**Applied Optimization for Wireless, Machine Learning, Big Data**
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**Lecture – 04**
**Inner Product Space and its Properties: Linearity, Symmetry and Positive Semi-definite**

Hello, welcome to another module in this massive open online course. We are looking at the mathematical preliminaries for optimization; let us continue our discussion with another concept namely an inner product space, ok.

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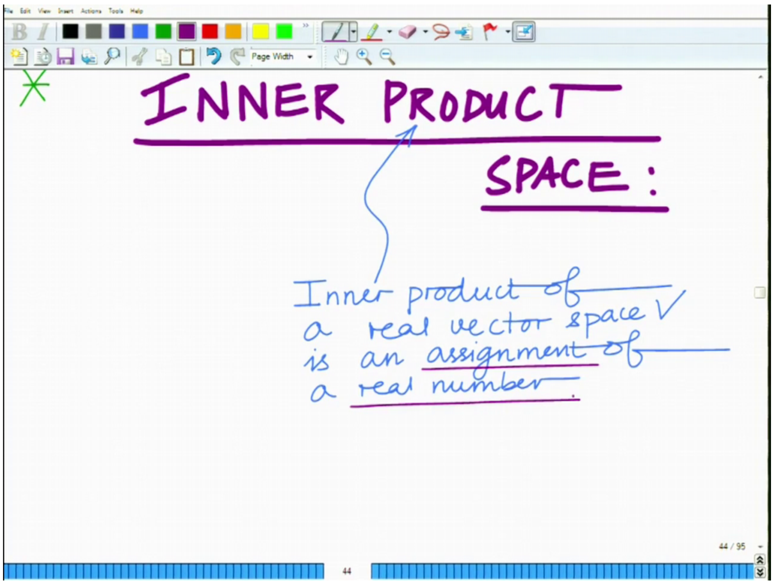


So, you want to start looking at the concept of what is known as an inner product space. Now, what is an inner product space? Now, an inner product space of a real vector; now the inner product of a real vector space; the inner product of a real vector space is an assignment of a real number, is an assignment of a real number.

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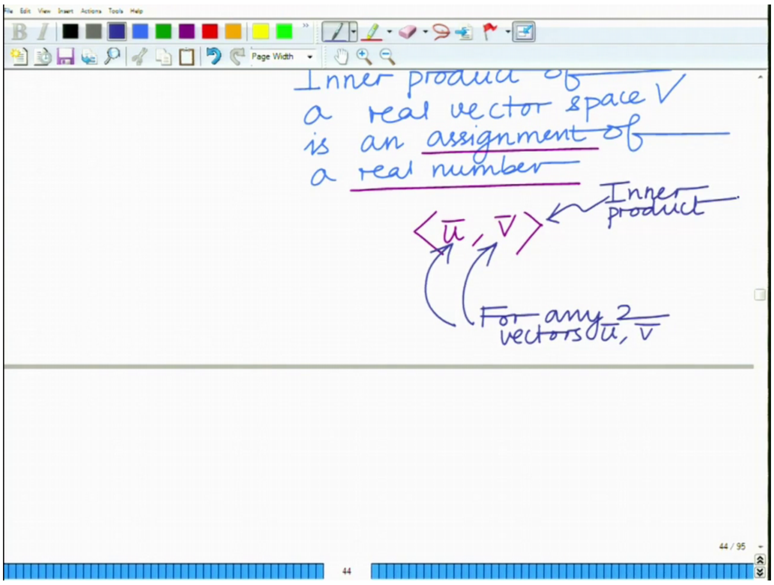


And for any 2 vectors u bar v bar that is, denoted by this notation u bar, the inner product of u bar comma v bar and this is defined for any 2 vectors u bar b bar in this case real vectors for any 2 vectors u bar comma v bar. And this is the inner product, which is a real number; in the case of a real vector space and which satisfies the inner product satisfies the following properties.

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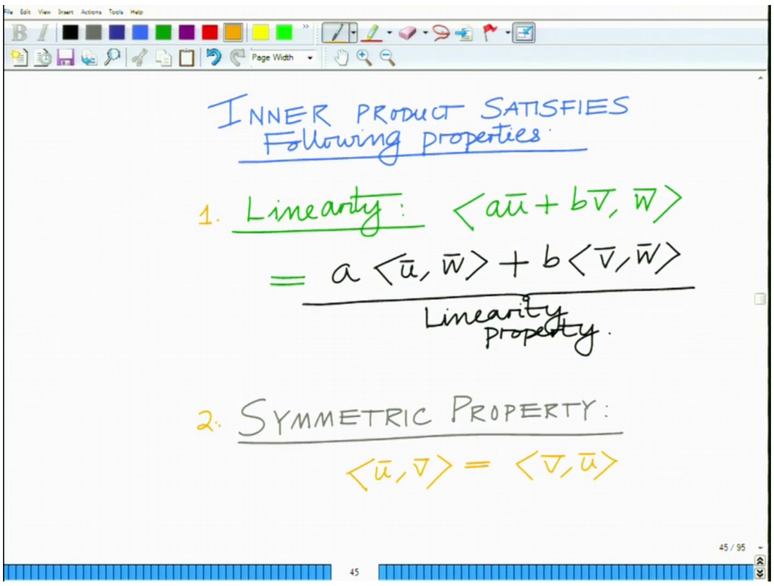


The inner product of the vector; satisfies the following properties. First property is linearity, which is a u bar plus that is the inner product of a linear combination a u bar plus b v bar is the linear combination of the inner products that is, this is a times u bar inner product of u bar w bar plus b times the inner product of v bar comma w bar ok.

So, this basically is the linearity property that is a linear, the inner product of the linear combination a u bar plus b v bar with the vector, w bar is the linear combination of the inner products a times the inner product u bar w bar plus b times the inner product of v bar comma w bar; this is the first property. Then we have the symmetric property or the symmetry. It is very simple that is the inner product of u bar v bar equals the inner product of v bar comma u bar. Then, we have so if you call this property number 1 linearity to symmetry then we have the positive semi definite property.

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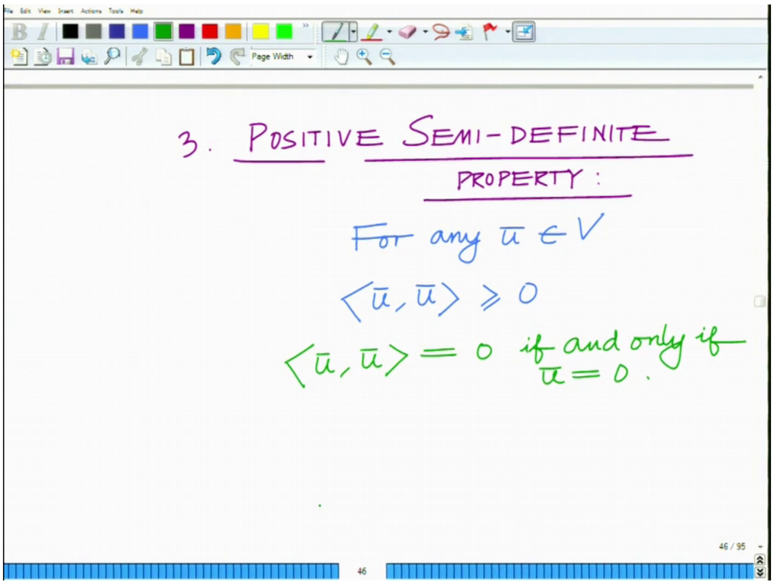


The positive semi definite property it must be the case that for any u bar, that is for any u bar element of the vector space v then the inner product of u bar with itself must be greater than or equal to 0. And more importantly and also u bar the inner product with itself equals 0, if and only if u bar as itself 0. So, inner product is 0 if and only if the vector u bar is 0.

