Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture - 40
Practical Application: Detailed Solution for Robust Beamformer Computation in Wireless Systems

Hello, welcome to another module in this Massive Open Online Course. So, we are looking at Robust Beamforming. And we also looked at the optimization problem for robust beam forming, and the solution the structure of the solution although we could not derive it in detail. What I am going to do in this module is I am going to derive the optimal robust beam former as I have already told you the derivation is slightly involved.

So, I advise you that if you are looking; if you are going through this for the first time, and if you are not interested in knowing the intricacies of the derivation, you can skip this module. If you are interested in delving deeper into this, you can follow the derivation and understand how this is the robust beamformer is derived all right. And I am going to illustrate the step by step procedure to derive this robust beamformer W bar that is the exact structure of the robust beamformer as well as the procedure to determine the Lagrange multiplier lambda all right.

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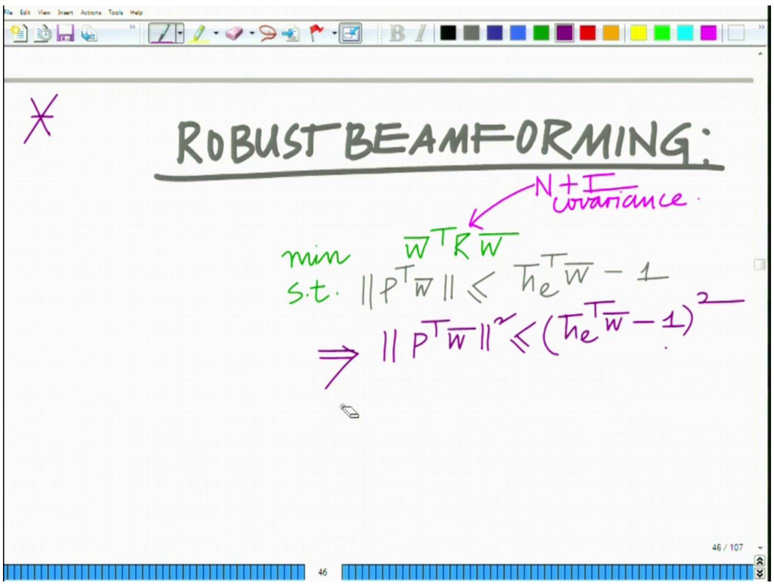

So, we start with the robust beam forming problem. So, again let us just title this; this is your robust beamforming minimize W bar transpose R W bar subject to the constraint that norm of P transpose W bar less than or equal to norm of h bar e transpose W bar minus 1, this is our robust beamform of problem we said. This is the quadratic constraint W bar transpose R W bar of course, R is a positive semi definite matrix is the noise plus interference covariance correct it is the N plus I noise plus interference covariance. And this is a conic constraint norm of P transpose W bar less than equal to h e bar transpose W bar minus 1.

Now, what this implies is first of all if we look at this, this implies that if you look at the constraint that implies that P transpose W bar norm square is less than or equal to h bar e transpose W bar minus 1 square which you can also say that.

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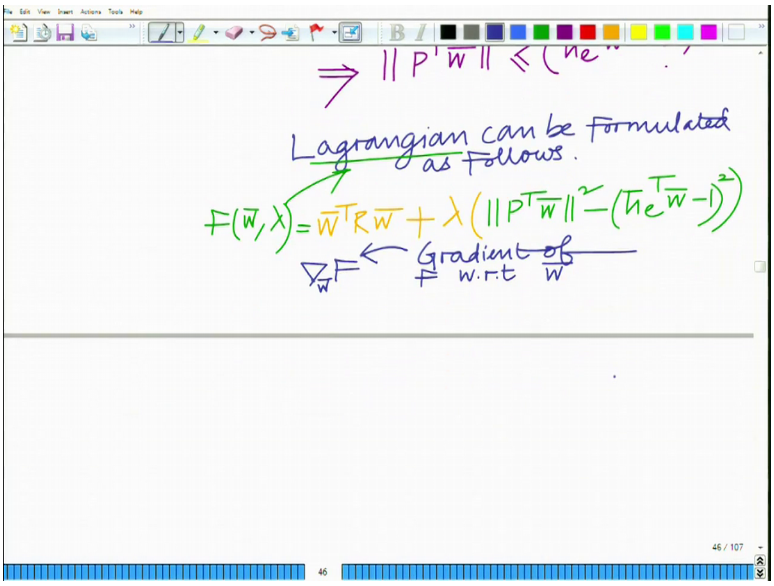

