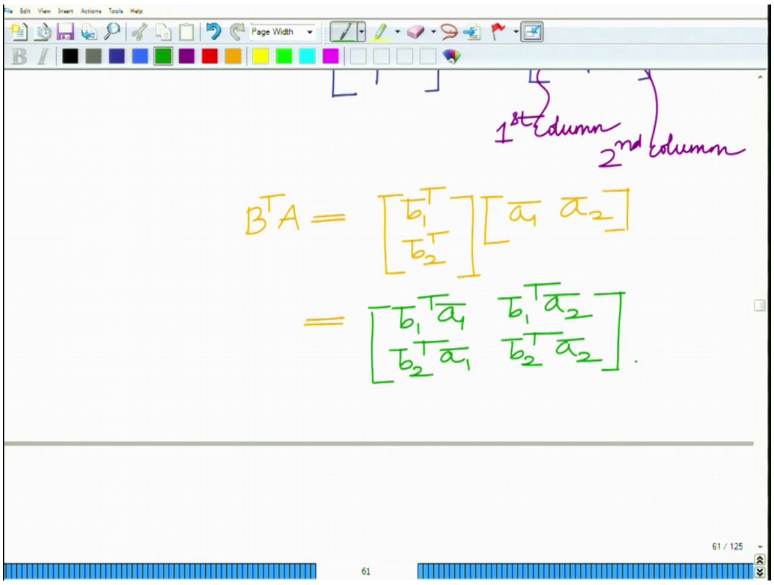

And that will be now look at this it will be a 2 cross 2 matrix. So, that will be first entry will be b 1 bar transpose a 1 bar b 1 bar transpose a 2 bar b 2 bar transpose a 1 bar b 2 bar transpose a 2 bar and you can check the various entries will be in this 2 cross 2 matrix the various entries will be b 1 1 a 1 1 plus.

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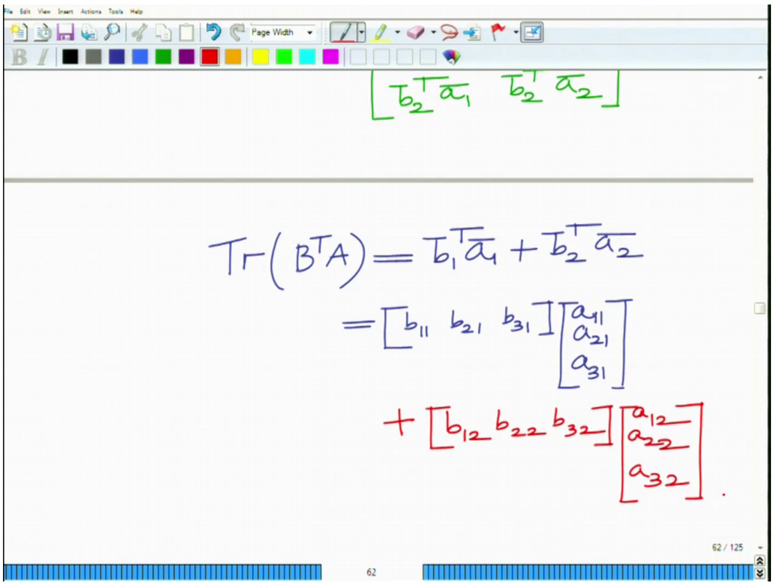

And therefore, now you can check the various entries, but if you take the trace of this that is if you trick the trace of this B transpose A that will be b 1 bar transpose a 1 bar plus b 2 bar transpose a 2 bar which will be nothing, but b 1 bar transpose is well b 1 bar transpose if you can look from above that is b 1 1 b 2 1 b 3 1 into a 1 bar that is a 1 1 a 2 1 a 3 1 plus b 2 bar that will be b 1 2 b 2 2 b 3 2 times a 1 2 a 2 2 and a 3 2.

