Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture - 35
Practical Application: Maximal Ratio Combiner for Wireless Systems

Hello, welcome to the Tunnel Mode, another module in this massive open online course. So, we are looking at the problem of beam forming that is to find the beam former or the combiner basically which maximizes the signal to noise power ratio for the user in a particular direction, right and we said the constraint for the signal gain.

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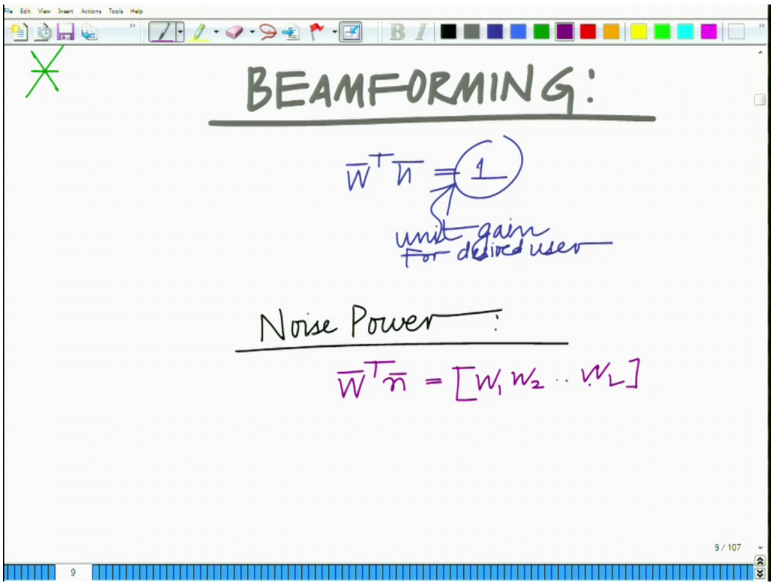

So, we are looking at the beam forming problem, and the constraint for the signal gain is w bar transpose h bar equals 1. This is unit gain for desired user or unit gain for signal. That is what we said we said the signal gain to be unity and minimize the noise power.

Now, coming to the noise power and the noise power can be calculated as follows. We know the noise component is w bar transpose n bar which is basically your row vector w 1 w 2 w l times the column vector of noise which is n 1 n 2 up to n l.

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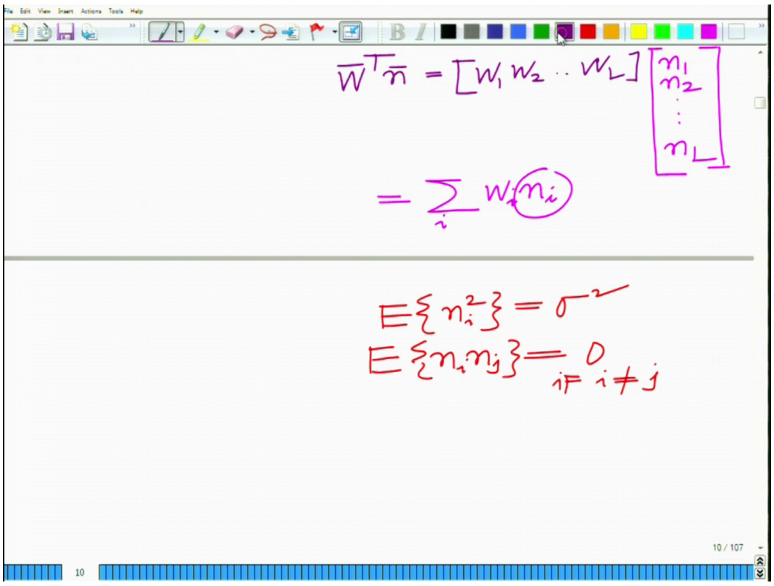

So, this is summation over i w i n i. Now, we have to assume something about they are not described properties of these noise samples, but as let us assume a very commonly employed model that is all the noise samples are statistically identical that is they have 0 mean and variance that is expected value of n i square equals sigma square. That is noise samples of variance of power. The noise power is sigma square, and they are 0 mean. These noise samples n i are 0 mean, and in addition let us also assume that if you take 2 distinct noise samples expected value of n i times n j, this is equal to 0 that is if i not equal to d, that is the distinct noise samples.

