Applied Optimization for Wireless, Machine Learning Big Data

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Lesson - 45
Least Norm Signal Estimation

Hello, welcome to another module in this massive open online course. So, we are looked at the least squares paradigm; let us look at it is analogue or a counterpart or something that interestingly related to that which is known as the Least Norm Paradigm and that can be described as follows.

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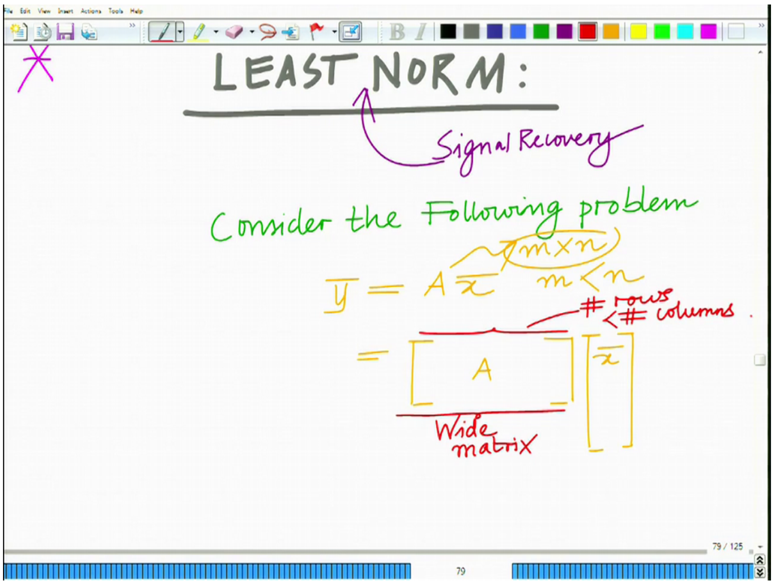

And these two always go hand in hand this is we so, far we have seen the least squares paradigm. What you want to do now is the least norm framework and this can also be used for Signal Recovery or you can also think of this as single estimation ok. And the least known paradigm is as follows so, consider the following problem where in we have again y bar equals A times x bar and A is an m cross n matrix similar to what we have seen previously.

But while previously m is greater than n in the least squares framework we will consider a framework where m is less than n. That is the number of rows is much lower than the number of columns so if you look at this, it will look like this which is this is your matrix

A this is your vector x bar this implies that number of rows is much less the number of columns. One can call this as a wide matrix; just like when you are number rows is more than the number of columns we call that as a as a tall matrix ok.

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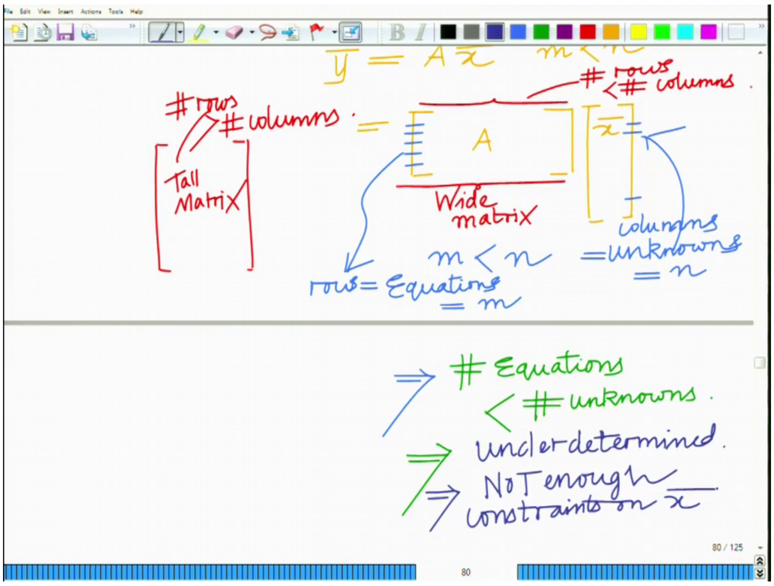

Remember previously in the least squares we had a tall matrix ok; number of rows is more than the, this is number of rows is more than the number of columns. Now we have a wide matrix that is number of rows that is m is less than the number of columns ok. Which basically implies number of equations as you remember this is the number of rows each row is an equation.

