Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture – 47
Convex Optimization Problem representation: Canonical form, Epigraph form

Welcome to another module in this massive open online course. So, we looked at various optimization problem sort of in formally, what we are going to or what we are going to do in this module. And the subsequent a few of the subsequent modules is to basically set out or basically lay down the sort of formal framework to basically state or to for the formulation of a Convex Optimization Problem, alright.

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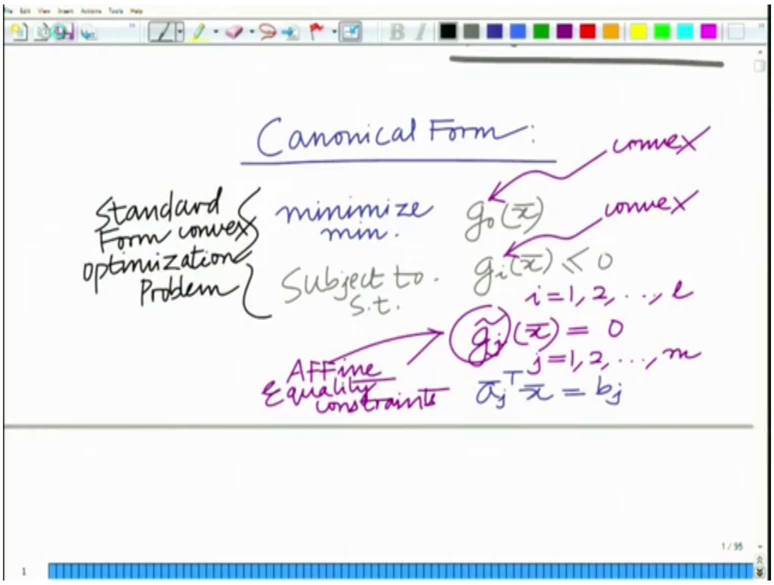

So, let us discuss let us begin our discussion so, as to speak on the formal framework of a convex optimization of a convex optimization problem. And now, a convex optimization problem as we have seen in the standard form, alright this can be thought of as a canonical form or the standard form of a convex optimization problem or a textbook convex optimization problem can be stated as follows; that is you minimise or this can also be sometimes written as min in fact, we frequently simply write as min period which means minimize an objective function, alright.

This is basically can be objective function vector can be objective function of a scalar. So, g naught of x bar subject to some constraints like we have seen so far or s dot d dot or t dot subject and these constraints can be g i of you can have any number of constraints g of x bar less than or equal to 0.

And now this objective function g naught has to be this has to be a convex remember for a convex optimization problem this objective function is to be convex I can have these constraints. Each of this constraint also is convex is a convex function I have i equals to 1, 2, up to l constraints and in addition I can have equality constraints g j tilde of x bar equals 0. And these have to be for instance j equals 1, 2, up to m and these have to be affine constraints so, equality constraints have to be affine in nature, is basically the implies that they are hyperplane.

So, you have to have constraints of the form a j bar transpose x bar equals b j. So, these are affine constraints or basically these are hyperplanes, so, the equality constraints. The affine the inequality constraints can simply be convex function and of course, the objective itself is a convex function, alright. So, this is a standard form of a convex optimization problem so, you can think of this as a standard form convex optimization problem.

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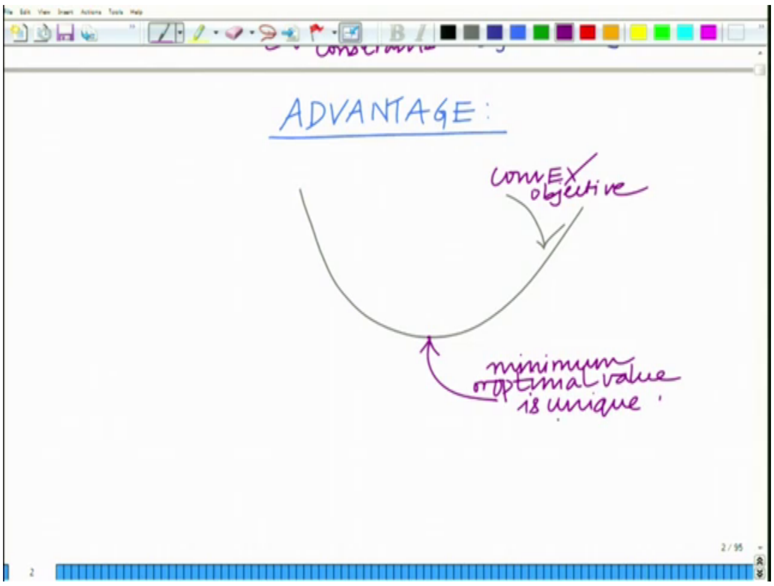