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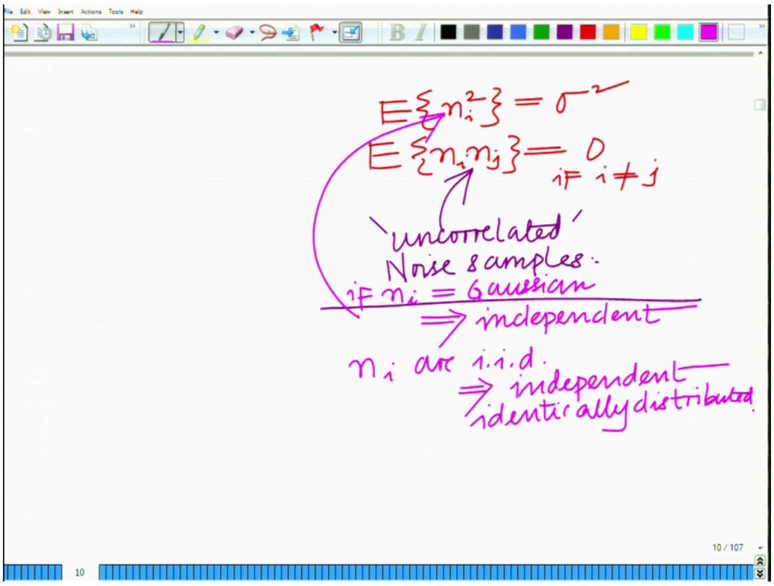

Noise samples are uncorrelated noise samples. In addition if these are Gaussian if n i are Gaussian, then uncorrelated implies that they are also independent, however this does not hold for non-Gaussian noise. Typically the gauss samples are assumed to be additive white Gaussian which means that the different noise samples at the different antennas are independent and identically distributed identically distributed in a sense that they have 0 mean and identical variance sigma square. So, we say the noise samples n i r i i d equals independent, identically independent and identically distributed.

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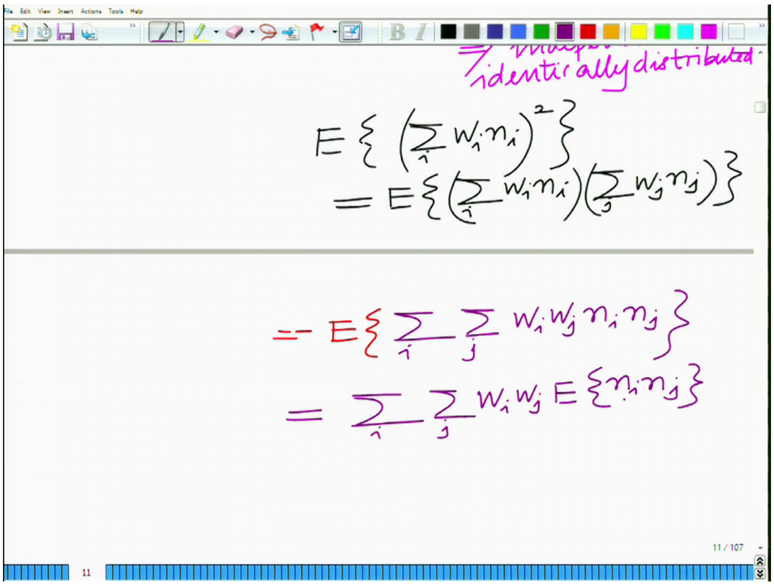

Now if you look at the noise power that is expected value of summation over i w i n i square, this becomes expected value of I can write it as a quantity times itself w i n i times summation over j w j n j which is equal to the expected value of well multiplying it out summation i summation j w i w j n i n j and now, it is easy to see what this reduces to. So, take the expected value inside summation over i j expected value of w i w j n i. In fact, w i w j are constant. So, you can simply write this as w i w j expected value of n i j, we have seen this is equal to 0 if i is not equal to j and equal to sigma square if and only if i is equal to j.