Remember row equals equation and each of this is an unknown each element of x because x is the unknown signal you can say is unknown. So, each column you can say each column of a correspondence to an unknown ok. So, rows equal equations equals m columns equals unknowns equals m. So, this implies that for this kind of system when m is less than n; this implies that number of equations is smaller than number of unknowns which implies that system is under determined, there are not enough constraints, under determined not enough constraints implies not enough constraints on x bar.

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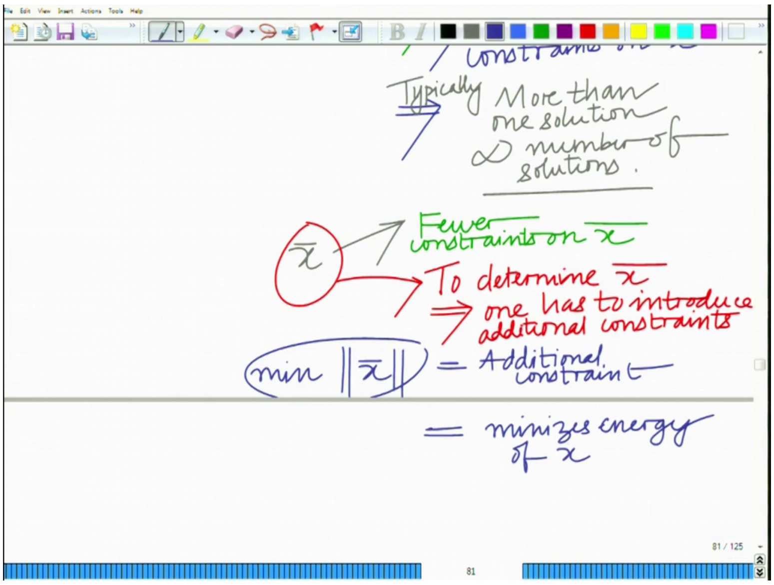

And when the system is under determined that is there is not enough constraints not enough equations there are only n equations, but n unknowns. This means that typically there is no unique solution or typically there are more than one possible there are infinite number of solutions. This implies typically again let me qualify this typically with verify propelled typically more than one solution or infinite or an infinite number of solution. Now, therefore, how to determine now therefore there is an infinite number of solution infinite number of possible solutions.

So, there are fewer constraints alright there are fewer constraints on x bar ok. And therefore, how do you determine x bar which means you have to additionally constrain x bar naturally if there are fewer constraints the only way to fix x bar or determine the possible value of x bar is to introduce additional constraints ok. So, therefore, to determine x bar, one have to introduce additional constraints, one has to introduce additional constraints. And therefore, one such difficult constraint is to find the energy efficient solution or if you look at norm x bar minimize the norm of x bar so this is your additional constraint.

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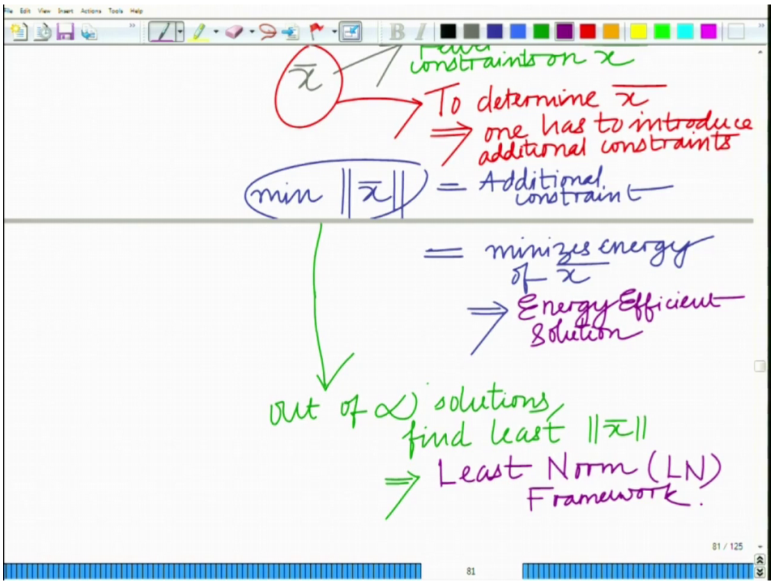

It basically minimizes the energy of x bar which implies that the solution is energy efficient; implies that you are trying to find an which implies you are trying to find a energy efficient solution. That is what is it is a out of all the so remember we said it is says infinitely many solutions all it x bar is not unique infinitely many possible values of x bar because it is an undetermined system. Out of all these x bar find the one that is minimum norm that has the minimum energy, that is the solution that we desired then that is how we are constraining this problem.