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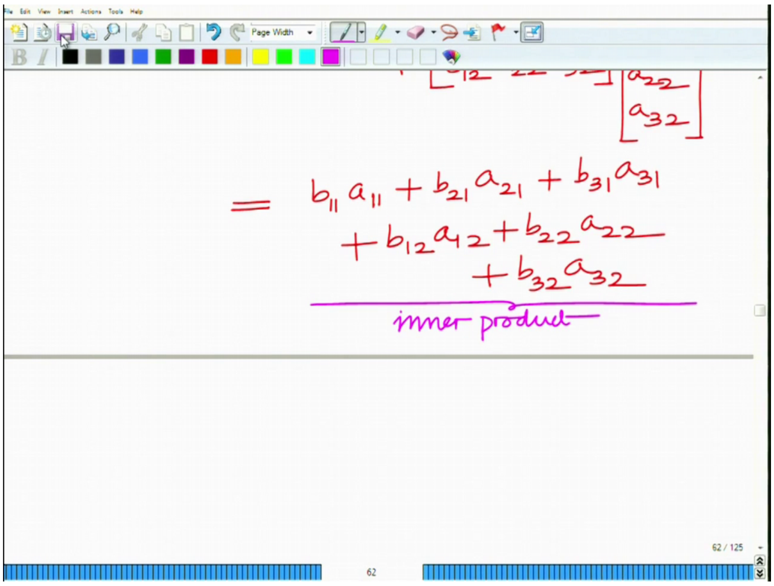

And this is equal to well if you simplify this what you will get is b 1 1 a 1 1 plus b 2 1 a 2 1 plus b 3 1 plus b b 3 1 times a 3 1 plus b 1 2 a 1 2 plus b 2 2 times a 2 2 plus b 3 2 times a 3 2 and well this is the inner product this is your inner product and now in general for an m cross n. So, this is the inner product which just a simple example illustrated this for a 3 cross 2 case in general this can be generalized for m cross n matrices.

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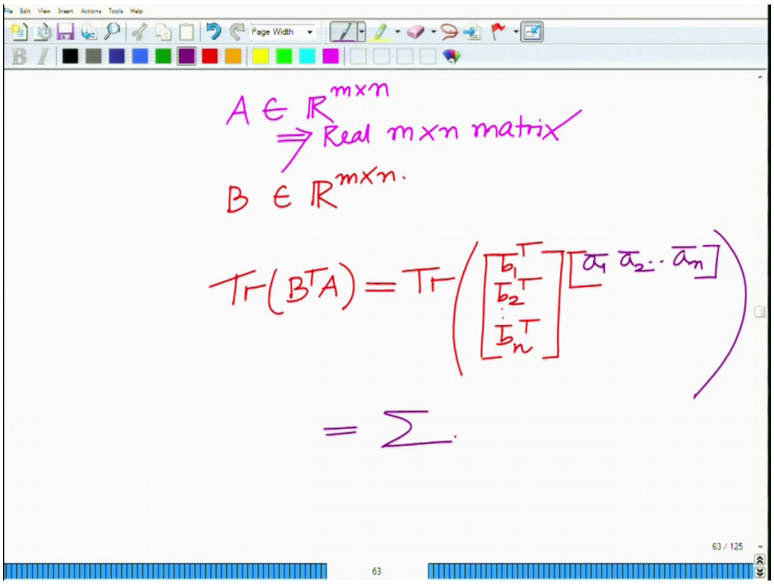

So, for instance we can have A element of m cross n real m cross n matrix further B belongs to m cross n these are two m cross n matrices then trace of B transpose a will be trace of that is the sum of the diagonal elements of you can look at this the columns will become row. So, m cross n matrices will have n columns of m element c.

So, this will be b 1 bar transpose b 2 bar transpose b if I am not mistaken be n bar transpose times a 1 bar transpose or times a 1 bar a 2 bar a n bar and that will be if you look at it summation if you take the trace of that that will be summation.