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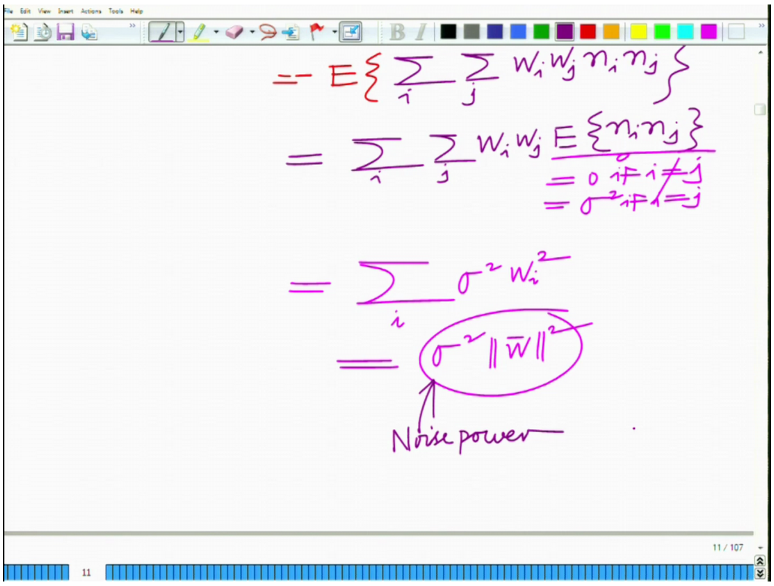

So, only the term when i is not equal to j expected value of n i n j is basically 0. There are samples of the two different antennas are uncorrelated, all right. So, only the terms where i equal to j survive and therefore, I can simplify this as since only terms where i equal to j survive and in that case expected value of n i n j sigma square.

So, this will be i sigma square when i equal to j w i to w j will be w i square. So, sigma square summation w i square are taking the sigma square outside. This will be sigma square summation w i square which is basically nothing worse nor w bar square and this is basically you are noise power. This is basically the noise power, ok.

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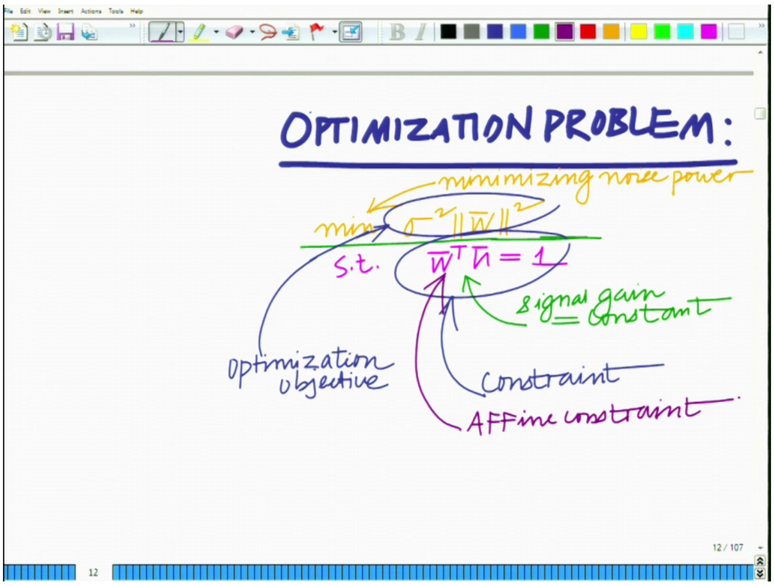

Therefore, now I can formulate my optimization problem. Now, my optimization problem will be the following. The optimization problem, remember the optimization problem for beam forming is to minimize the noise power. We have just found the noise power that is minimized sigma square nor w bar square that is what are you doing is, you are minimizing the noise power, minimizing the noise power subject to the constraint w bar transpose h bar equal to 1. That is your signal gain equals constant or constant gain in the direction of desire. So, what you are doing is you are minimizing the noise power keeping constant signal power. So, what does that does is that maximizes the signal to noise power ratio that is what we said and therefore, if you look at this optimization problem, it has two parts. First what you are trying to minimize. This is termed as the objective or the optimization objective or simply the objective function and this is termed as the constraint because remember we have to ensure w bar transpose h bar equals 1. This is term constraint or in fact, you can have more than one constraint, you can have constraints. And if you see this constraint is an affine constraint it is an affine equality constraint and if you look at the objective, the objective is convex sigma square norm w bar square.