So, these are so this is the definition it is an assignment of a real number for a real vector space it satisfies the linearity, symmetric, asymmetry and the positive semi definite properties ok. Let us look at a simple example to understand this for instance the standard product, let us consider the standard dot product between 2 vectors alright.

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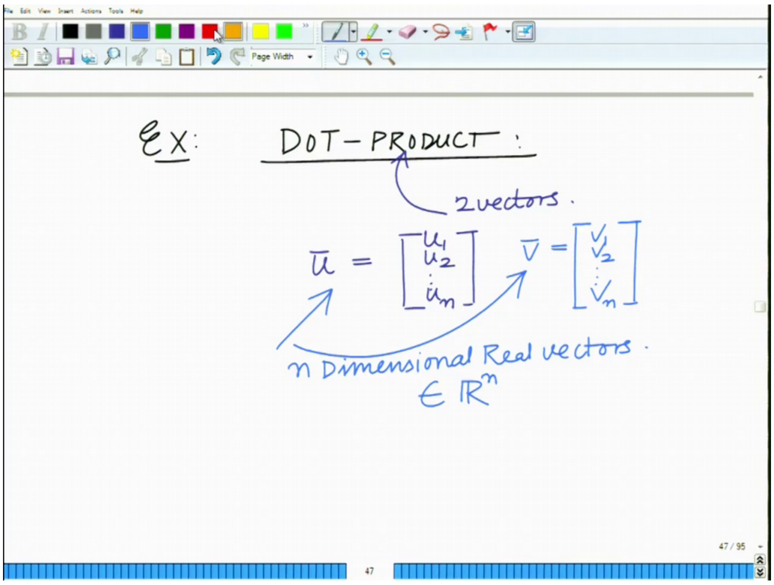


Consider 2 vectors the dot product is defined for between 2 vectors consider u bar equals u 1, u 2 up to u n and v bar equals v 1 v 2 up to v n. Then now these are real vectors n dimensional vectors that is you can say that belong to the space of n dimensional real vectors denoted by this bold R, R to n. And this is also termed as the Euclidean n space.

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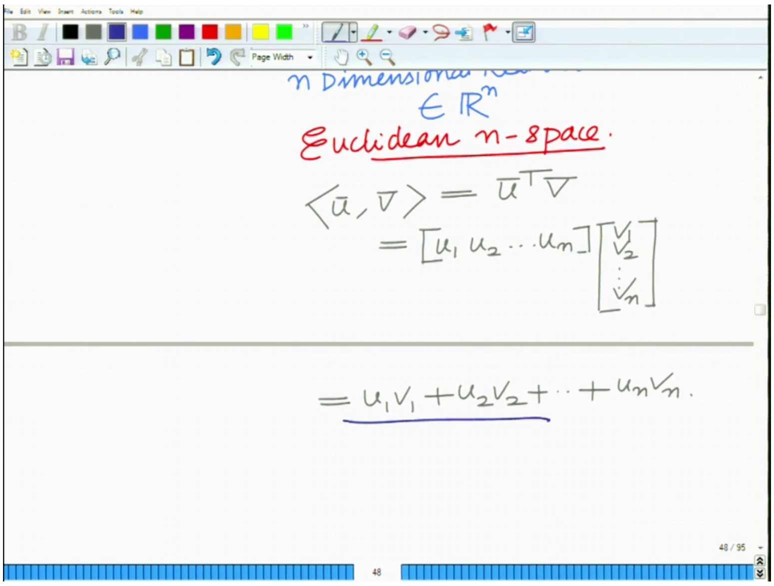


This is also termed as the Euclidean n space and the inner product in this Euclidean n space between vectors u bar and v bar is defined as u bar transpose v bar, which is basically if you look at this that is u 1, u 2 u n. Row vector times the column vector v 1, v

2, v n. Which is basically u 1, v 1, u 1, v n ok. So, this is the definition of the dot product ok. So, this is the dot product between 2 n dimensional real vectors.

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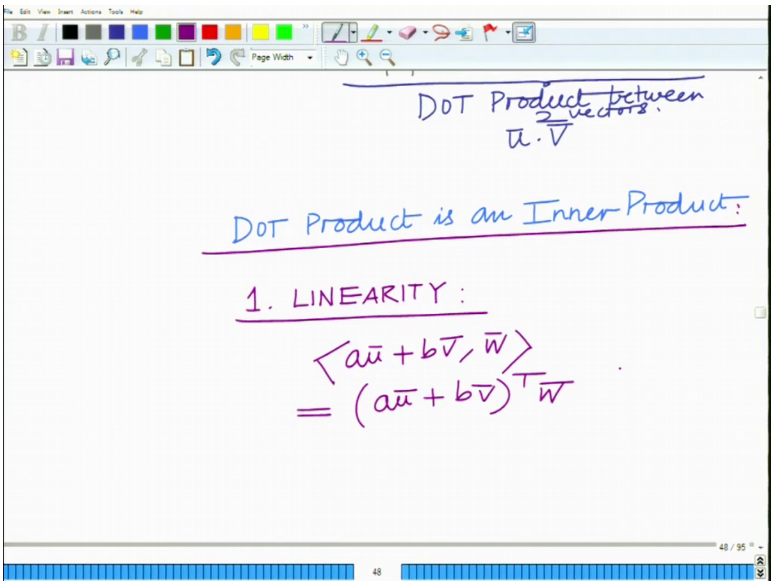


Something that you might have well see in your in your vector space or vector calculus class in high school all right. This is a standard dot product which is also denoted by the dot operator that is u bar dot v bar ok. This is the dot product between 2 vectors. In fact, more specifically to real n dimensional vectors, now we will show that dot product is an inner product.

Now, that is very simple to see, first let us look at the linearity property. If you look at a u bar plus b v bar dot product with w bar that is simply a u bar plus b v bar transpose times w bar, as we have seen that is the definition of the dot product.