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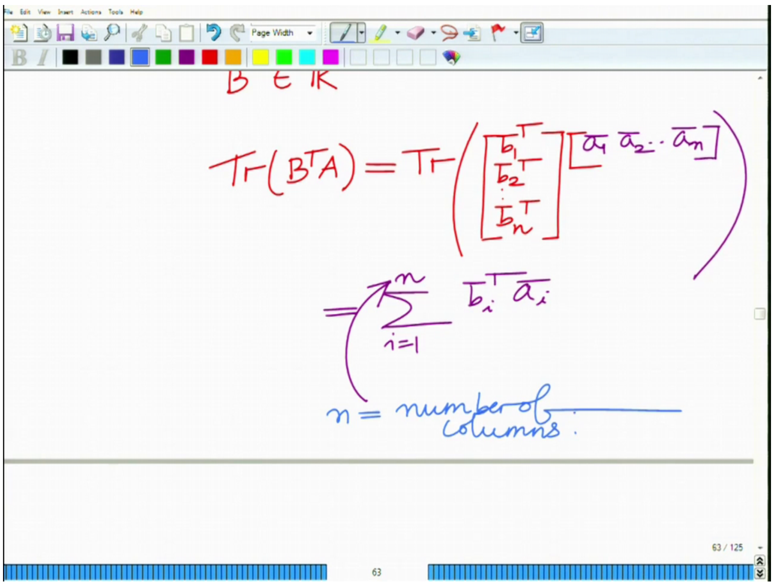

I equals one to n b i bar transpose a i bar and this is n remember n equals number of columns n equals the number of columns.

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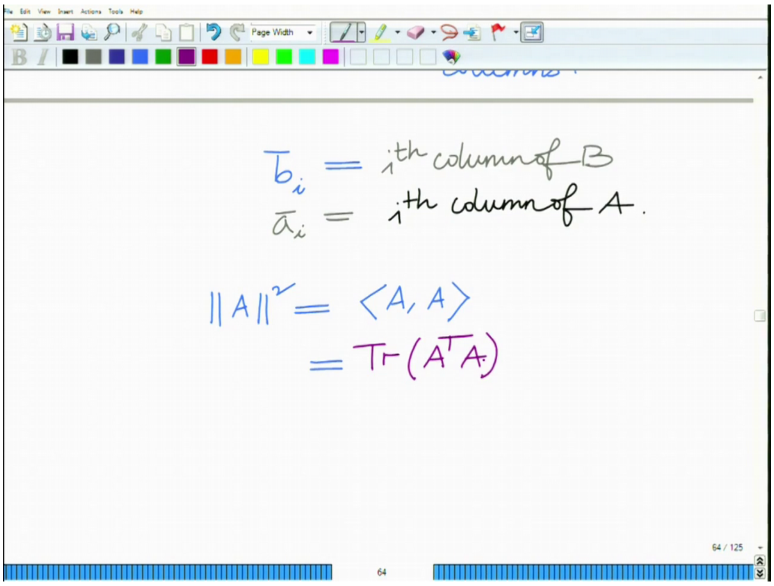

The various quantities b i bar each b i bar b i bar equals the ith column of matrix B and a i bar equals the ith column of matrix A equals the ith column of matrix A and further now. And now, let us come to the norm that is induced by this inner product remember every inner product can be every inner product induces a norm all right and the norm induced by this inner product that is the norm you can be used to define as the norm of the matrix that is norm A square is inner product of A comma A that is basically your trace of A transpose A.

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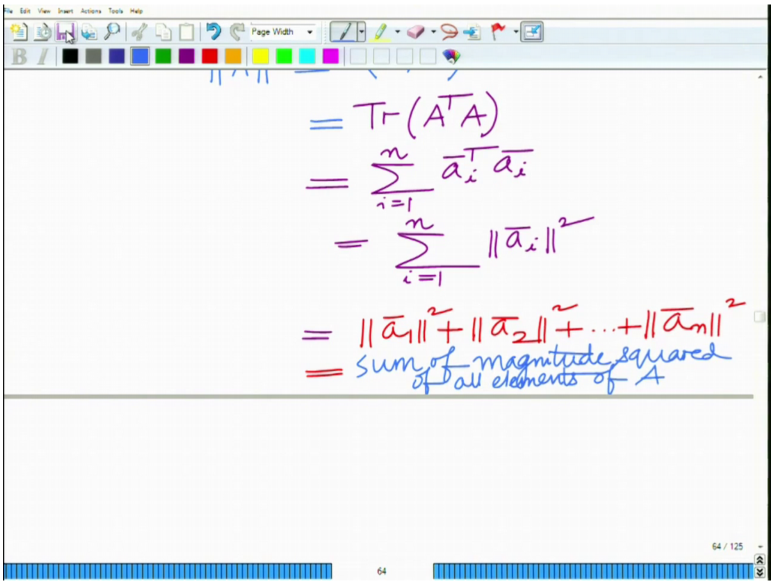