And, what is advantage of a convex optimization problem that we also formulated and the important advantage as you might already know for a convex optimization problem is a following that is when you look at a convex objective function and you minimise it the minimizer for so, this is your convex objective. And the minimizer is or the optimum value is unique, the minimizer need not be unique, but optimal value or you can think of this as the minimum or optimal value is unique ok.

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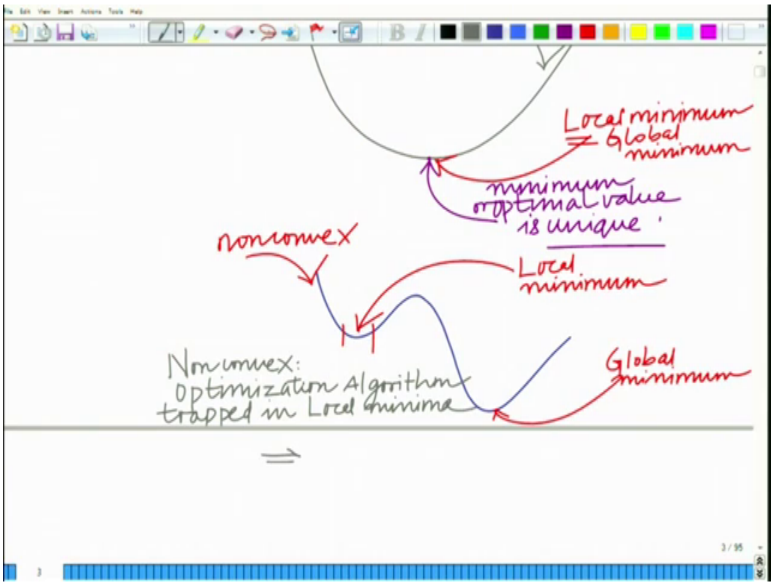

Now, as against if it is nonconvex for instance, alright, so, if it is nonconvex now what happens here is you have the concept of what is known as a low for instance. If we look at this here, this is also it appears like a minimum, but this only minimum locally that is in a local neighbourhood in a certain neighbourhood it is for instance if you look at this neighbourhood it is the minimum. But, it is not the minimum globally; the global minimum that is minimum over the entire domain is this.

So, this you have the concept of a global minimum and this is the local minimum ok, when the objective is nonconvex. However, here local minimum any local minimum is the global minimum ok. So, that is the advantage of convex that is any local minima is global minima. Here, in nonconvex there can be very many local minima and only one global minimum, alright. So, the problem is that the algorithms that is the optimization algorithms that you employ can get trapped in this.

So, they can get trapped in this local minima and they can yield spurious solutions which are not actually the minimum, which are not actually the optimal values of the objective function. So, for nonconvex you have this problem that for nonconvex the optimization routine or optimization algorithm is trapped that is a terminology used frequently, trapped in local minima.

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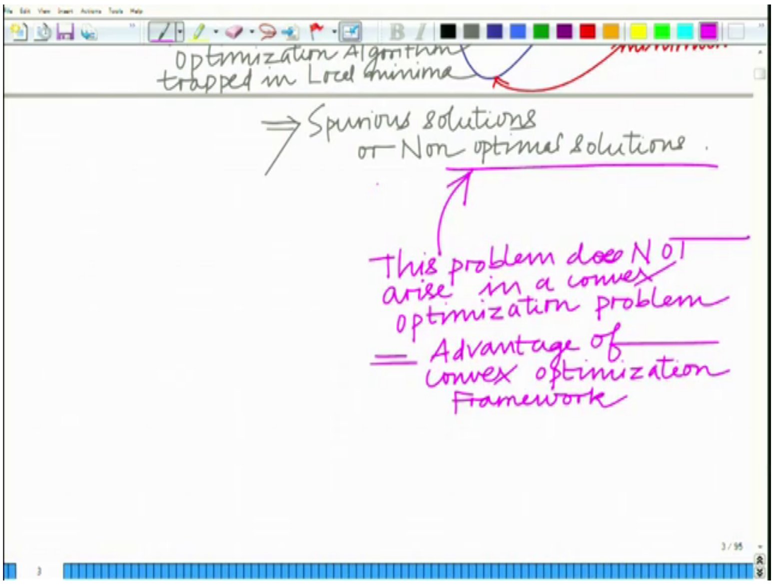

Implies there is a you have spurious solutions or non optimal solutions and this is precisely because, there is only a whatever is a local minimum for a convex optima objective function is a global or a convex optimization problem is a global minimum. So, this problem of spurious minima or local getting trapped in local minima is entirely avoided by a convex optimization problem.