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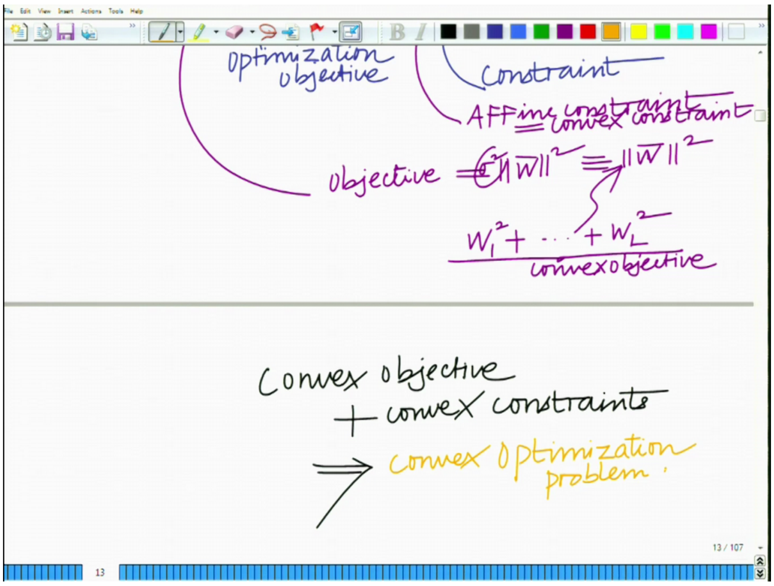

Well, minimizing sigma square norm w bar square, this is equivalent to minimizing norm w bar square because sigma square is a constant. So, if I minimize not w bar square I will also minimize sigma square norm w bar square and now, look at this norm w bar square is simply w 1 square plus so on w l square sum of convex functions. So, this is a convex objective affine constraint is also a convex constraint. So, you have a convex objective function and you have a convex constraint. You can have more than one convex cancer; you can have a set of convex constraint. So, convex objective plus convex constraints that makes a convex optimization problem that is a special subclass of optimization problems which we are going to focus on.

So, this is going to be the template of a convex optimization problem. So, convex objective plus convex constraints in it lies implies a convex optimization problem. So, this implies convex also we have a convex objective.

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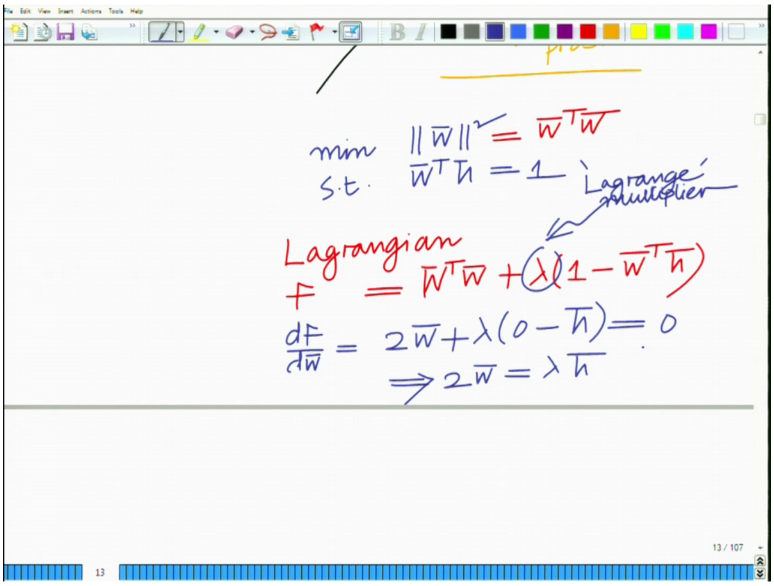