So, now what we are going to do is, so the Lagrangian can be formulated as follows. The Lagrangian can be formulated as follows so, that will be W bar transpose R W bar plus lambda times we write the constraint that is norm P transpose W bar square minus h bar e transpose W bar minus 1 whole square. So, this is the Lagrangian, Lagrangian it is obviously a function of you can write this as a function of F of W bar comma lambda. So, this is your Lagrangian. Now, differentiate take the gradient with respect to W bar. So, gradient of F with respect to regular W bar with respect to W bar.

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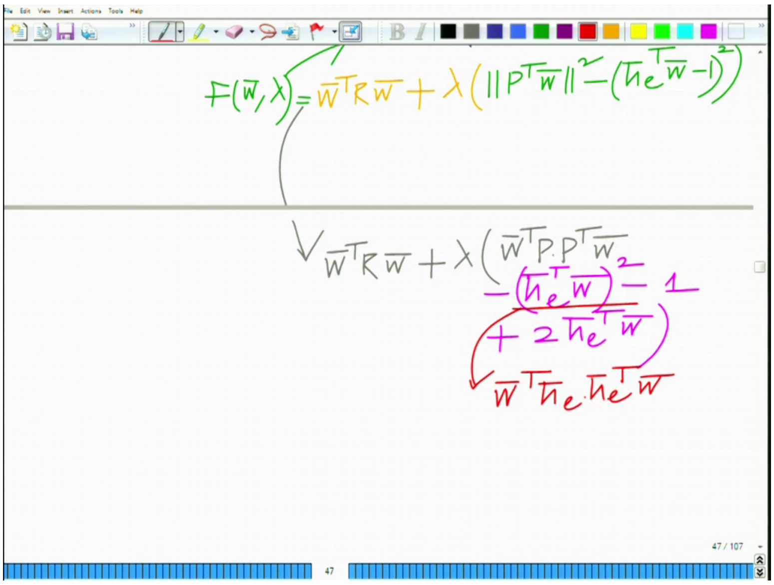

And what this is W bar transpose R W bar. So, this we know this is twice R W bar. Now, even before we take the gradient let us simplify this further. So, this will be before you take the gradient, let us simplify this a little bit further. So, this will be W bar transpose R W bar plus lambda times now norm of a vector square P transpose W bar square is the transpose of a vector times itself, so that will be W bar transpose P into P transpose W bar minus square of this quantity. Square of this will be well it will be h bar transpose e W bar whole square minus 1 plus twice h bar e transpose W bar.

And further now this h bar e transpose, now this is scalar quantity h bar e transpose W bar, so I can write this as W bar transpose h bar e this transpose of the vector times itself. So, W bar transpose h bar a times itself which is again h bar e. So, basically you can also say this is the magnitude square of this both these quantities are equal. So, I can write this a scalar quantity. So, I can write R s transpose of the quantity times itself, so that is W bar transpose h e bar times h bar transpose W bar.