Which is equal to and you can see from the this thing above that is simply equal to norm a 1 bar square plus well you can write this as follows that is summation of i equals 1 to n a i bar transpose a i bar which is now remember a i bar transpose a i bar is nothing, but norm of a i bar square which is sum of the squares of the norms of all the columns which is basically norm of a 1 bar square plus norm of a 2 bar square plus one plus norm of a n bar square. The sum of squares of norms of all columns which is also equal to you can say sum of magnitude squared of all elements you can also check clearly that this is sum of magnitude square of all; sum of the magnitude square of all the elements of A. And this is termed as the Frobenius norm.

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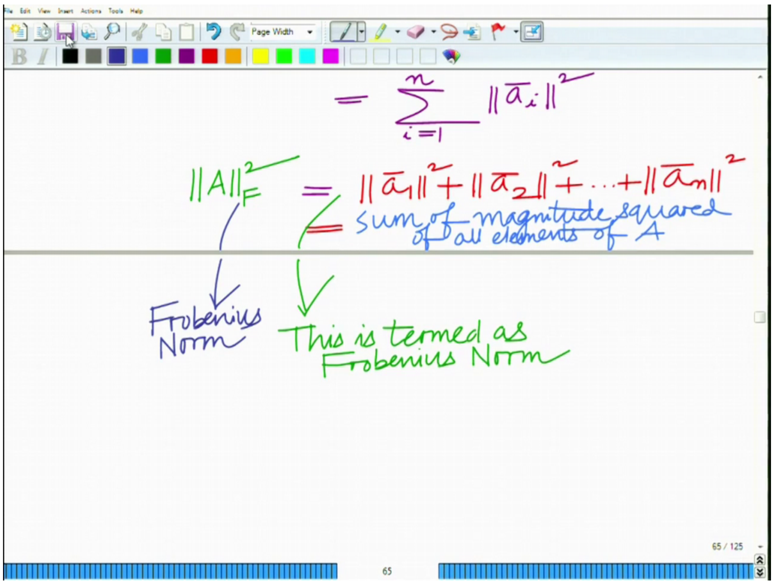

This is termed as the matrix Frobenius. In fact, this is termed as the matrix Frobenius norm square. So, this is basically your this thing this F denotes the Frobenius norm, this is the matrix Frobenius norm which is basically nothing, but the sum of the norm squared of all the columns of the matrix or sum of the magnitude squared of all the elements of the matrix. And this is the norm that is induced by this particular definition of the inner product corresponding to matrices. The next important aspect in this is what is known as the Cauchy Schwarz inequality that is the inner product satisfies the Cauchy Schwarz inequality.

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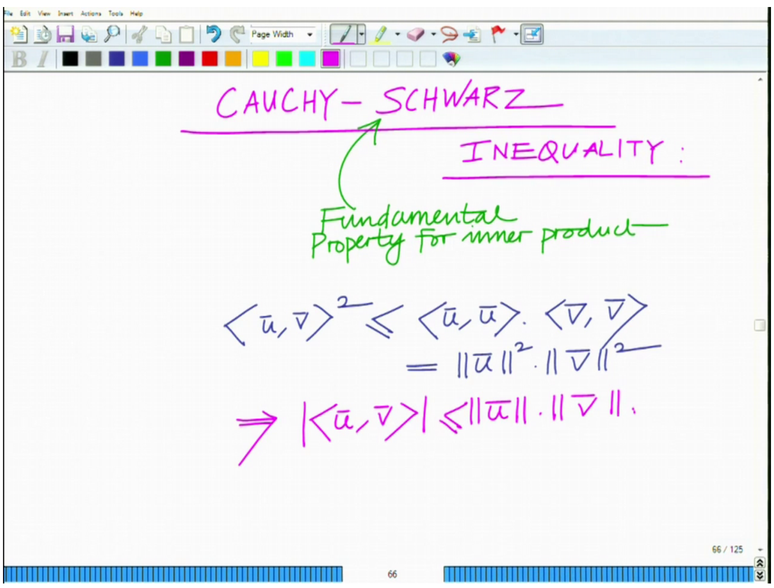

These are fundamental property and arises frequently, what the Cauchy Schwarz inequality that this is the fundament this is very fundamental property of a inner product is a fundamental property of an inner product.