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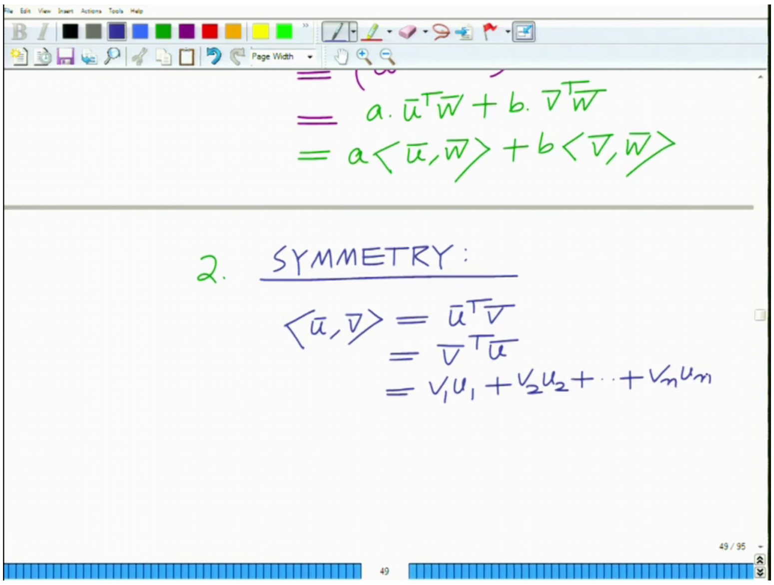


Which is a times u bar transpose w bar plus b times v bar transpose w bar which is a times the inner product of u bar with w bar plus b times the inner product of v bar with w bar ok, so it satisfies the linearity property. Now 2 now coming to the symmetry property, we have u bar dot product v bar which is equal to u bar transpose v bar, which is basically v bar same as v bar transpose u bar which is v 1 u 1 plus v 2 u 2 plus v n u n which is v bar u bar and 3.

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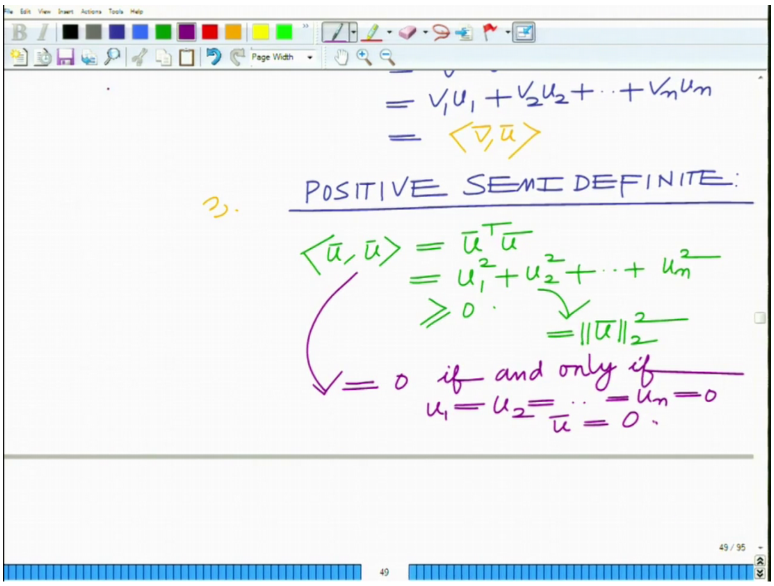

So, symmetry satisfies a symmetric property, now we have to look at the positive semi definiteness. Now positive semi definite property, now you can see that the inner product of u bar with itself is u bar transpose u bar, which is u 1 square plus u 2 square plus u n square; which is greater than or equal to 0 further. In fact, we have seen this is nothing but this is used to define the 2 norms square that is, this is the l 2 norm square norm u bar square which is greater than or equal to 0 in fact, equal to 0 if and only if some of the squares is 0, which means each of the components is 0 u 1 equals u 2 equals u of n equals 0 which means u bar equals 0.

So, it is positive semi definite it is positive semi definite that is u bar inner product of u bar with itself is always greater than equal to 0 it is 0 only when u bar the vector is identically 0 all right. And therefore, the standard the dot product is an inner product. And in fact, this is also termed as the standard inner product, this is also termed as the standard inner product on R n that is the Euclidean n space. Or the n dimensional set of n dimensional space of real vectors Euclidean n space or the n dimensional space of real vectors the dot product is an inner product.

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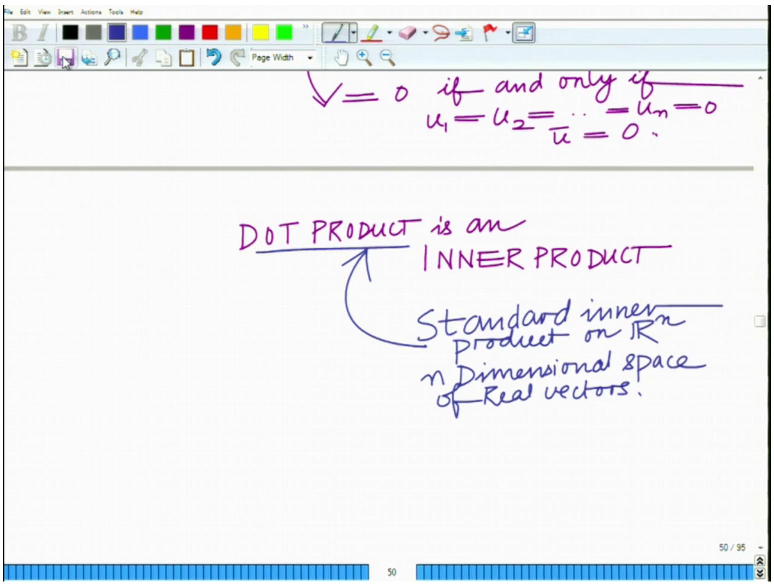


Let us now consider another example for 2 dimensional vectors instance.

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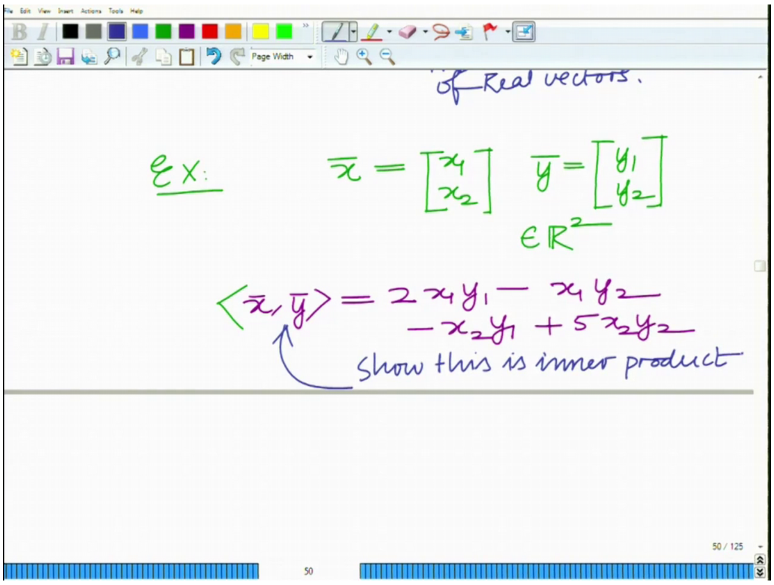


We have x bar equals x 1 x 2 and y bar equals y 1 y 2. So, both of these are basically 2 d vectors that is there belong to the 2 dimensional Euclidean space. And let us define this assignment x bar y bar for this 2 2 d vectors as twice x 1 y 1 minus x 1 y 2 minus x 2 y 1 plus 5 x 2 y 2. Now what we want to do is we want to show that this is an inner product; this is a valid this assignment is a valid inner product and this can be shown as follows as usual we start with the linearity property.