So, this problem does not and that is the advantage of a convex. So, this problem does not arise in a convex optimization problem and that is the advantage of the convex optimization framework. That is the advantage of the convex optimization framework, that the algorithm because there are there are no local minima, alright you only have global minima so, the algorithm does not get trapped in local minima.

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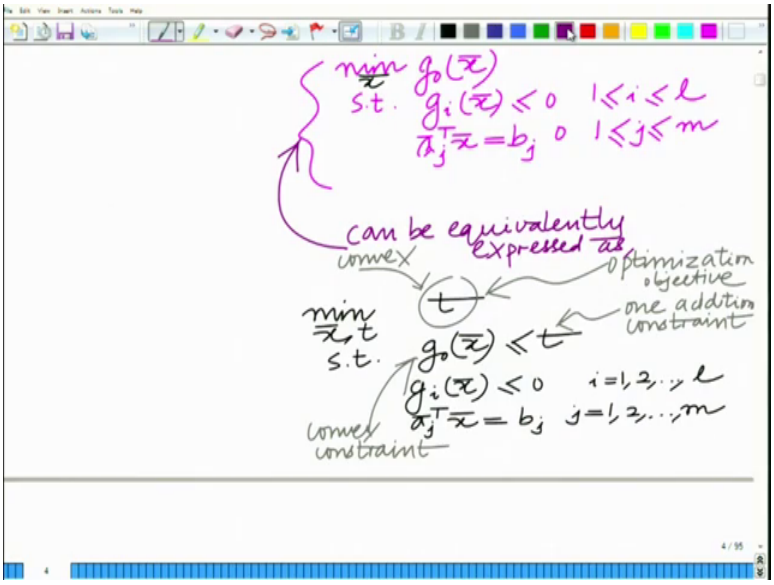

Now, a related form in that something convenient reformulation you can think of you can think of it is a convenient reformulation of a convex optimization is what is known as the epigraph form which I am going to discuss shortly. So, a convex optimization problem can be recast in what is known as it is epigraph form and that is the following thing that is remember we said you have a convex optimization problem minimized optimization objective g naught x bar subject to the constraints g of x bar less than equal to 0 for 1 less than equal to i less than or equal to l.

And g j tilde x bar these are affine constraints or I can write this directly as in fact, a j bar x bar equals b j a j bar transpose x bar these are hyperplanes 1 less than equal to j less than equal to m. And, now I can write this in epigraph so, this can be equivalently expressed as follows. What I am going to do is I am going to introduce an additional variable optimization. So, this optimization here is with respect to x bar I am going to introduce an additional optimization variable t that is minimise x bar comma t.

Now, I am going to minimise t and in the constraint now, I am going to add an additional constraint that is g naught of x bar less than or equal to t ok. And, the rest of the constraints remain the same that is g i of x bar less than or equal to 0, i equals 1, 2, up to l and a j transpose x bar equals b j; j equals 1, 2 up to m. So, these constraints remain same so, my optimization objective now become simply t, optimization objective simply becomes t and I have one additional constraint.

And the point is the this is the convex optimization problem because, if you look at this is simply linear this simply t. So, this is convex simply function of t this is convex and we already said the optimization objective g naught x bar is a convex function, right. So, g naught x bar less than or equal to t that is a convex constraint alright convex function less than equal to t, alright. So, that is allowed in a convex optimization problem.

So, this is a convex constraint and therefore, this is still a convex optimization problem. It is a very simple and elegant reformulation that simplifies many complex convex optimization problem.