At this it is precisely known as the least norm problems. So, this means out of infinitely out of infinite solutions find the one that has least norm. So, therefore, this is known as the least norm framework least norm or minimum norm, least norm you can also say min norm least norm the thing is the previous one was least square as remember we had no solution. So, find the one that minimizes the approximation error y bar minus A x bar or y bar minus h x bar. Here we have more than one solution we have infinitely many solutions. So, find the one that has minimum norm ok. And I think I am using yeah the matrix A and strainley straight forward. And therefore, the relevant optimization problem for this least norm solution can be formulated as follows.

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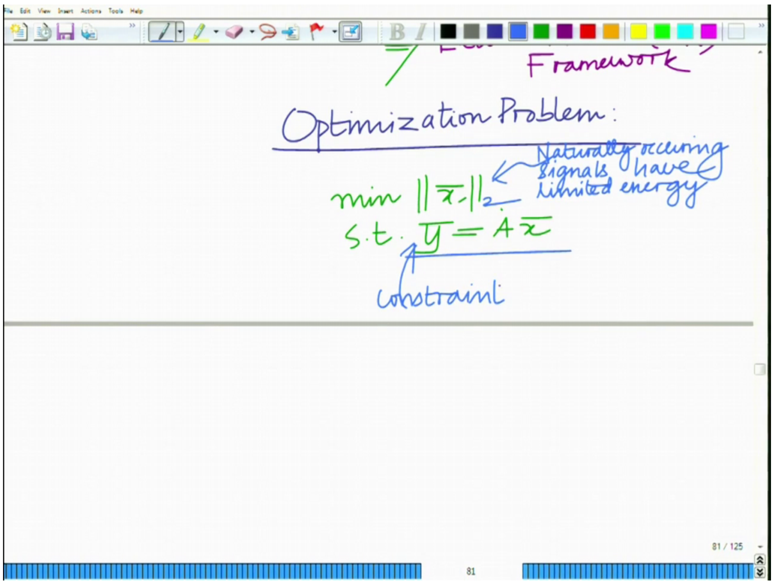

The relevant optimization problem for this is as you have already seen the objective function is to minimize norm x bar and the constraints now is y bar equals A x bar. Even also justify this minimum normal that naturally occurring signals have do not have an infinite number of energy infinite amount of energy, they are typically limited in terms of energy. So, therefore, we want to make sure that the signal corresponds to something that is naturally occurring which means such energy is bounded.

So, this is justified because naturally occurring signals have limited. In fact, we will see an interesting version of this later when naturally occurring signals will say have a sparsity. They are naturally sparse in nature, but for us to begin with let us look at the minimum norm solution. In fact, this is the minimum two norm and this is basically this linear system this is your constraints; so, minimum norm that is objectives. So this is the constraint for our optimization problem.

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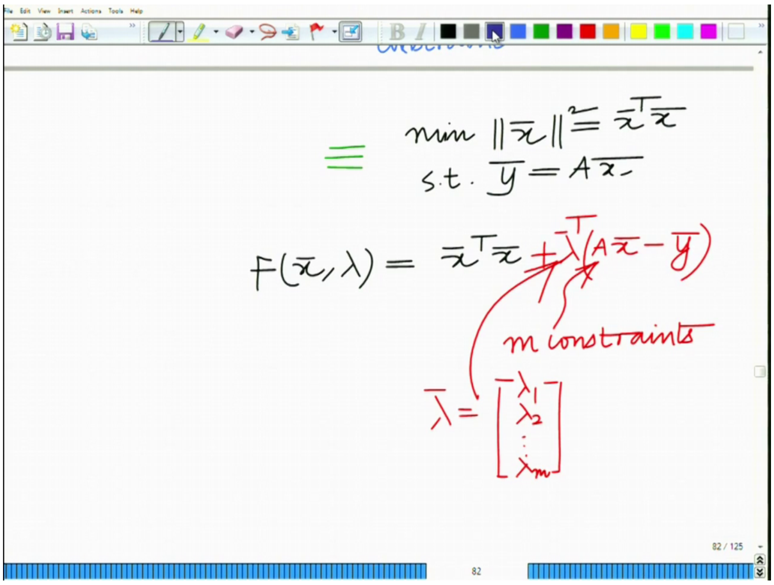

And this can be solved as follows so what I can do is I can write this equivalently solving this is fairly straightforward we can use earlier techniques minimum norm x bar square subject to the constraints y bar equals A x bar. Therefore, one can form the lagrangian which is equal to objective function x bar norm x bar square is nothing, but x bar transpose x bar plus lambda times A x bar minus y or y minus A x bar A x bar minus y bar. In fact, this has to be lambda bar transpose remember because how many constraints we have we have m constraints; each row is an equation. So, there are m equation so m constraints so there has to be one Lagrange multiplier for each constraints. So, your lambda bar will in fact be a vector so that is basically lambda bar.