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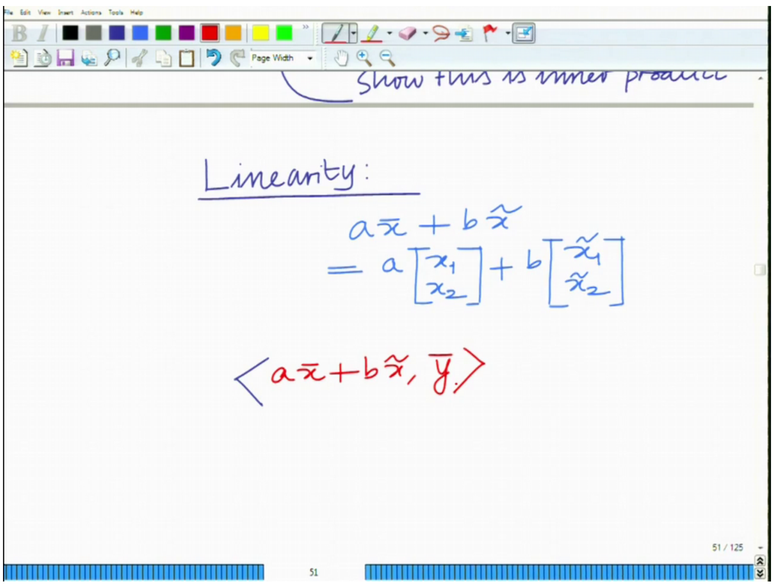

Now let us consider a vector that is a x bar plus b x tilde that is; a times x 1 x 2 the 2 dimensional vector plus b times x 1 tilde x 2 tilde all right. So, we are taking a linear combination of the 2 vectors x bar and x tilde. Now for this to be an inner product let us consider now a x bar plus b x tilde the inner product with any vector y bar.

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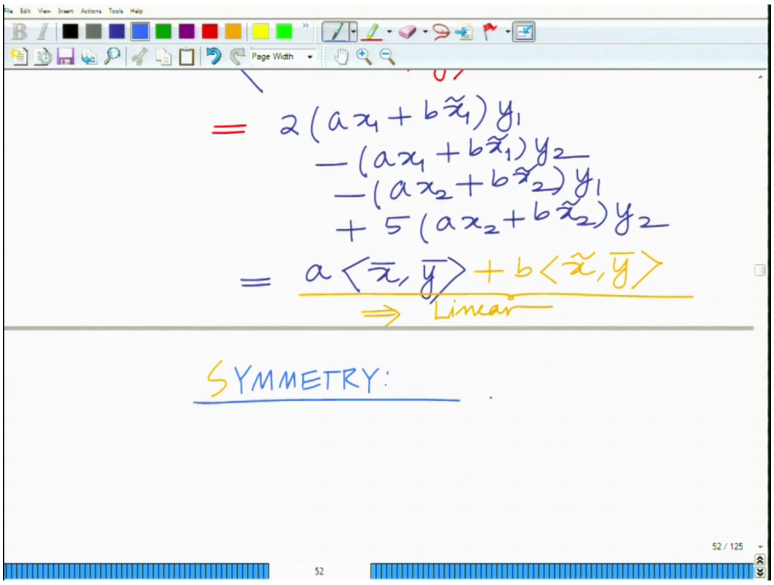


Now, this can be shown as now, we can see this is equal to twice a x 1 plus b x 1 tilde into y 1 minus a x 1 plus b x 1 tilde into y 2 minus a x 2 plus b x 2 tilde into y 1 plus 5 times a x 2 plus b x 2 tilde into y 2. And now you can clearly see this is linear that is this can be written as a if you separate the components and write it you can easily see that this is a times x bar comma y bar plus b times x tilde comma y bar and therefore, it is linear ok. So, satisfies implies that this is linear alright. So, we have shown that the assignment is linear, now let us know symmetry the symmetric property and this can be shown as follows.

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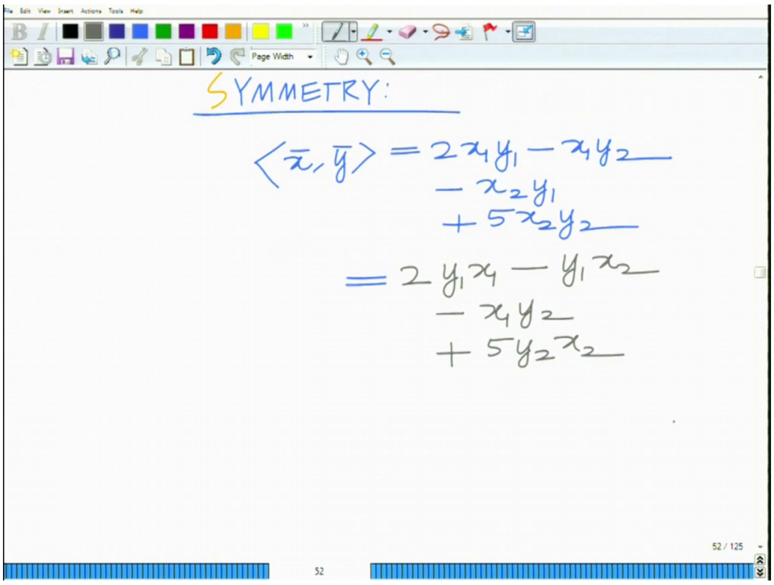


So, coming now to the aspect of symmetry you can see that x bar y bar this is equal to twice x 1 y 1 minus x 1 y 2 minus x 2 y 1 or minus y 2 x 1, I am sorry minus x 2 y 1, minus x 2 y 1 plus 5 x 2 y 2 ok. And now this can also be written as without any effort you can see this is twice x y 1 x 1 interchanging the terms the second inter term this will be minus y 1 x 2 minus x 1 y 2 plus 5 y 2 x 2. And you can readily see that x 1 y 2 I will write as minus y 2 x 1.

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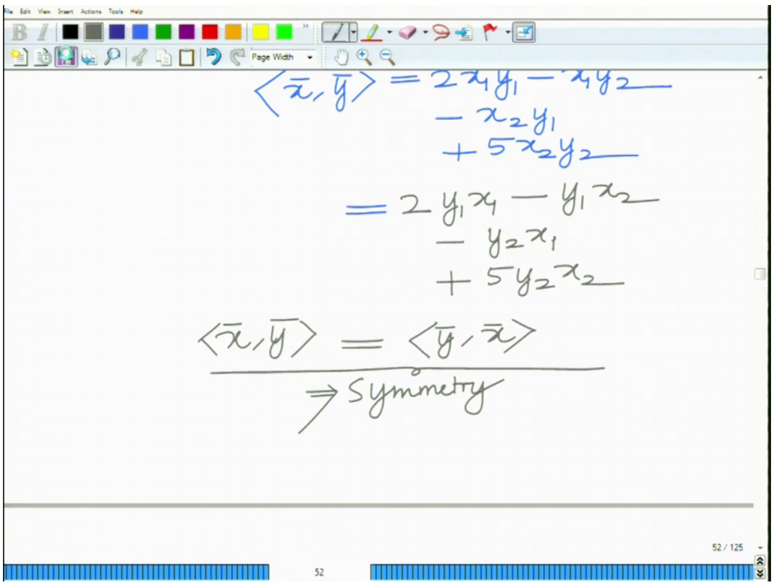

And you can readily see that this is nothing but the assignment of y bar comma x bar. So, we have x bar y bar equals y bar comma x bar. And hence, in this implies that that it satisfies the symmetry property and finally, now coming to the positive semi definite as positive semi definite property.