And what it states is that u bar comma v bar norm square is less than or equal to that is the inner product of u bar v bar square is less than or equal to the inner product of product of the inner product of u bar with itself, times the inner product of v bar with itself which is basically nothing, but the norm of u bar square into norm of v bar square. Which basically implies that if you look at the magnitude of the inner product of u bar v bar that is less than or equal to norm of u bar times norm of v bar this is the Cauchy Schwarz inequality.

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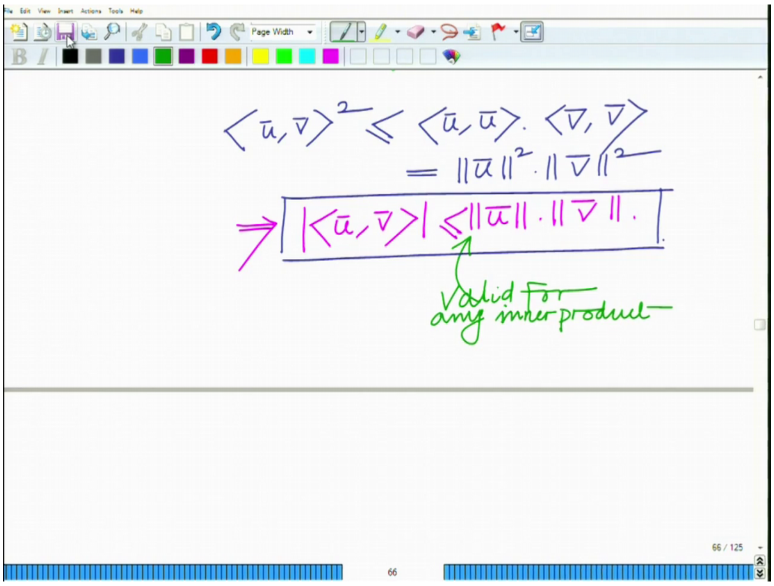

This is can be applied for a valid for any inner product that is it can either be the inner product for the vectors or inner product for the functions that we had defined previously correct we had consider continuous functions on the interval a to b case valid for that inner product it is also valid for the inner product of the may of matrices right m cross n matrices that we have defined above and so on. So, this is valid for any general definition of the inner product.

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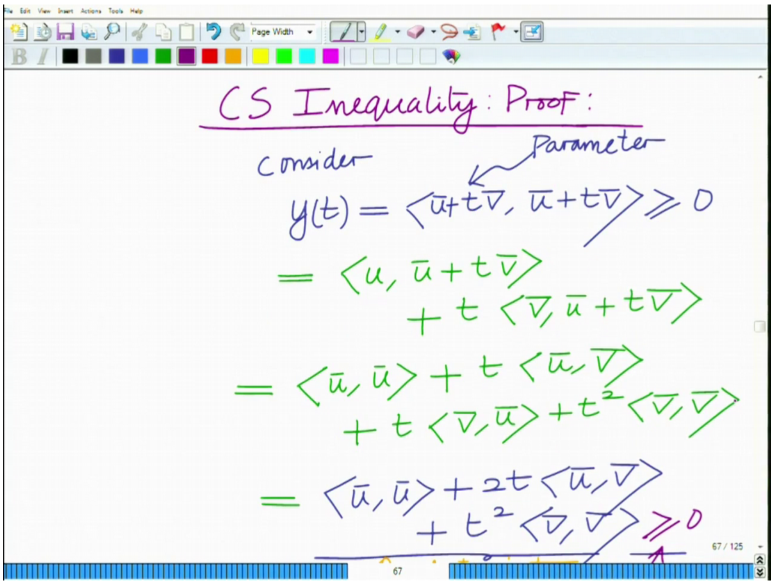