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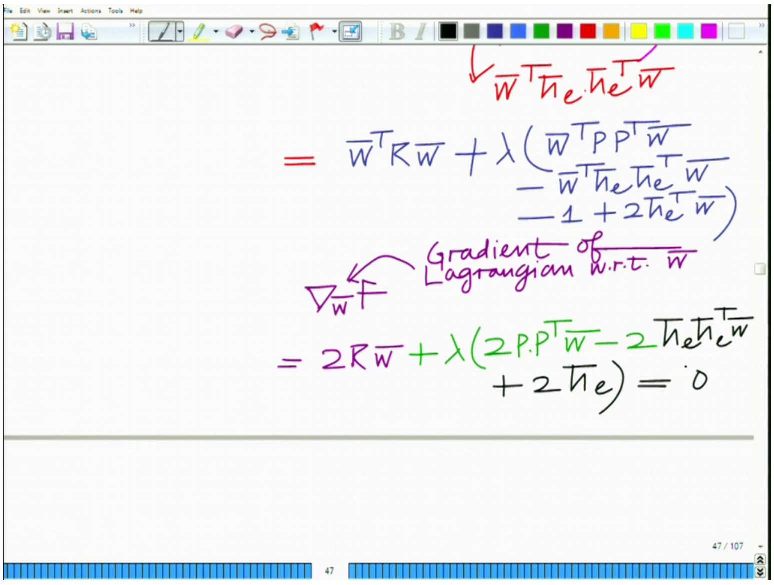

So, this is again equal to, so the objective function again as I have said slightly involved can be simplified as one has to be careful to write each and every term at every stage. So, W bar transpose P P transpose W bar minus W bar transpose h e bar h bar e transpose W bar minus 1 plus 2 h bar e transpose h bar e transpose W bar. And now what we will going to do, we are going to take the gradient with respect to gradient of the Lagrangian with respect to F, for gradient of the Lagrangian with respect to W bar.

And this will be equal to the gradient of the Lagrange plus R W bar you can clearly see this is twice R W bar we know this plus lambda 2 P transpose W bar minus twice, again W bar transpose h bar e h bar e transpose W bar. So, this is minus twice again h bar e h bar e transpose W bar minus of course, when you differentiate minus 1, it will be 0 plus twice c bar transpose W bar that is c bar twice h bar e, this is equal to 0.

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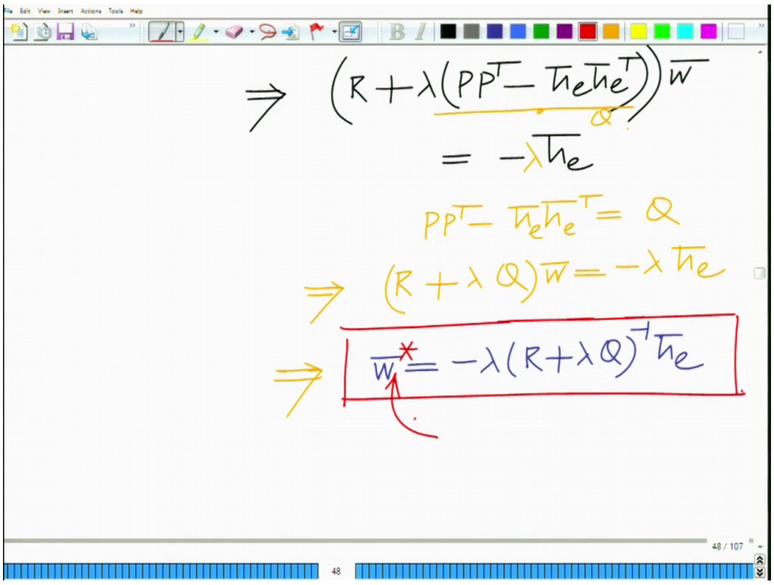

And now if you simplify this, so this basically implies now setting it equal to 0 gradient all right considering the gradient of the Lagrangian with respect to W bar and we are equating this to 0. Now, if you solve this thing, this will be twice R W bar or the factor of 2 this will cancel. So, you can remove this factor of 2. So, this will be R W bar or R plus lambda P P transpose minus h bar e h bar e transpose, this whole thing into W bar equals minus h bar e.