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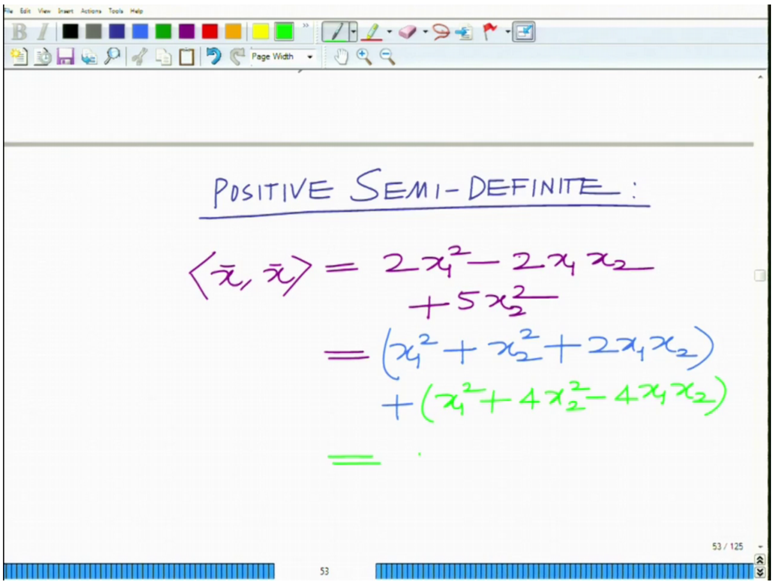


Coming now to the positive semi definite property, now if you look at x bar x bar that will be twice x 1 square minus 2 x 1 x 2 plus 5 x 2 square, which is equal to now I can write this as the sum of 2 terms x 1 square plus x 2 square plus 2 x 1 x 2 plus x 1 square plus 4 x 2 square minus 4 x 1 x 2.

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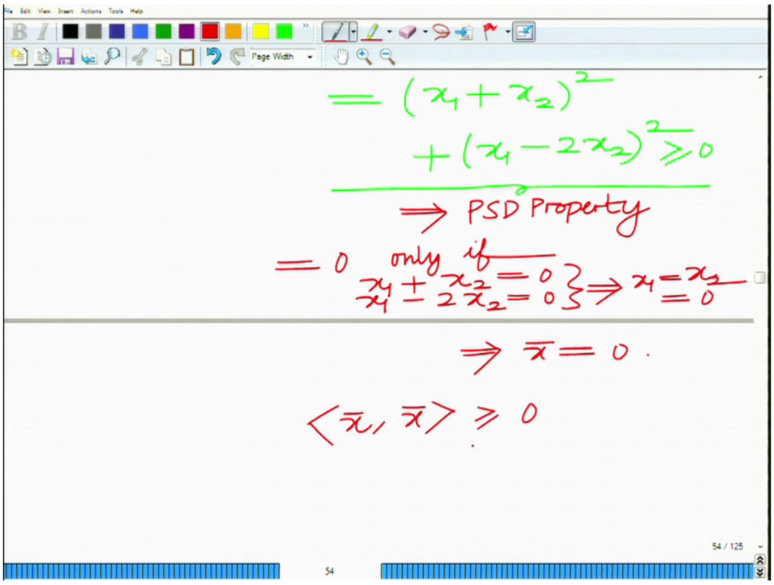


And this is equal to well x 1 plus x 2 whole square plus x 1 minus 2 x 2 square, which is greater than or equal to 0. So, sum of 2 perfect squares this is greater than equal to 0. So, implies this satisfies the positive semi definite property. This satisfies the PSD property and that you can see this is equal to 0 only when it is equal to 0; only if x 1 plus x 2 both must be 0 because it is sum of perfect squares. So, both must be 0 x 1 minus 2 x 2 must also be 0, and this implies that x 1 equals x 2 equals 0. So, it is only 0 when x 1 equal to x 2 equal to 0 that implies x bar equals. So, we have the PSD property which is x bar x bar the assignment is greater than or equal to 0 and equal to 0 only if x bar equal to 0.

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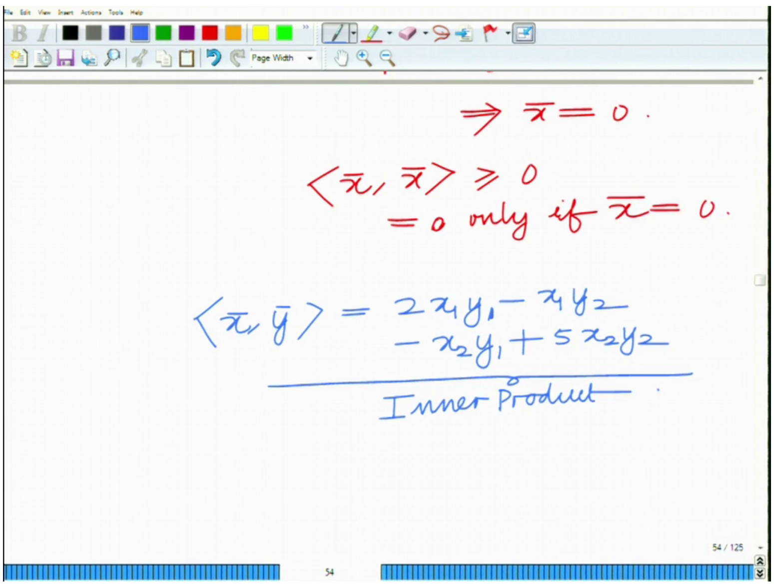

That is what we have shown therefore, it also satisfies the PSD property, hence because it satisfies the linearity symmetry and PSD properties, this is an valid inner product all right. So, finally, what we have is that x bar comma y bar equals 2 x 1 y 1 minus x 1 y 2 minus x 2 y 1 plus 5 x 2 y 2 we can now claim that this is a. And this is not a coincidence. In fact, such an inner product can be constructed by observing the following thing I can write this thing as.