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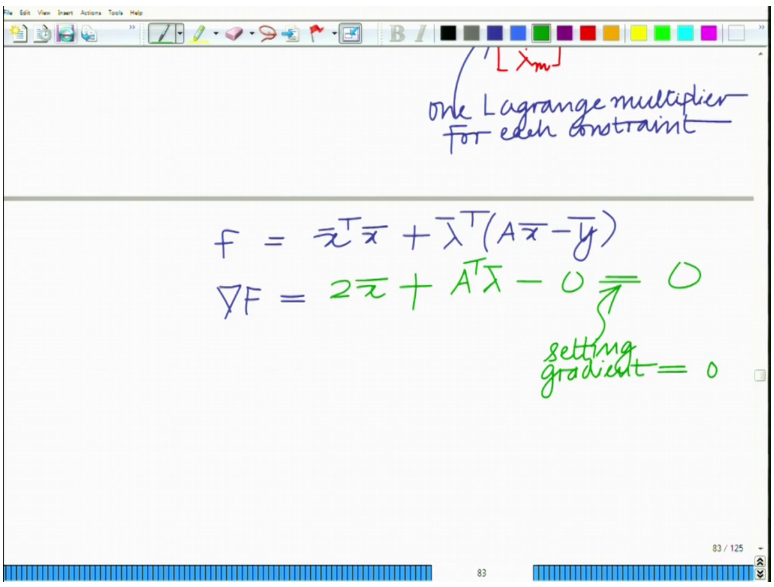

So, this is one lagrange multiplier for each constraint it is one lagrange multiplier for each constraint. And now when you take the gradient of this so your F equals x bar transpose x bar plus lambda bar transpose A x bar minus y bar. Now we take the gradient we have done this before x bar transpose x bar is nothing, but x bar transpose I identity types x bar. So, this is twice x bar plus lambda bar transpose A into x bar is c bar transpose x bar so the gradient of this is c bar. So, this will be a transpose lambda bar a transpose lambda bar minus lambda bar transpose y bar gradient with respect to x bar is 0 and this is equal to 0 setting gradient. So, you are setting the gradient setting the gradient equal to 0.

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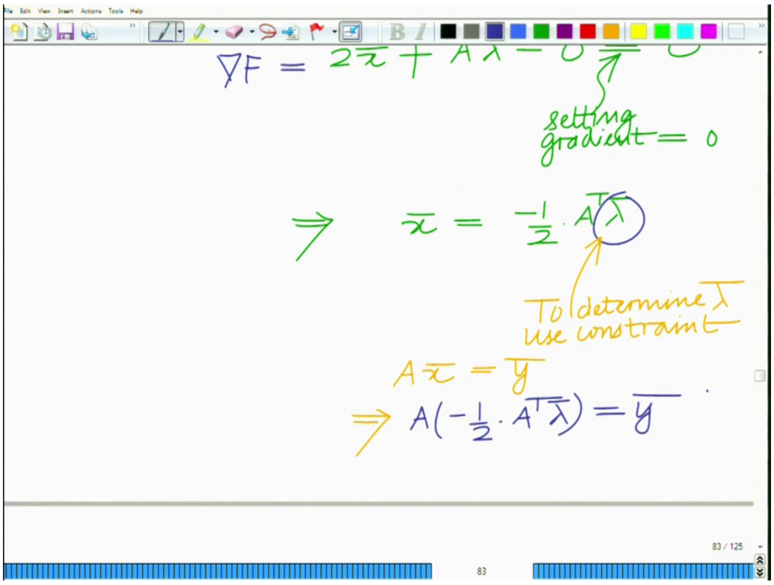

And once you solve this thing this implies you will get something interesting this implies that x bar equals minus half a transpose lambda bar remember lambda bar is a vector. So, I cannot simply get it or manipulate it otherwise so I simply have to write x bar equals minus half lambda minus half a bar transpose a transpose lambda. How to determine lambda bar? Use the constraint this similar to what we have to determine lambda bar use the constraint. Remember or constraint is A x bar equals y bar substitute x bar which implies A minus half A transpose lambda bar equals y bar which implies minus half A, A transpose lambda bar equals y bar which implies that lambda bar equals minus twice AA transpose inverse into y bar.