And, now if you set this matrix as Q, so we have P P transpose minus h bar e h bar e transpose equals Q. So, this implies R plus lambda Q into W bar; I am sorry there is going to be another factor of lambda over here so minus lambda h bar e. So, this is equal to minus lambda h bar e. And this implies that and therefore, that gives us the equation which is W bar equals R plus there is going to be minus lambda minus lambda times R plus lambda Q inverse minus lambda into R plus lambda Q inverse into h bar e. This and this you can see, this is therefore you can write this as star this is the optimal robust beamformer which we are already seen.

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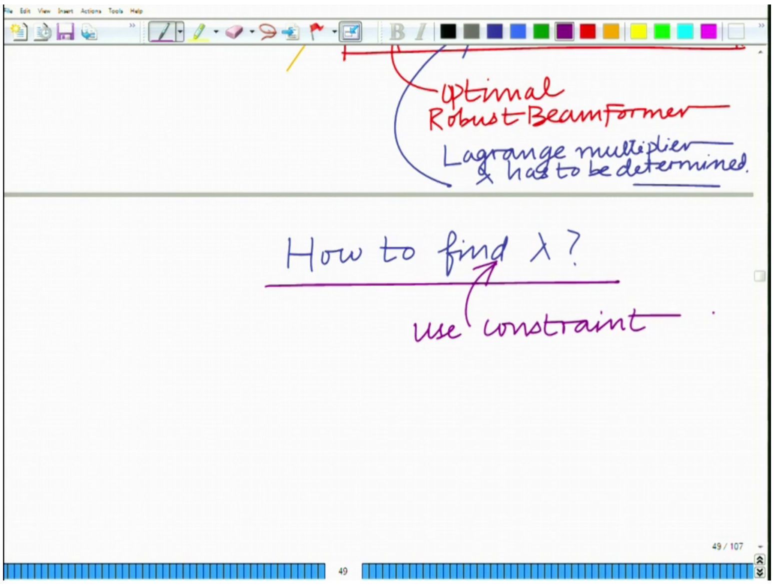

So, this is the formal derivation for there. This is the optimal robust performer all right. And this of course, depends on the Lagrange multiplier lambda. And therefore, what we have to do is that we have to determine the Lagrange multiplier lambda to complete this derivation. So, and that is a little bit involved so, we have to derive the Lagrange multiplier. So, Lagrange multiplier m lambda has to be determined. How to find the Lagrange multiplier lambda, for that use the constraint for this we use the constraint.

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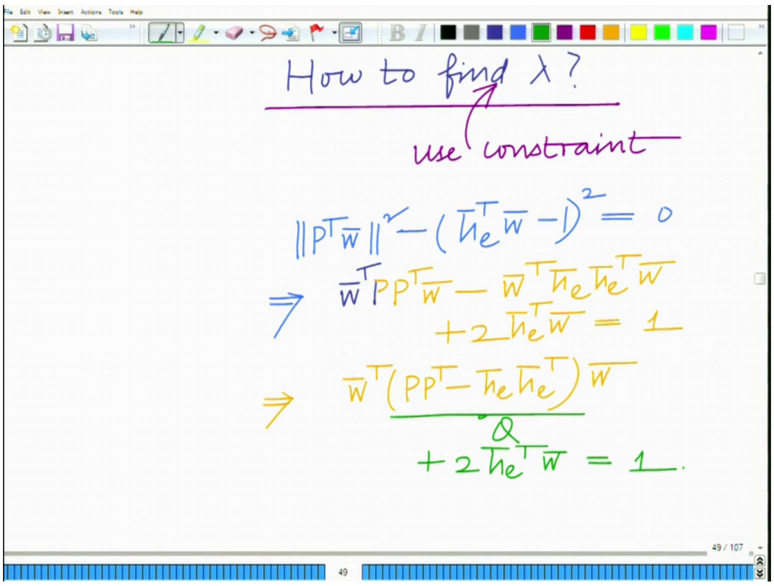

And what is the constraint? Remember our constraint is that norm P T W bar square minus h bar transpose e into W bar minus 1 whole square equal to 0. This implies that W bar transpose P P transpose W bar minus W bar transpose h bar e h bar e transpose W bar plus twice h bar e transpose W bar equal to 1. So, basically this is the constraint, now again you can combine this P P transpose minus h bar h bar e transpose. So, what you get is W bar transpose P P transpose minus h bar e h bar e transpose W bar. And this is nothing but our matrix Q that we had seen plus of course, the other terms are there h bar e transpose W bar equals 1.