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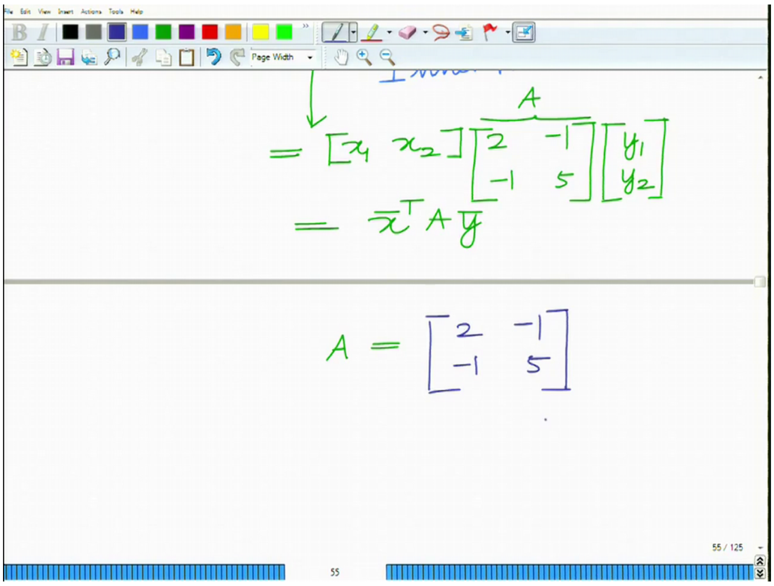

This is equal to x 1 x 2 twice minus 2 minus 1 minus 1 comma 5 y 1 y 2 which is nothing but; x bar transpose A y bar where a is this matrix you have A is the matrix 2 cross 2 matrix which is given as 2 minus 1 minus 1 5 you can see that this is nothing but; identical to the this is basically another way of writing the inner product that we have just defined. And now you can show in a very interesting property of this matrix in fact, this matrix A is a positive semi definite positive definite matrix all right. So, you can see that this matrix A is a positive definite matrix.

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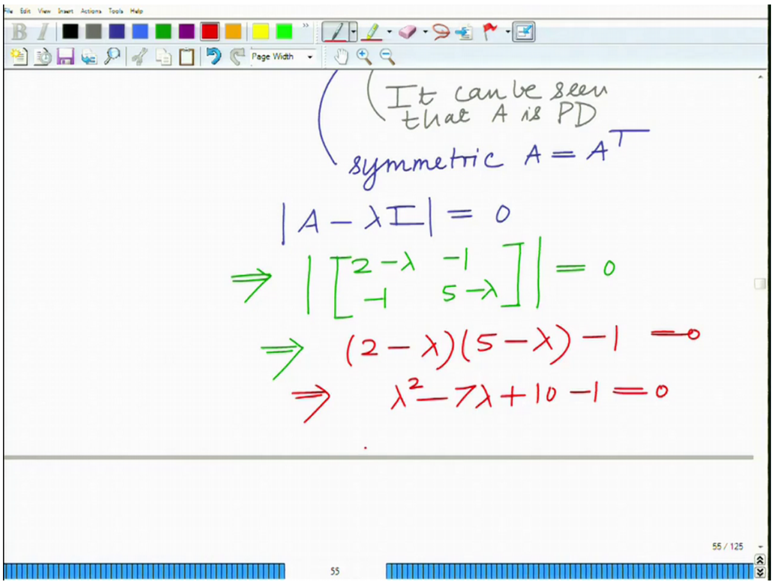


And you can see this as follows remember we said this matrix is first see that this matrix is symmetric. We have A equals A transpose, further if you look at the Eigen values a minus that is if you said the determinant of a minus lambda equal to 0, then what we have is you have the determinant of 2 minus lambda minus 1 minus 1 5 minus lambda equal to 0; this implies 2 minus lambda into 5 minus lambda plus 1 equal to 0. This implies now if you simplify this lambda square minus 7 lambda, I am sorry this determinant is minus 1 minus lambda square minus 7 lambda plus 10 minus 1 equal to 0.

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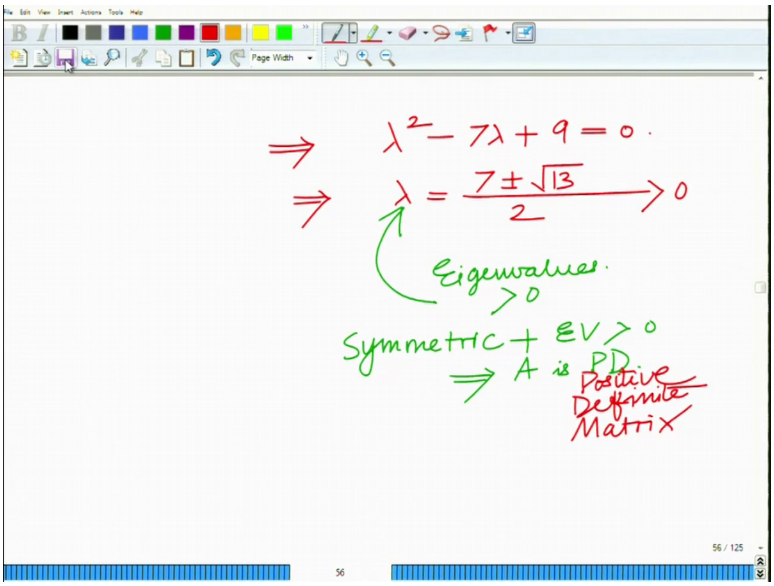

Implies lambda square minus 7 lambda plus 9 equal to 0 implies; lambda equals well 7 plus or minus square root of 7 square 49 minus 9 36 7 plus or minus 13 divided by square root of 13 divided by 2. And you can see both the Eigen values are greater than 0 all right it has 2 Eigen values and the Eigen values are strictly greater than Eigen values are strictly greater than 0. So, symmetric plus the Eigen values greater than 0 implies A is positive definite ok. So, A is a positive definite matrix.

So, A is a positive definite matrix and therefore, hence x bar transpose. So now, in fact an interesting property and you can easily show this that is; if you look at 2 vector if we define a inner product that x bar between x bar and y bar as x bar transpose a y bar, where A is positive definite this is a inner product that is, this is a is an this is a inner product. Where A is a symmetric positive let me also just write this a is a where A, A is a inner product.

This is an inner product x bar transpose A y bar, where A is a symmetric positive definite matrix is an inner product all right. And one of the other interesting aspects of the inner product is that it can be also be used to define a norm, and that is one of the most interesting and important aspects of the inner product it induces an norm and that norm is given as follows.

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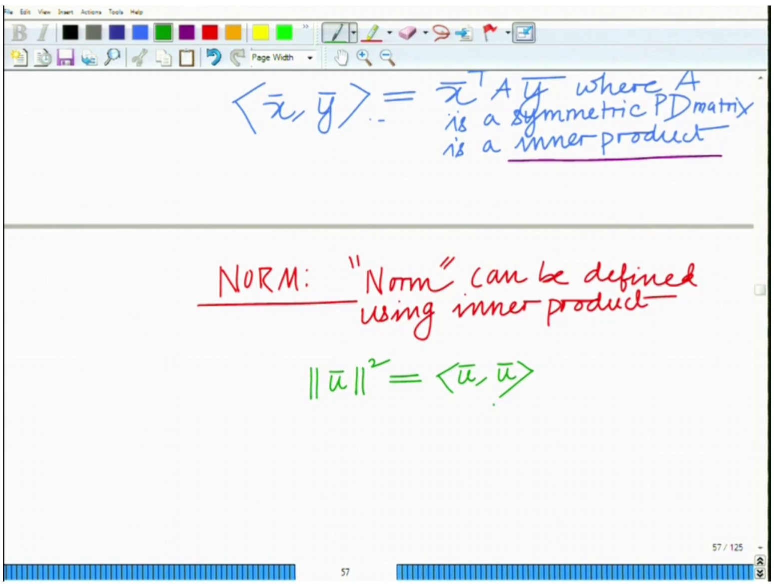

So, the norm is so inner product can be used to define a norm. So, the norm can be defined using this concept of inner product. And in fact, it can be defined as follows we have norm of u bar square this is equal to the inner product of u bar with itself.