And now this can be proved as follows the Cauchy Schwarz inequality; the Cauchy Schwarz inequality can be proved as follows that is I can consider a scenario consider a function y t equals the inner product it is a function of t which is considered any two vectors u bar v bar. So, I consider the inner product of u bar plus t v bar with I am using a parameter t. So, this is the inner product on forming the vector u bar plus t v bar and I am considering the inner product with itself. So, this t is a parameter and now remember this inner product of a vector with itself u bar plus t v bar. So, this is always greater than or equal to 0 and 0 only if u bar plus t v bar equals 0 and therefore, and now this can be simplified as remember u bar comma u bar t v bar plus the inner product of t v bar with u bar t v bar which is basically t times the inner product of v bar with u bar plus t v bar which can now be simplified as the inner product of u bar with u bar plus twice the inner product you can manipulate this to get twice the inner product of u bar with v bar plus t square times the inner product. Or let me write one more step to simplify this u bar with u bar plus t times the inner product of u bar with v bar plus t times the inner product of v bar with u bar plus t square times the inner product of v bar with v bar.

And now you can see inner product of v bar u bar is nothing, but the inner product of u bar v bar.

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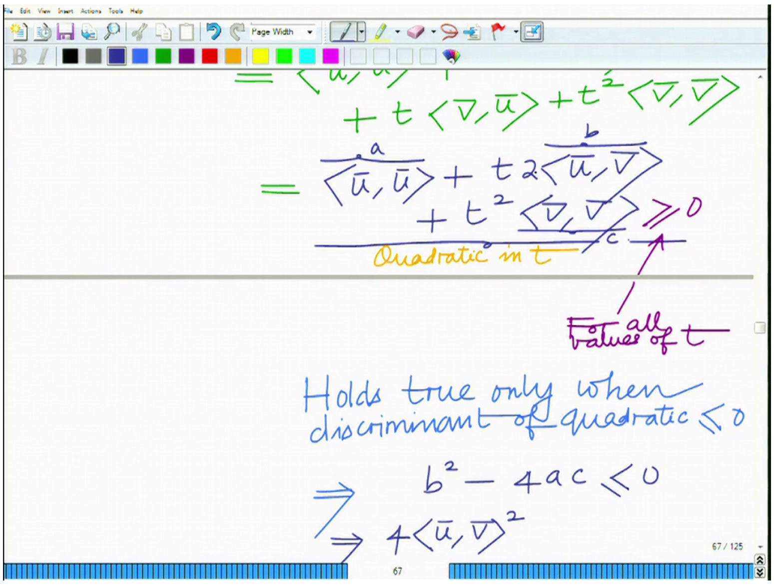

And therefore, this is equal to inner product of u bar comma u bar plus twice t the inner product of u bar with v bar plus t square times inner product of v bar plus v bar which if you view this as a function of t if for fixed u bar vectors u bar and v bar that is if you view this as a function of t then this is a quadratic expression in t. So, this is quadratic in t now observe that this is a quadratic in t and this is always greater than or equal to 0 for all values of t right. Because that is what we have seen this is the inner product of u bar plus t v bar with itself inner product of an a vector with itself is always greater than equal to 0 all right.

So, this is greater than equal to 0 this quadratic in t is greater than or equal to 0 for all values of t which basically implies right and this holds true only when the discriminant of the quadratic equation is less than or equal to 0. So, this holds true, this holds true only when discriminant of the quadratic is less than or equal to 0 this implies if you calculate the discriminant of this quadratic that will be b square minus 4 a c less than or equal to 0 that implies you have your remember this is your for this quadratic this is your a this is your b and this is your c.

So, we have b square minus 4 a c less than or equal to 0 which means u bar comma v bar inner product square. In fact, 4 because b is twice u bar comma v bar we can write it as follows we can write it as t into twice. So, twice u bar comma.