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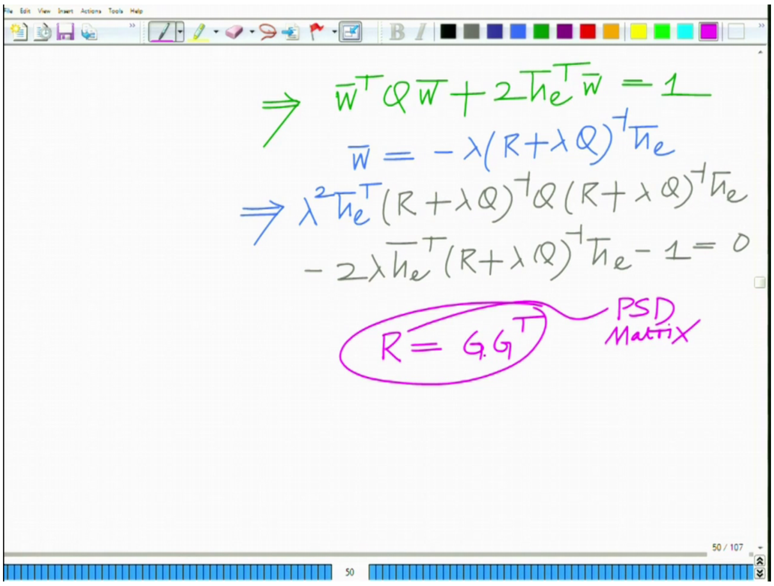

This so now, you can substitute this, this implies that W bar transpose Q W bar plus twice h bar e transpose W bar equals 1, very good. Now, what we are going to do is we are going to substitute the solution for the optimal d beam formal that is we have remember W bar optimal is minus lambda R plus lambda Q inverse h bar e. Now, once you substitute this w, but this value that is this expression for the optimal v former what you will get is the following expression. You will get lambda square h bar e transpose R plus lambda Q inverse Q R plus lambda Q inverse into h bar be h bar e. It is a slightly lengthy expression minus twice lambda. However, you can easily verify it by substituting it h bar e transpose R plus lambda Q inverse h bar e minus 1 this is equal to 0.

Now, what we are going to do is we are going to substitute R equals G G transpose. Now, employing this is remember R is the noise plus interference covariance matrix. So, G G transpose is a decomposition all right, every positive semi definite matrix because this is a PSD matrix, this is a covariance matrix. So, this is PSD matrix every PSD matrix can be decomposed as G some matrix G times G transpose.

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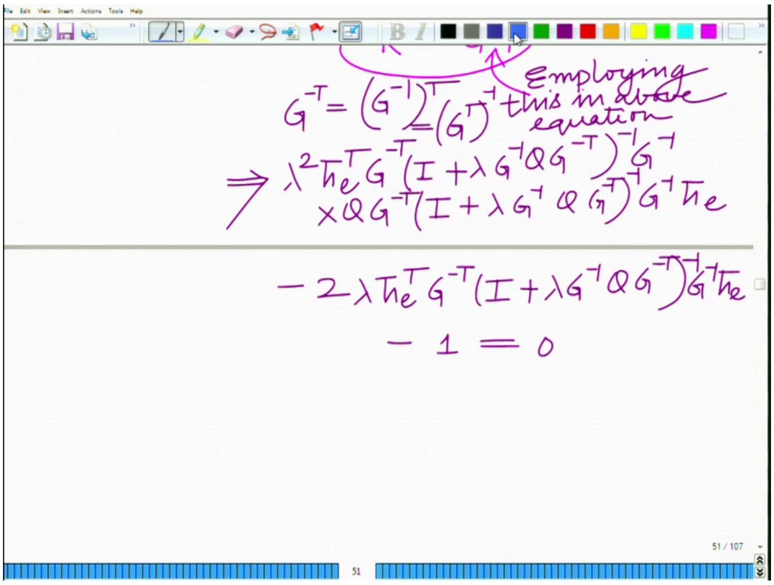