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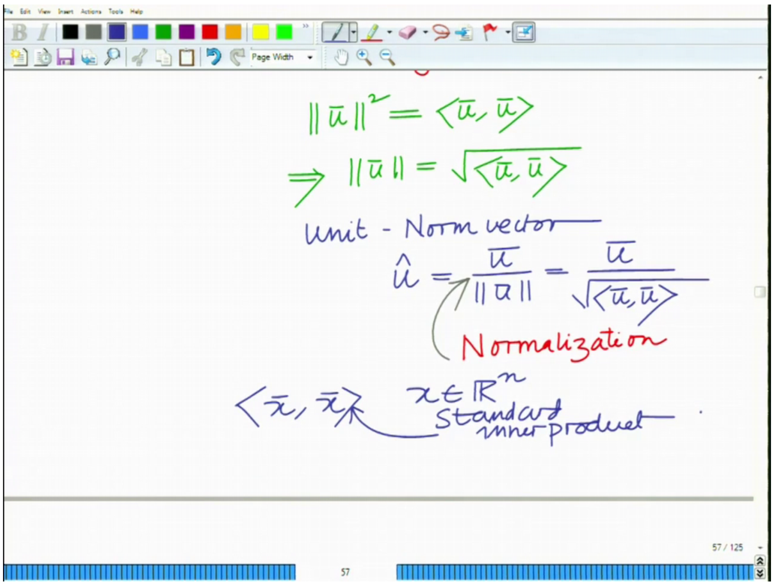


Which means, basically the norm of vector u bar is equal to square root of the inner product of u bar with itself. And in fact the unit norm vector can now be defined as u hat equals u bar divided by norm of u bar that is u bar divided by square root of u bar u bar and. In fact, what we have seen is and this is also this process also termed as normalization that is, when you divide a vector by it is norm you can also say that the vector is normalized. And you can also see this is true for the standard inner product that is the standard inner product on R n that is if you look at x bar comma x bar where x bar belongs to R n and this is the standard inner product.

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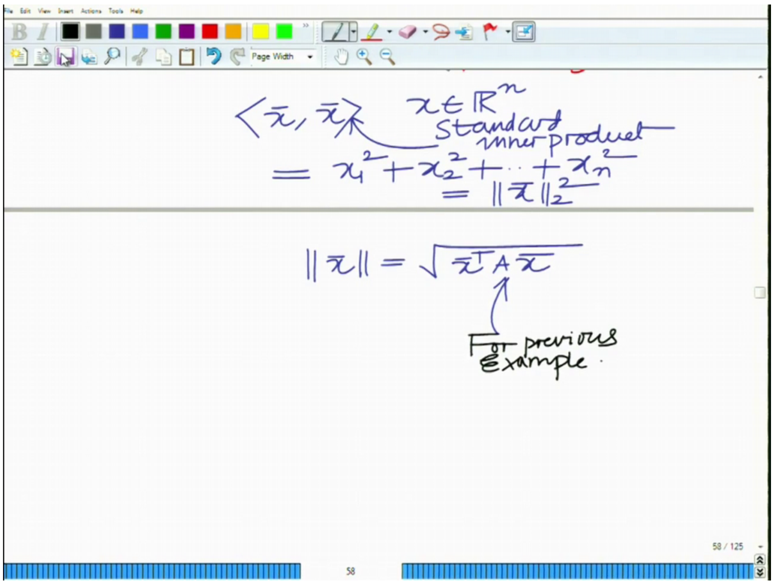


We already see in that inner product of x bar with itself is basically this is x 1 square plus x 2 square plus x n square which is nothing but the l 2 norm square, ok. And therefore, this is the square of the norm and we have already seen this for the standard inner product, what this result is that; not just for the standard inner product which is given with the dot product of 2 vectors.

But any inner product on these 2 vectors x bar y bar can be used to define a norm that is; the norm is given as the square root of the inner product of x bar with itself for instance in the previous example we have seen x bar transpose a y bar all right, which means the norm of x bar under that inner product is given as square root of x bar transpose A x bar ok. So, the norm of x bar is equal to square root of x bar transpose A x bar and this is for the previous example. Let us look at other examples of inner products.

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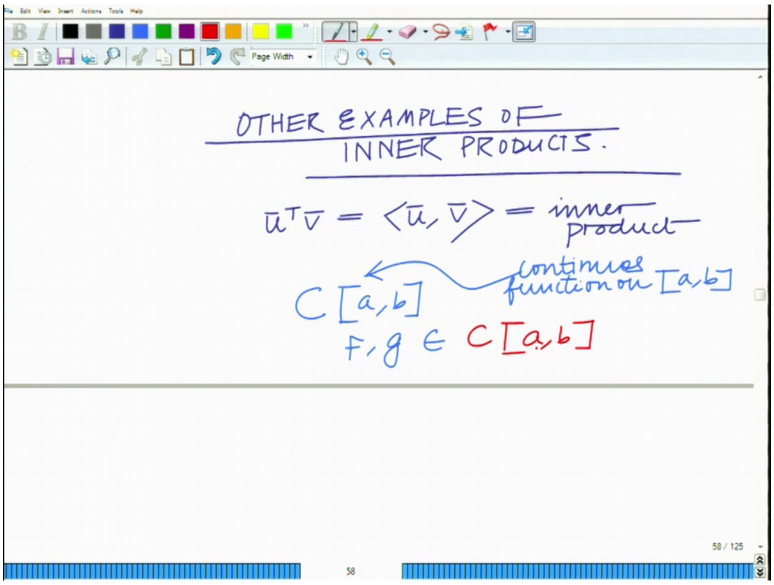


So, we have some other examples we are already seen, u bar transpose v bar that is the standard norm an R n this is an inner product this we already seen. Now another interesting application of inner product is let us consider the space of continuous functions on an interval a comma b denoted by c of a comma b, that is the set of continuous functions on a comma b on interval a comma b. And let us say we have 2 functions F comma g which belong to C of a comma b that is their continuous functions on the interval a comma b.