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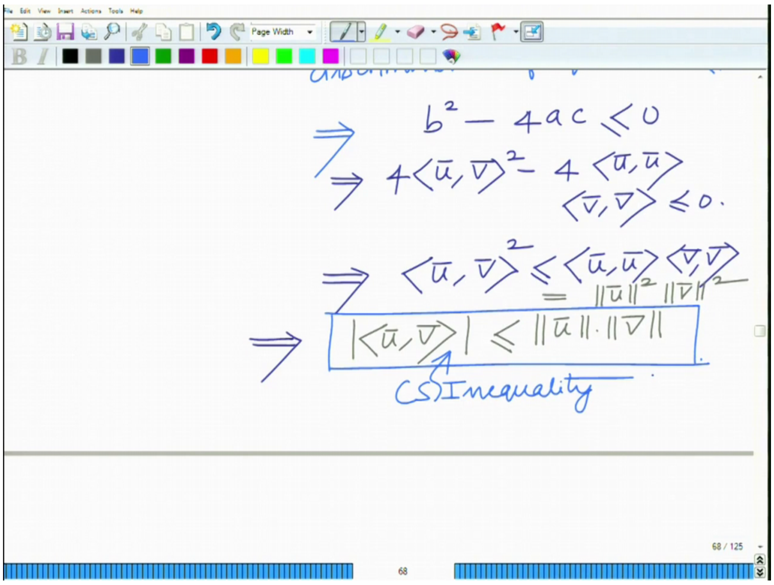

So, 4 u bar v bar square minus 4 into u bar comma u bar into v bar comma v bar less than or equal to 0 this implies u bar comma v bar less than or this implies u bar comma v bar square less than or equal to u bar comma u bar times v bar comma v bar and this implies u bar comma u bar u bar comma v bar and this implies basically now taking the square root this implies that magnitude of u bar comma v bar less than or equal to. So, this is basically your norm u bar square into norm v bar square. So, magnitude u bar comma v bar less than or equal to norm u bar times norm v bar and this is basically your CS inequality a Cauchy Schwarz inequality.

So, that basically proves the Cauchy Schwarz inequality which says the important property that the magnitude of the inner product between two vectors u bar v bar is less than or equal to the product of the norms that is norm of u bar into norm of v bar all right. You might have also seen this in the context of vectors in a high school on vector calculus which states that.

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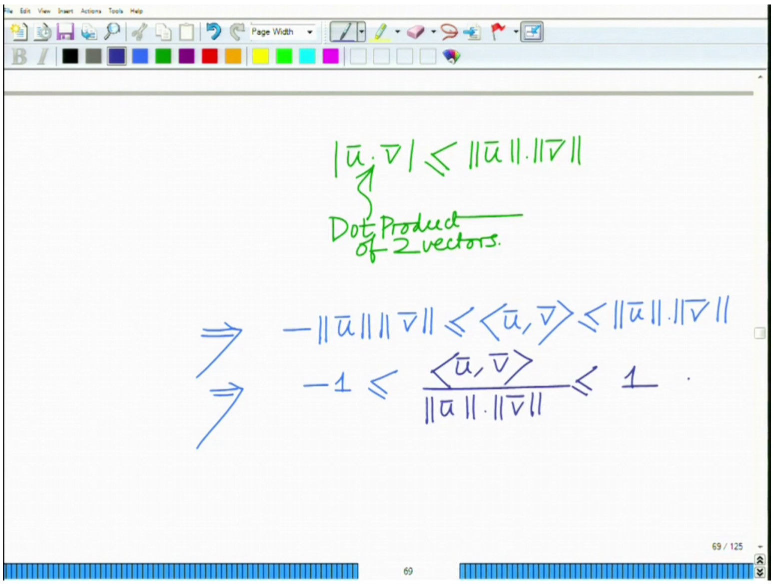

The dot product that is if you look at the magnitude of the dot product magnitude of u bar v bar is less than or equal to the product of the norms of the two vectors that is norm u bar into v bar. So, this is your standard dot product all right.

And in fact, this also implies from the above this thing you can also conclude that minus norm of u bar into norm of v bar less than or equal to u bar comma v bar less than or equal to norm u bar into norm v bar which implies dividing overall by minor norm u bar into norm v bar this implies minus 1 is less than or equal to u bar comma v bar divided by norm u bar comma norm v bar less than or equal to 1 and this quantity look at this it lies between minus 1 and 1.