Now, substituting this above or employing this in the above equation, what we have is this implies gets further involved lambda square h bar e transpose G minus transpose which means transpose of G inverse. So, you can think of G minus transpose or there was G transpose inverse of G inverse transpose both of them are the same, G transpose inverse G minus transpose I now plus lambda you can verify this G inverse Q G minus transpose inverse G inverse times Q G minus transpose I plus lambda G inverse Q G minus transpose inverse G inverse into h bar e. And it continues minus twice lambda h bar e transpose G minus transpose I plus lambda G inverse Q G minus transpose inverse into G inverse into h bar e minus 1 equal to 0. So, this is the equation that you get after you substitute R equal to G G transpose.

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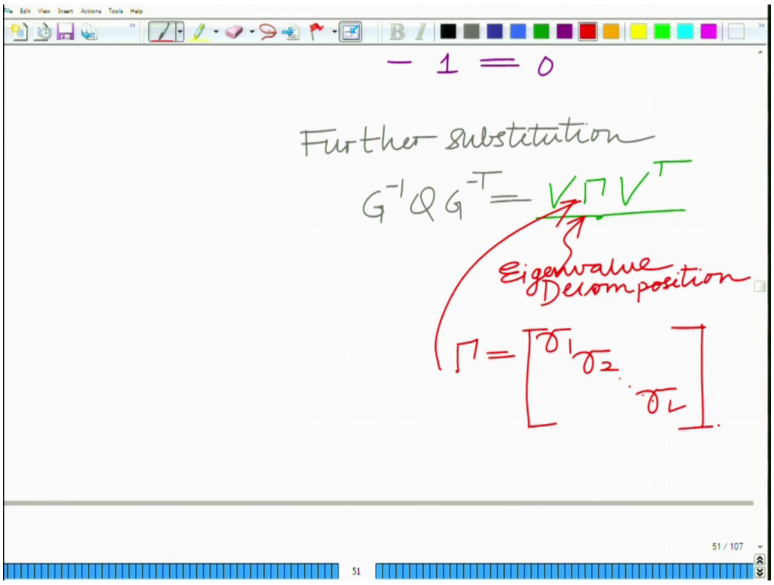

Now, we are going to do a further substitution all right, which will simplify this actually. And we are going to see how. So, now we are going to employ a further substitution. And the further substitution is if you look at this matrix looks like this matrix is the key, because this appears repeatedly. So, G inverse Q G minus transpose, we employ its eigenvalue decomposition as follows that is V gamma V transpose. So, this is basically the eigenvalue decomposition of this matrix G inverse Q G inverse transpose. So, this is eigenvalue decomposition and this matrix gamma is the diagonal matrix of eigenvalues. So, this will be gamma 1, gamma 2 and so on and so forth up to whatever is the dimension which is gamma L. So, this is diagonal matrix of eigenvalues.

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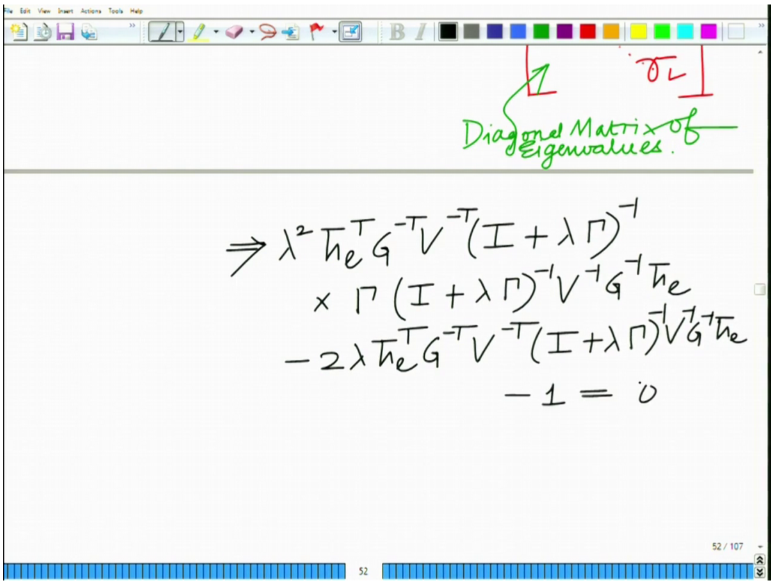