We have a convex, we have convex constraints. This implies we have a convex optimization problem. So, in fact they were constrained convex optimization problem. No constraints are an implicit part I mean constraints are this is a part. So, this is you have objective function and not just the objective function, but you also have a set of constraints that the desired solution has to satisfy. How do we solve this? Let us formulate the objective power, let me rewrite the optimization problem. So, I am dropping the sigma square because it is a constant. So, I am going to simply minimize non w bar square subject to the constraint w bar transpose h bar equals 1.

What I am now going to now do is something that you must have seen in an early course on calculus that is to solve a constraint or constrained optimization problem, one needs to use Lagrange multipliers. So, I am going to form and by the way this is equal to norm w bar square equals w bar transpose w bar here which is basically f. I can denote this as w bar transpose w bar plus lambda times 1 minus w bar transpose h bar, and now what? So, this quantity lambda, this is the Lagrange multiplier, this is the Lagrange multiplier, and this has to also this is a new parameter and optimization problem and this Lagrange multiplier also has to be determined as the solution to the optimization problem, ok. So, now I am going to differentiate this with respect to w bar and set it equal to 0 and that gives us w bar transpose w bar differentiate that gives us 2 w bar plus lambda time differentiate one that gives 0 minus differentiate w bar transpose h bar derivative of that is h bar. So, set it equal to 0.

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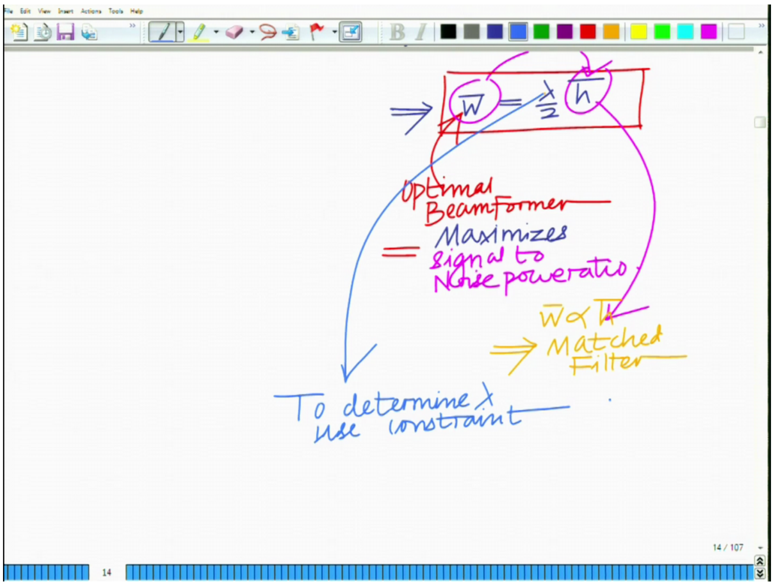

So, this implies 2 w bar equals lambda h bar which implies w bar equals lambda over 2 h bar that is w bar equals lambda over 2 times h bar and this is the optimal beam form. Why is it the optimal beam former? It is because it maximizes the signal to noise power ratio maximizes the signal to noise and you can see that the optimal beam former is proportional to h bar. So, therefore this is also like a matched filter. In fact, the spatially matched filter in space right because typically you have a matched filter in time. This is a matched filter across the antennas corrects with some spatially matched filter. So, w bar is proportional to h bar which implies that this is an analogue. Analogue is to a matched filter that you employ in a digital communication system.

Now, how do we determine lambda? To determine lambda, this Lagrange multiplier we have to use the constraint.