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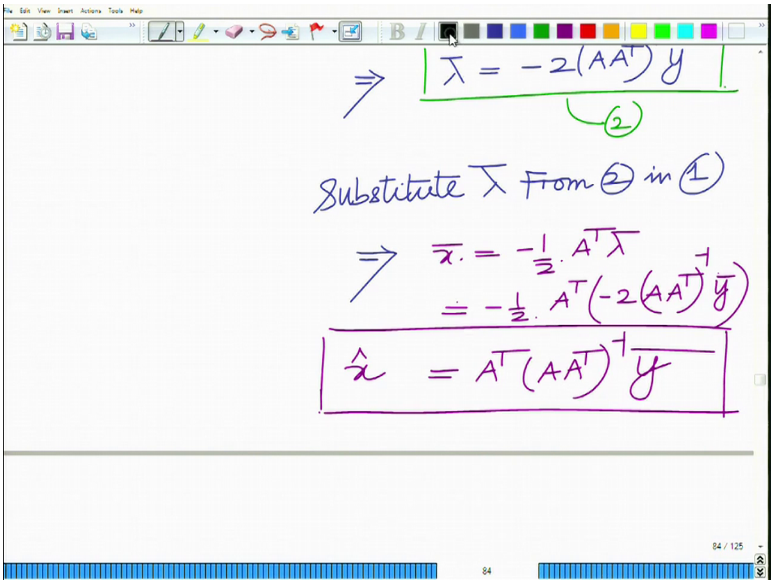

So this is basically so, if you call this as 2 this is expression for lambda bar and if you call this as 1. Now what we able to do substitute lambda bar from 2 in 1 and what we get here is that lambda bar equals we have already seen this x bar equals x bar equals minus half a transpose lambda bar substitute lambda bar minus half A transpose minus 2 AA transpose inverse into y bar so the minus half and minus 2 cancel. So, this will be a transpose A transpose inverse into y bar this is your x hat, for your signal estimate that has the signal estimate that has the least norm correct. So this is basically your least norm.

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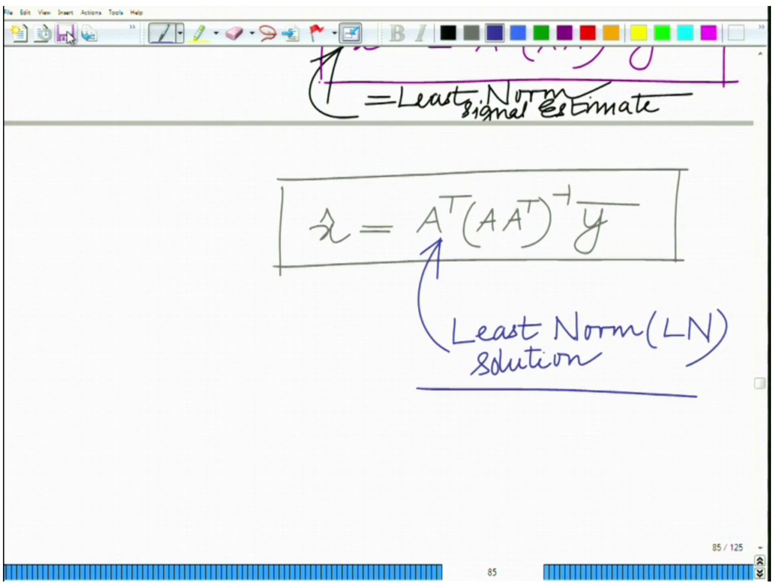

What we have obtained is the least norm this is the least norm signal estimate and this also known as the least known solution. So, you can write this as x hat equals A transpose AA transpose inverse into y bar ok. So, this is also known as the least norm solution or minimum norm we can also there are many names. So, this is also known as the least norm solution alright that gives you the solution x hat which has the minimum 2 norm. And as I already told you this is suitable or well suited for scenarios where there are it is a under constraint system.

That is your pure equations than unknowns your more unknowns which means there infinitely possible infinitely many possible solutions. So, we have to constraint we have to introduce additional constraints we want to find the solution one the one which has the minimum norm of the minimum energy. And this expression gives you close form expression for the minimum norm solution alright we will stop here.

Thank you very much.