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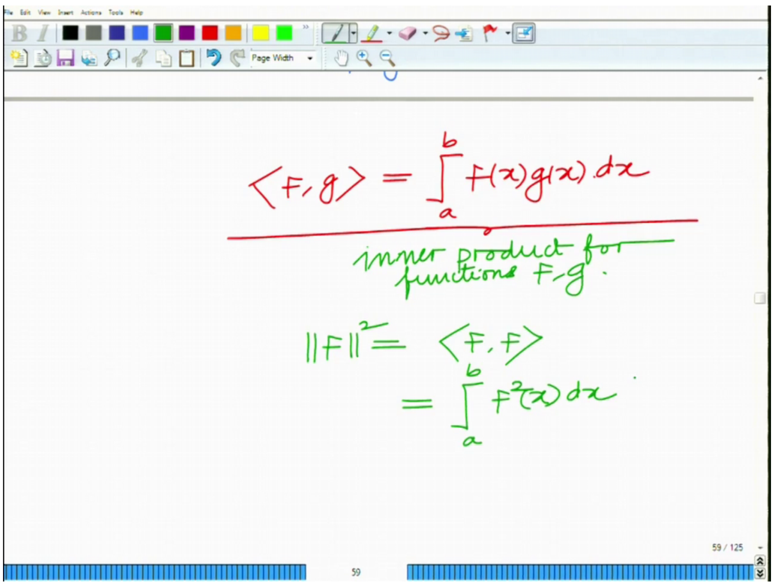

Then the assignment defined as F of g equals integral over the interval a comma b F of x g of x or F t d t that is this integral F of x g of x d x this is an inner product like this is a very interesting application. In fact, this can now be used to define a norm so this is an inner product for functions F comma g, and in fact, the norm that arises is basically nothing but norm of F is integral a comma b or norm of F square a is basically inner product of F with itself that is integral of a comma b, integral on the interval a comma b F square x dx or this is nothing but the energy of the signal; that is, if you replace if you look at this as signal in time.

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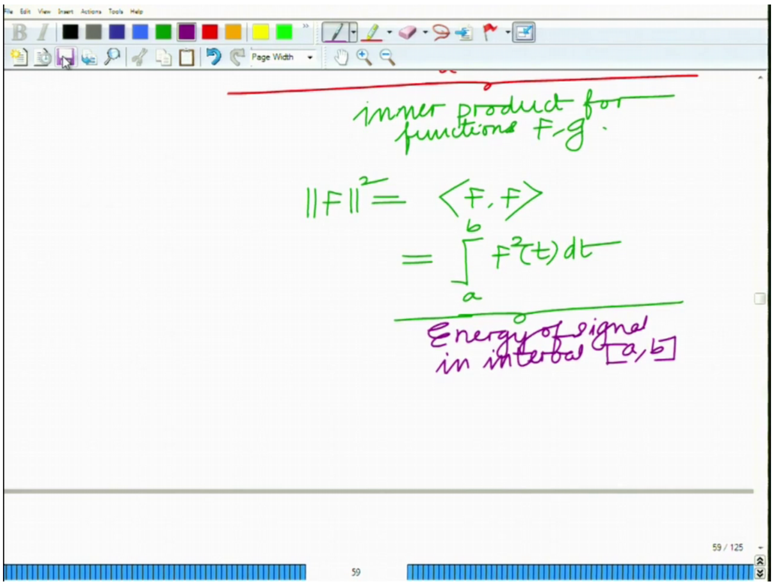


If I think of this as a signal in time, this is basically the energy of the signal in interval, this is the energy of the signal in the interval a comma b all right. So, that is another so it can also be used. So, basically one can also define an inner product correct, one can also define an inner product on the space of continuous functions all right and therefore, define the norm of the norm and as well as the norm square. In fact, the norm square of the function is based or the signal is the energy of the signal in that particular interval. Another interesting example of this inner product space consider the space of m cross n matrices.

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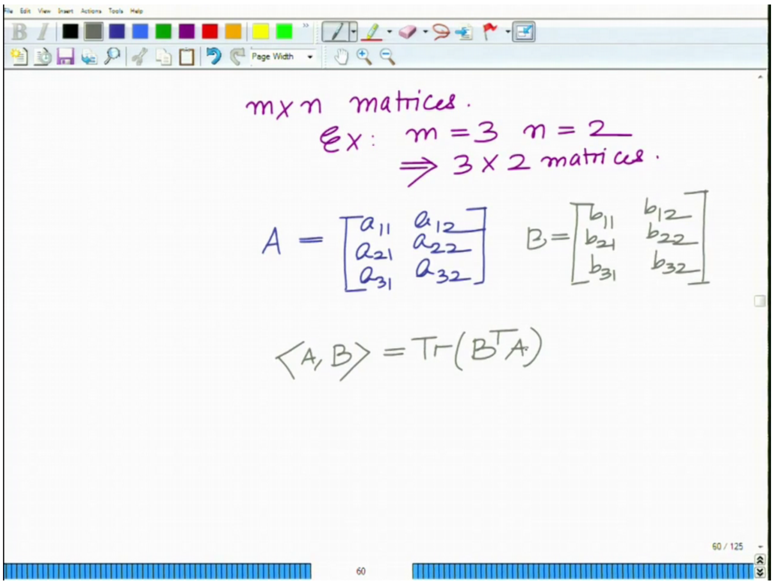


So, we have the space of m cross n matrices example m equal to 3 n equal to 2 implies; we have 3 cross 2 matrices. And for instance the 3 cross 2 matrix is are equals a 1 1 a 1 2 a 2 1 a 2 2, a 3 1 a 3 2 and we have B the matrix B equals b 1 1 b 1 2, b 2 1 b 2 2 b 3 1 b 3 2. And the inner product A B defined as trace of B transpose A that is, the now this is an interesting concept that is trace of a square matrix.

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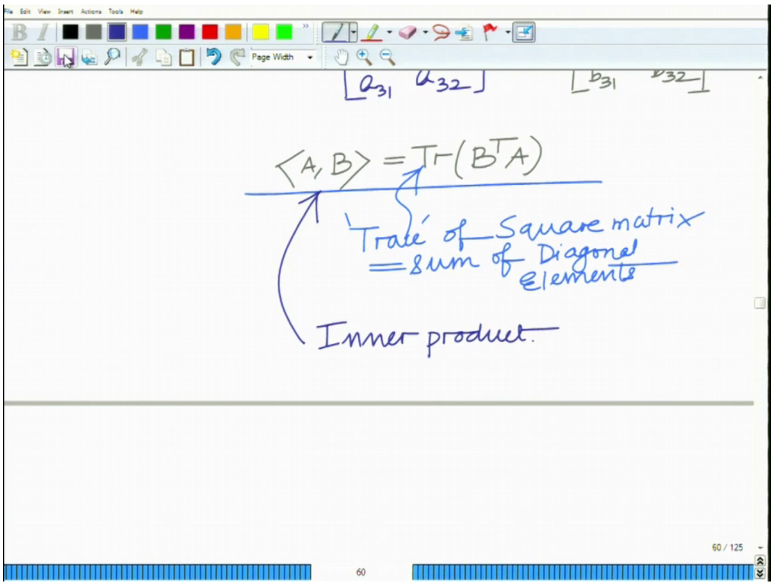


This is the trace of a square matrix the trace operator this is defined as equal to is equal to sum of the diagonal elements of a square matrix, some of the diagonal elements of a square matrix that is the trace of the square matrix. And this can be shown to be an inner product; this can also be shown to be an inner product. Trace of this can also be shown to be an inner product all right. So, what we have done in this module is we have looked at the inner product it is definition the various properties or when, is an assignment and inner product and several examples. We will continue this discussion in the next module.

Thank you very much.