**Applied Optimization for Wireless, Machine Learning, Big Data**
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**Lecture – 25**
**Test for Convexity: Positive Semidefinite Hessian Matrix, example problems**

Hello, welcome to another module in this massive open online course. So, we are looking at convex functions and test for convexity. We have seen the test for a function of a single variable correct when y equals F of x and F of x is a one-dimensional variable correct. Let us now extend the test for functions of a vector or a multidimensional variable ok.

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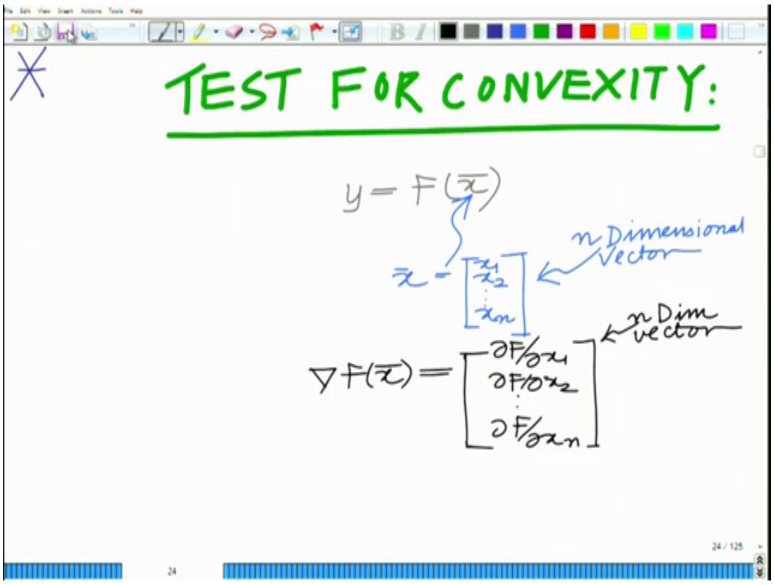


So, we want to find the test or convexity ok. And we want to consider y equals F of x bar. So, this is a vector now ok. So, x bar equals this is an n dimensional vector x 1, x 2 up to x n. So, this is an n dimensional vector.

And so we want to find the test for convexity for this function of a vector. And for this we want to first define the gradient of this that is the gradient with respect to x, I am just going to simply write this as gradient of F with F x bar since its clear from the context that the function is of the n dimensional vector x bar. This is simply defined as the vector which contains the partial derivative with respect to each component of the; with respect to each component of x n ok. So, this is also an the gradient is also an n dimensional vector correct.

So, what is the gradient of F of x bar that it contains the partial derivative of F with respect to x 1 partial derivative of F with respect to x 2 so on and so forth until the partial derivative of F with respect to x n. So, it is also an n dimensional vector of partial derivatives ok, one with respect to each component of the vector x bar.

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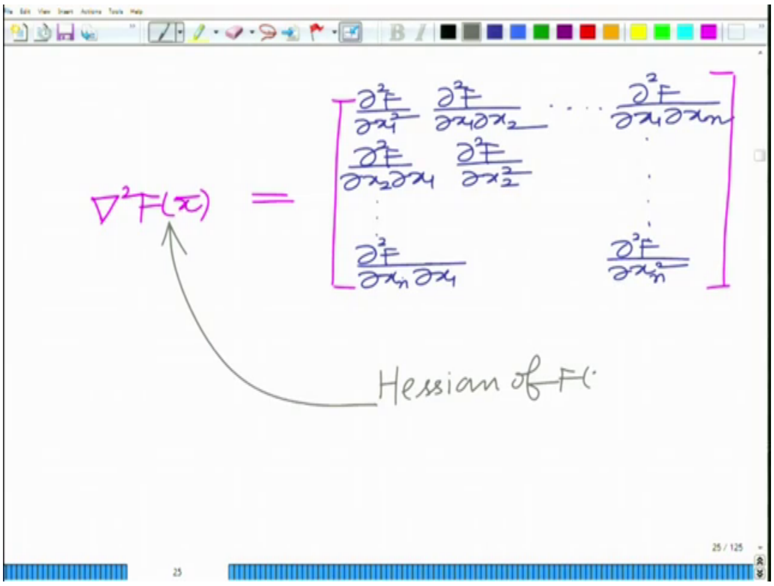


Now, what is the Hessian, we want to consider the next that is the Hessian ok. So, this is denoted by square of x bar. This is a matrix of partial derivatives second order partial derivatives dou square F by dou x 1 square dou square F or let me write it here. The Hessian of F is dou square F by dou x 2 square dou square F by dou x 1 partial with respect to x 2 dou square F by dou x 2 dou x 1 dou square F by dou x 2 square dou square F by dou x n dou x 1 dou square F by dou x 1 dou x n and so on and so forth dou square F by dou x n square. And this is basically the Hessian ok. So, this is you think of this as a second order derivative, this is the Hessian.