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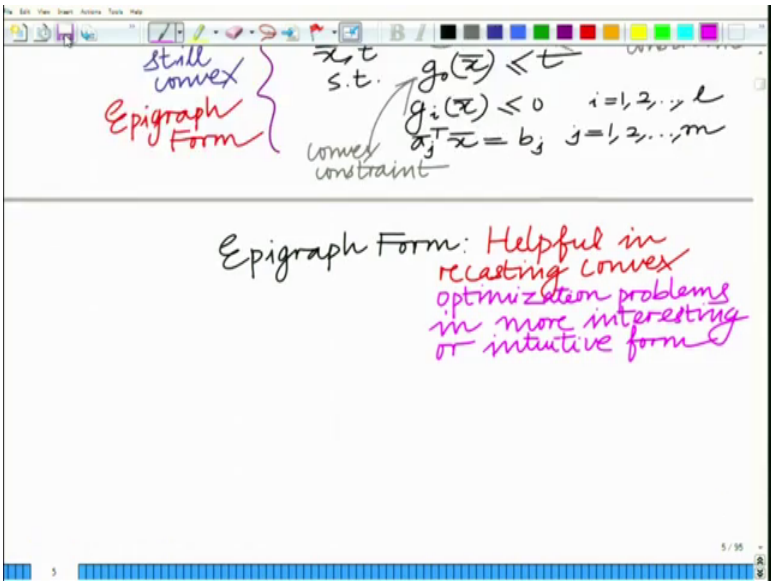

So, this modified optimization problem this is still convex and it yields the same solution that is the solution x bar of this problem optimal value that is optimal solution x bar of the second problem of this of the second problem is the same as the optimal value x bar that you obtain from the first problem. However, the second problem you are optimising both with respect to x bar and t and this is termed as the epigraph form, this is termed as the epigraph form of the problem ok.

And, in the epigraph form of the problem and as I already told you epigraph form it is helpful in recasting this convex optimization problem in a more interesting or intuitive form. So, if the epigraph form the advantage of epigraph form is basically it is helpful in recasting in recasting convex optimization problems in more in a more interesting or intuitive form in a more interesting or intuitive form.

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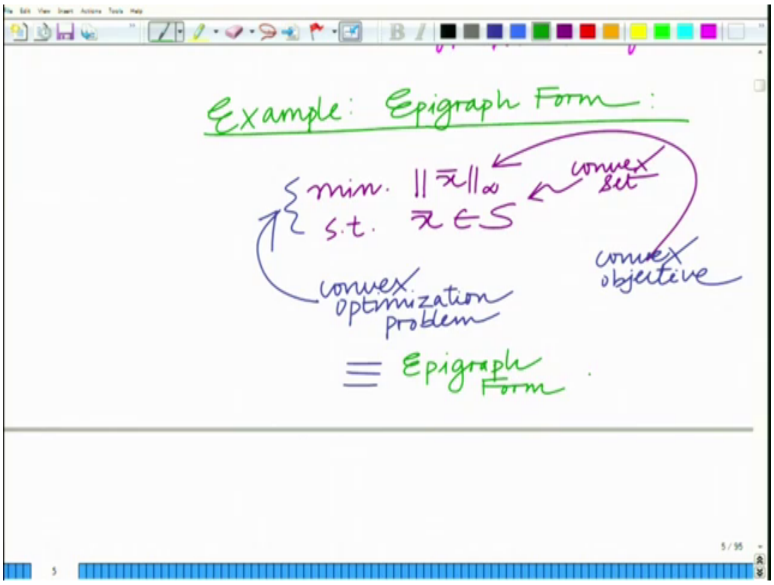

Now, let us look at a simple example to understand this better let us look at a simple example for this epigraph form and a simple example can be the following thing which is basically I want to. So, let us consider this problem minimize sometimes we can even it is frequently also written even omitting this period after min. So, this stands for minimise norm x bar the infinity norm subject to the constraint let us say there is some constraint there is x bar belongs to the set S that is it is a combination of linear and affine constraints.

Let us say some constraints this x bar must belong to the set S which basically a convex set, I am not too worried about this constraints. So, this is are of now look at this the set is convex. So, the constraints are convex and remember this is a norm infinite norm this is a convex norm so, the objective this is a convex objective ok. As a convex objective and therefore, this is in fact, a convex optimization problem ok. Now, this is in fact, a convex optimization problem and now what we want to formulate is the equivalent epigraph form? We want to formulate the equivalent epigraph form and that can be derived as follows.