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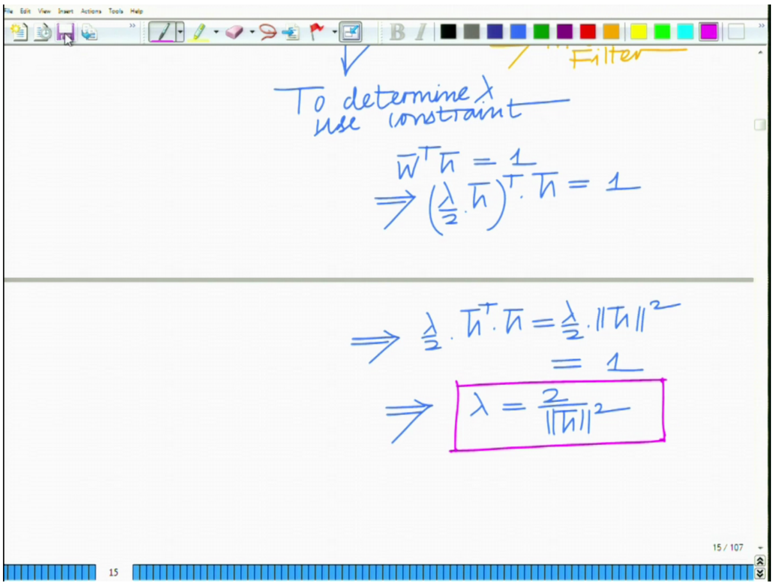

What is the constraint? Remember the constraint is w bar transpose h bar equals 1. Substitute the value of w bar that is lambda by 2 h bar transpose h bar equals 1 which implies that lambda by 2 h bar transpose h bar equals lambda by 2 nor h bar square equals 1 which implies lambda equals 2 over norm h bar square. So, lambda equals 2 over norm h bar square correct, and that is what we have.

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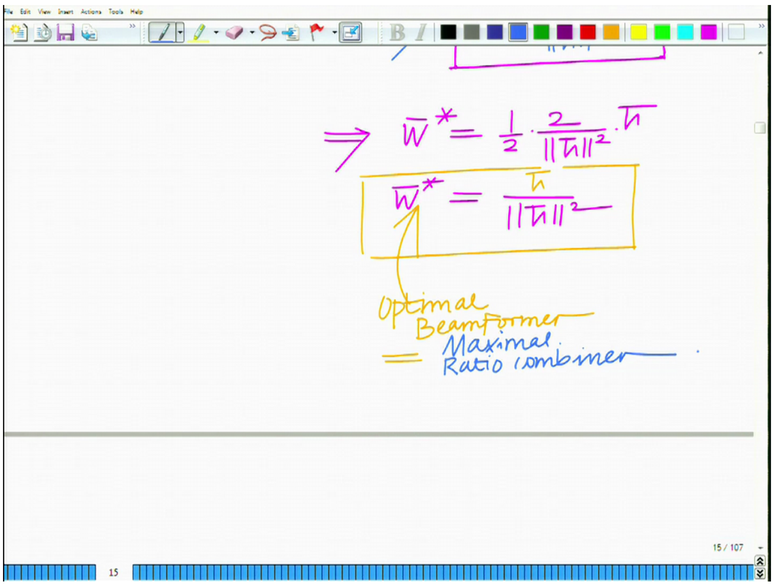

Therefore, it implies the optimal beam former which you can now denote as w star w star equals well lambda by 2. So, that is half lambda 2 over norm h bar square into h bar lambda by 2 or 2 h bar. So, this is h bar divided by norm of h bar square.

So, this is your optimal beam former. So, this is the optimal beam former h bar divided by. This is also termed as the maximal ratio combiner because this maximizes the signal to noise power ratio. It is also termed as the maximal ratio combiner, ok. So, this is also termed as the ok.