This is the diagonal matrix of eigenvalues. And once you employ this substitution this implies. So, once you employ the substitution then that is the substitution that we outlined about G inverse G G minus or G transpose inverse Q G transpose inverse or G inverse G inverse transpose equals V gamma V transpose. This reduces to, well this reduces to lambda square h bar e transpose G inverse minus transpose V minus transpose I plus lambda gamma inverse times multiplied by gamma I plus lambda gamma inverse V inverse G inverse h bar e minus twice lambda h bar e transpose G inverse transpose V minus transpose I plus lambda gamma inverse into V inverse G inverse h bar e minus 1 equal to 0.

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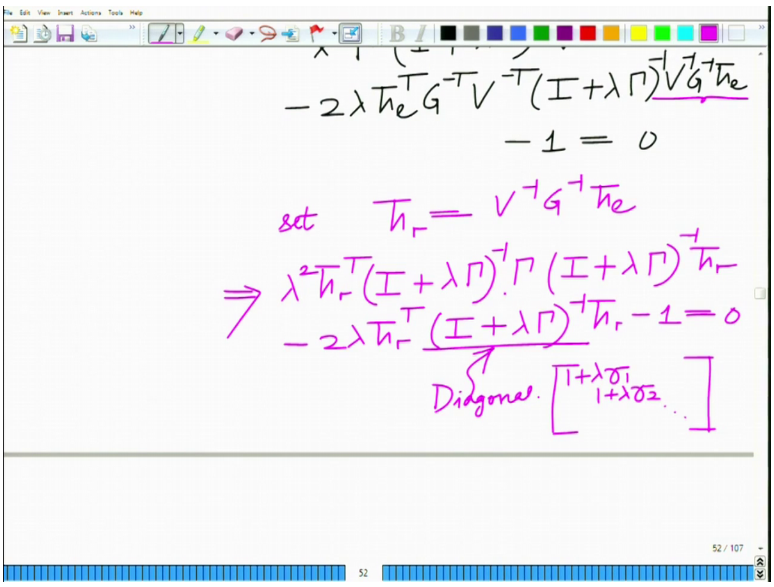

Now, this can be simplified as follows. If you look at this vector we set it as h r. So, we set h bar r so, set h bar r equals V inverse G inverse h bar e. And this implies, so the equation above reduces to the following lambda square h bar r transpose I plus lambda gamma inverse gamma I plus lambda gamma inverse h bar r minus twice lambda h bar r transpose I plus lambda gamma inverse h bar r minus 1 equal to 0.

And now you can see this is a very simple structure because if you look at all these matrices, all these are diagonal matrices I plus lambda gamma. If you look at this, this is a diagonal matrix gamma is a diagonal matrix this is a diagonal which is in fact, it is entries will be 1 plus lambda gamma 11 plus lambda gamma 2 and so on. So, all these matrices are diagonal matrices, so you can basically multiply it out and you can simplify this.