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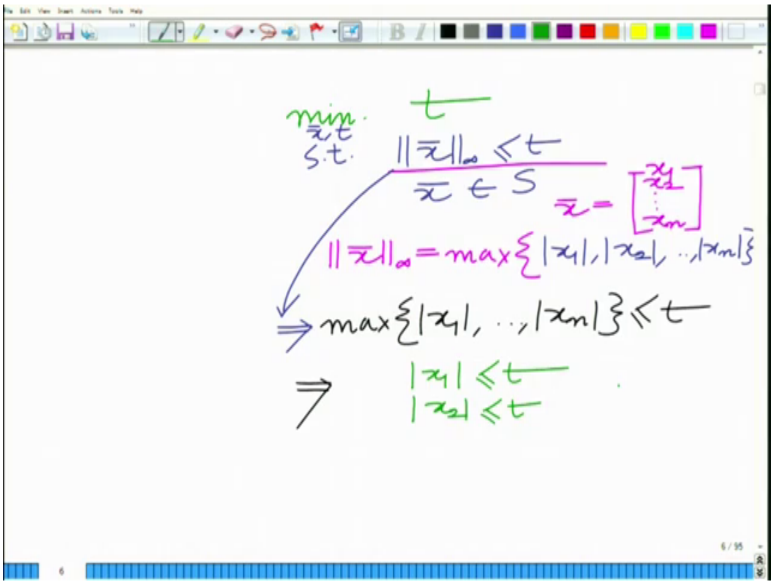

So, I can write this as the epigraph form minimise remember the epigraph form is straight forward minimize t, objective function is always t subject to the objective original objective. Of course, this minimization is over x bar comma t norm x bar infinity is less than equal to t and the original constraints remain that is x bar belongs to this set S so this is the epigraph.

Now, I can modify it slightly now if you look at this norm of x bar infinity. Let us say x bar is a vector ok, n-dimensional vector norm x bar infinity is nothing, but the infinite from is nothing, but maximum of magnitude x 1 comma magnitude x 2 magnitude x n. And when we say infinite infinity norm is less than equal to t so, this implies so, this constraints here this basically implies that the maximum of magnitude of x 1 up to magnitude of x n, this is less than equal to t.

Now, the maximum of n components or n quantities less than equal to t, that is possible only if each of the quantities is less than equal to t. So, this in turn now as leads to something interesting so, this implies that magnitude x 1 less than or equal to t magnitude x 2 less than equal to t so on and so forth magnitude x n less than equal to t.

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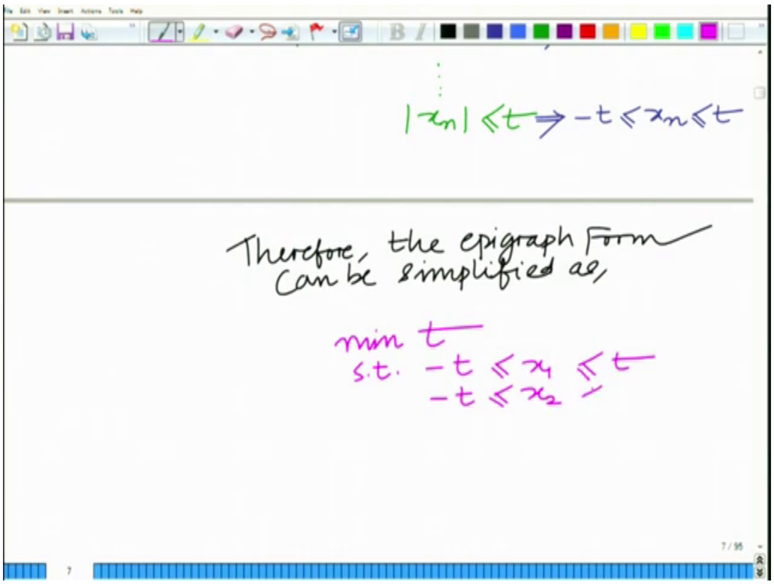

Now, magnitude x 1 less than equal to t this implies that minus t less than or equal to x 1 less than or equal to t. Similarly, magnitude x 2 less than equal to t implies minus t less than or equal to x 2 less than or equal to t and so on magnitude x n less than or equal to t implies minus t less than equal to x n less than equal to t.

And therefore, now the optimization problem above therefore, the epigraph form can be simplified as minimise of t minimise with respect to t subject to minus t less than or equal to each x i and right. Let me just write it explicitly to illustrate minus t less than equal to x 1 less than equal to t minus t less than equal to x 2 less than equal to t.