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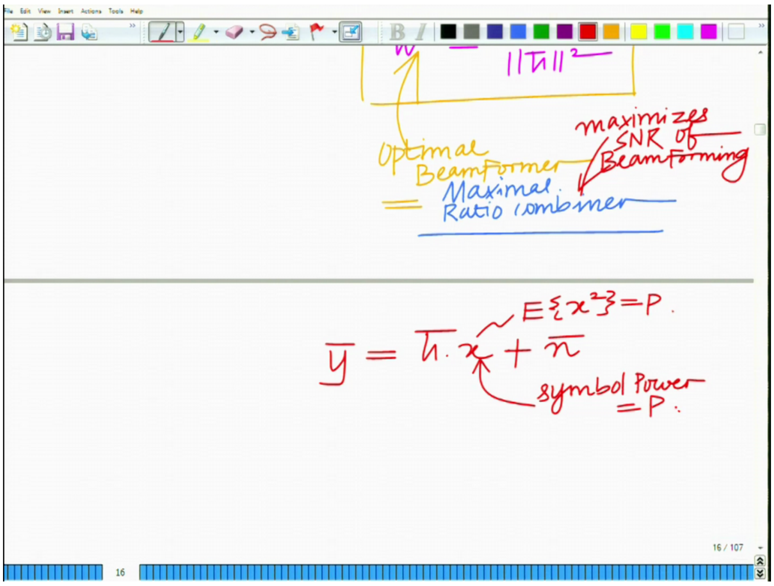

So, that gives us the optimal beam former which is h bar divided by norm h bar square that is a maximal ratio combiner and this maximizes the. It is known as the maximal ratio combiner because it maximizes SNR, correc. Maximizes the SNR employing the beam forming maximizes SNR at the output of the beam former. What is SNR? That is easy to see if you have y bar equals h bar times x plus n bar let this b p symbol power equals p that is expected value of x square symbol power equals p, ok.

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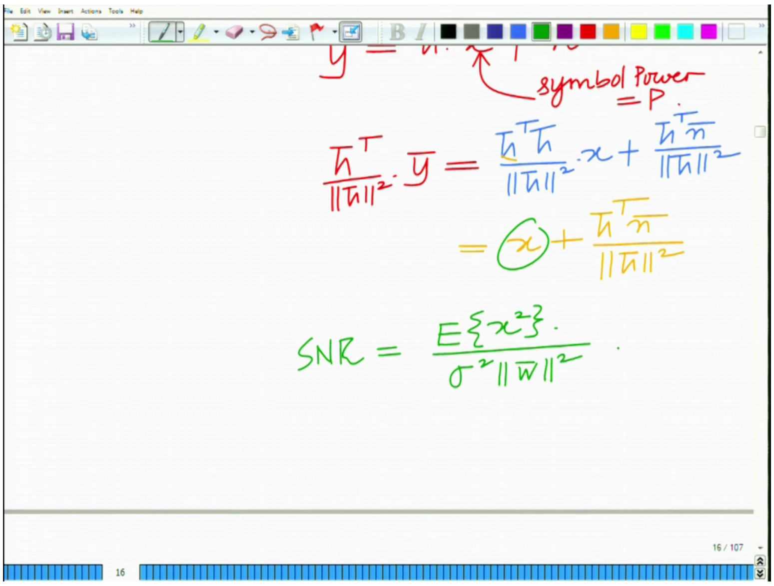

Now, what am I doing? I am beam forming with the maximal ratio combiner that is h bar transpose divided by norm h bar square into y bar. Remember this is a maximal ratio combiner which is h bar transpose h bar divided by norm h bar square times x plus h bar transpose n bar divided by norm h bar square. Now, if you see this h bar transpose h bar that is norm h bar square divided by norm h bar square which is 1 and that is indeed true because we ensure that the signal gain is 1. So, this is h plus h bar transpose n bar divided by norm h bar square. Therefore, the SNR at the output of the maximal ratio combiner is simply signal power that is expected value of x square by noise power, but we know noise for is sigma square times norm w bar square. We have already divided derived that for any combiner.