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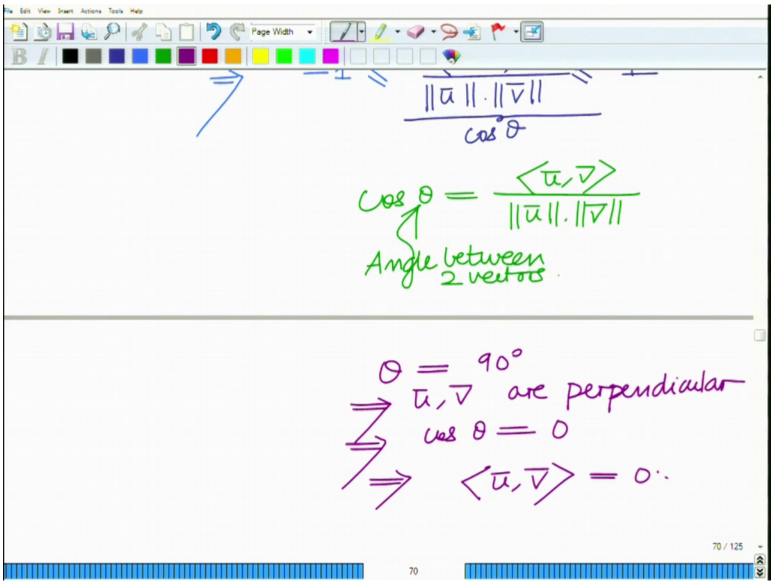

So, this quantity can be defined as the cosine of an angle, cosine theta because cosine theta lies between minus 1 and one and this is basically defined as cosine theta. So, we have cosine theta equals cosine theta equals u bar comma v bar divided by norm u bar norm v bar and this is the angle between the two vectors this is the definition of the; this is a definition of the angle between the two vectors all right.

So, now you can also see that if theta equal to 90 degrees that implies the vectors are perpendicular if the angle between two vectors is 90 degrees then we have that the vectors are perpendicular this implies u bar comma v bar are perpendicular. This implies that cosine theta equals 0 and this implies that the inner product between u bar comma v bar equal to 0 and therefore, the interesting property is that two vectors are perpendicular if they are u bar is perpendicular to.

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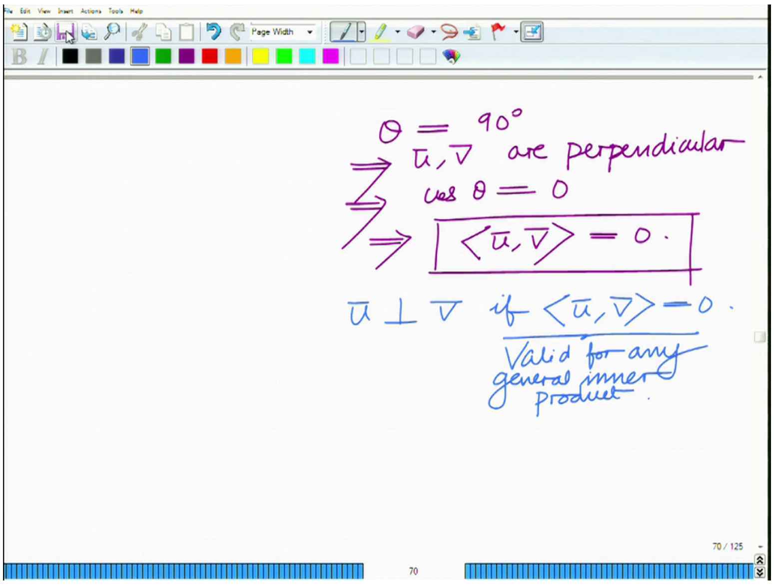

Remember this is the perpendicular symbol u bar is perpendicular to v bar if the inner product u bar comma v bar equal to 0 and this is again once again a concept that is valid for any general inner product, this is valid for any general inner product and therefore, this can once again be used to define for painting. So, we have a notion we know when two vectors are perpendicular this can be similarly be used to define a notion of perpendicularity for functions as well as matrices that is when the inner product between two functions is 0 the functions are perpendicular or the inner product between two matrices is 0 then the matrices are perpendicular and so on. So, this concept of a inner product is a very interesting and powerful concept which has a large number of applications and yields several interesting insights. So, we will stop here and continue with other aspects in the subsequent modules.

Thank you very much.