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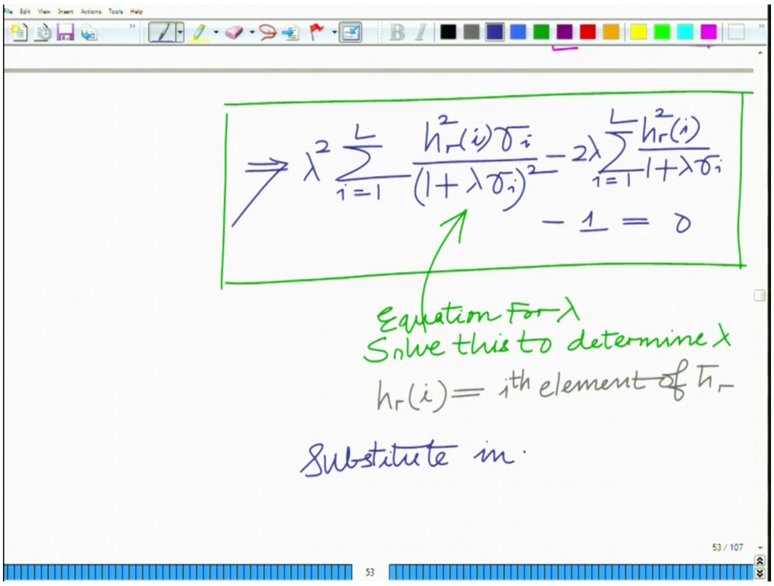

And once you simplify it, the resulting equation that you get will be the following. And you get the final equation for the Lagrange multiplier lambda and that is given as interestingly that is given as this lambda square summation I equal to 1 to L h r I am going to explain these terms in a minute. So, h r i square, this ith component of the vector h i h r divided by gamma I 1 plus lambda gamma i this is 1 plus lambda gamma square minus twice lambda i equal to 1 to l h r square i divided by 1 plus lambda gamma all this 1 plus lambda gamma I terms are coming from the i plus lambda gamma inverse. You can clearly see that minus 1 equal to 0. And this is the equation for lambda, this is the equation.

So, you solve this equation determined lambda, we solve this to determine lambda. And h r i as I already explained equals ith element of the vector h bar r which we defined above. Once you find lambda, so solve this equation you find lambda substitute. Now, you have to once you substitute in basically your earlier equation for the optimal beam former which is nothing but W bar star equals minus lambda R plus lambda Q inverse h bar e.

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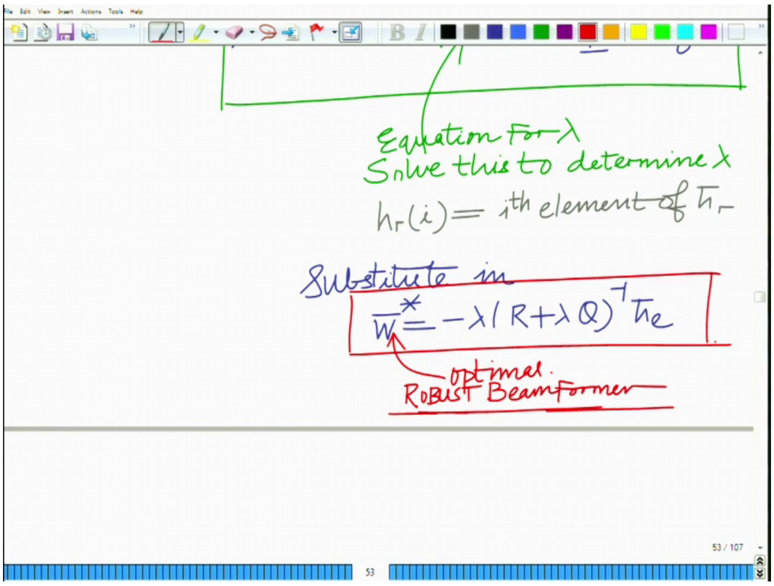

And this will give you gives this, so this is the optimal beam former. This is the equation for the optimal beam so that basically gives the equation for lambda, you substitute this to get the optimal robust beamform. In fact this is not just any beam former this is the optimal robust beam former which is robust to the uncertainty in the channel state information or uncertainty in the knowledge of the uncertainty in the knowledge of the channel coefficients or channel vector in the multi antenna system. All right so, we will stop here and continue with other aspects in the subsequent modules.

Thank you very much.