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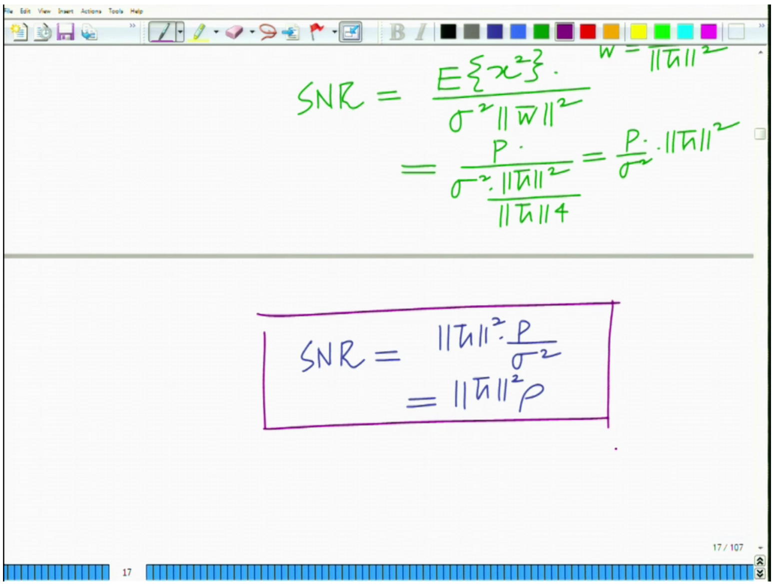

So, expected x square is p divided by sigma square norm w bar square remember w bar equals h bar by norm h bar square norm w bar square is norm h bar square divided by norm h bar power 4 equals p divided by sigma square into norm h bar square. So, SNR at the output is norm h bar square into p divided by sigma square. You can also write this as norm h bar square into rho where rho equals p divided by rho equals the transmitted.

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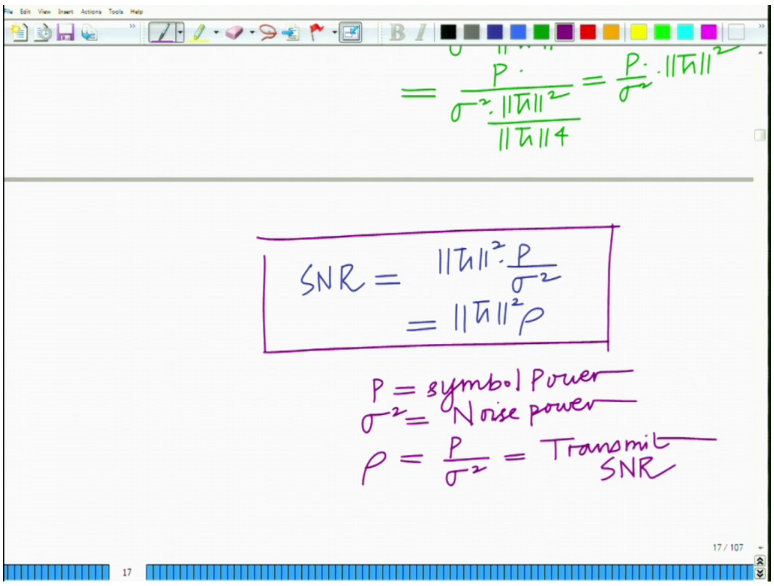

So, p equals the symbol power sigma square equals noise power and rho equals p divided a signal power by noise power. You can think of this as the transmit power because this is the power of the transmitted symbols. So, rho equals I am sorry transmit SNR p over sigma square which is the transmitted SNR. So, this is the interesting thing. So, this is the first convex optimization problem which is rather simple application of the optimization framework which we have seen and you can clearly see it is very powerful and very handy, right. You can formulate a neat optimization problem where in you are trying, today you are trying to design the optimal beam former. You set the beam forming gain or the gain of the beam former in the desired direction and the direction of the signal to be unity that gives you your constraint and you minimize the noise power which is sigma square norm w bar square to basically essentially maximize the signal to noise power ratio. We used the Lagrange multiplier framework to formulate the Lagrangian basically differentiate it set it equal to 0. This is also known as the KKT frame, but will look at it in more detail as we go in the subsequent modules, but initially if this illustrates to you in a very simple fashion through a practical example how the optimization framework and more specifically the convex optimization framework can be used to solve practical problems alright.

So, we will stop here and continue in the subsequent modules.

Thank you so much.