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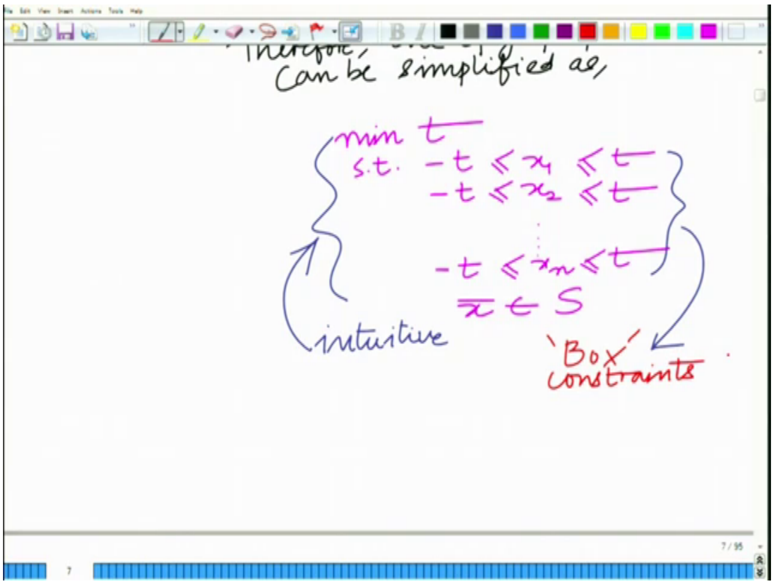

And, so on minus t less than or equal to x n less than or equal to t and original constraints are already the are always there that is x bar belongs to this set s which is the original constraint. And, this is something that is more intuitive and these are in fact, if you look at these are some sort of box constraints you can think of this constraints x bar to lie in a box of dimensions 2t.

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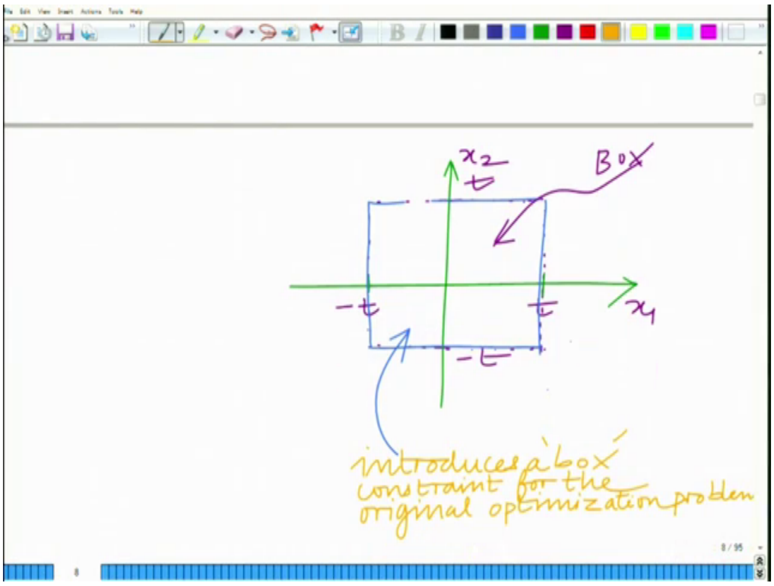

In fact, that is what you will see. If you consider a 2-dimensional scenario; that means, your x 1 so, if this is your x 1 and this is your x 2. So, x 1 has to lie between minus t and t x 2 has to lie between t and minus t. So, that effectively limits your area that is your x 1, x 2 has to lie in this so, this is the box ok so, this is the box in which your x 1, x 2 has to lie, alright. So, this is basically your you can think of this as your box constraints alright so, basically introduce introduces a box constraint for the original optimization.

So, introduces a box constraint you can say that it introduces a box constraint for the original optimization problem ok. So, it sort of introduces a box constraints for the original optimization problem. And, this is something that is it gives you an interesting either gives you better intuition or it also gives you an interesting interpretation for the original optimization problem which is in fact, an identical it is an equivalent optimization problem. But, it is sort of opaque, alright one can it is not a minable to derive insights, alright.

So, then this modified optimization problem is something that is more interesting and it is easy to interpret and probably also analyse sort of without using regress analytical tool sort of analyse it more or analyse or interpret it sort of simply by looking at optimization problem, alright. So, this is an important in fact, we one can use this and we are going to also use it from time to time to simplify or recast optimization problems, alright. So, we will stop here and continue in the subsequent modules.

Thank you very much.