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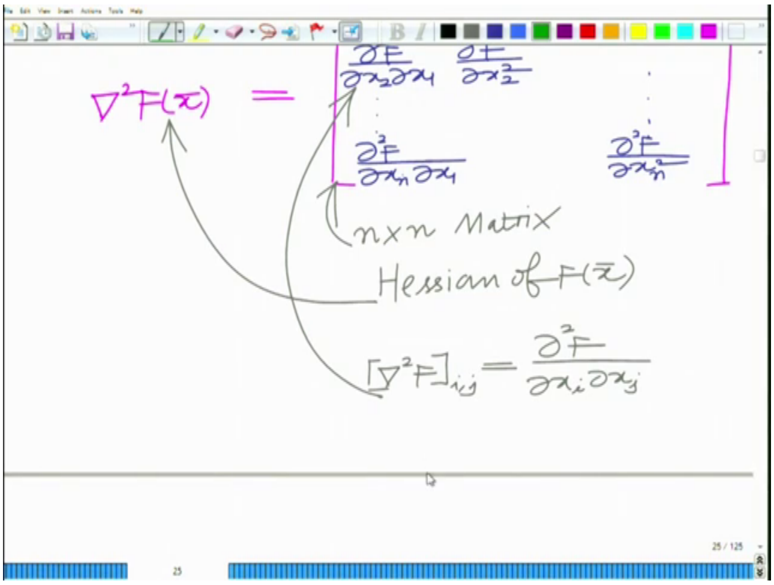


Where the ijth entry if you look at any ijth entry of the Hessian, the ijth entry correct delta square F, if you look at the ijth entry that is the second order partial of F with respect to x i and x j that is the ijth entry. So, it is a matrix with the ijth. And naturally this is going to be a needless to say this is an n cross n matrix for an n dimensional vector x bar. So, this is an n dimensional if you consider a vector x bar to be n dimensional, then this Hessian is an n cross n matrix with the ijth matrix entry corresponding to the partial second order partial derivative of F with respect to x i and x j.

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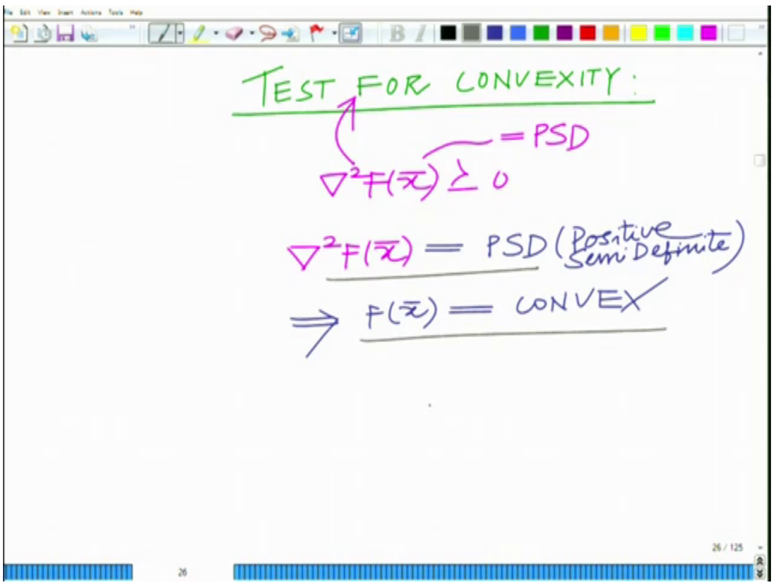

Now, the condition the test for convexity the test for convexity of this vector x bar is that we have to have the Hessian with respect to x bar this has to be remember this symbol this denotes a positive semi definite matrix. So, if this is positive semi definite that is delta square that is the Hessian equals a positive semi definite matrix. This implies that F of x bar is convex. So, if the Hessian is positive semi definite, we have F of x bar is a convex function of the vector x bar. So, we have to compute the Hessian of this function multi dimension this function of as a vector x bar and check if the Hessian is positive semi definite. And if the it is positive semi definite, one can conclude that F of x bar is convex.

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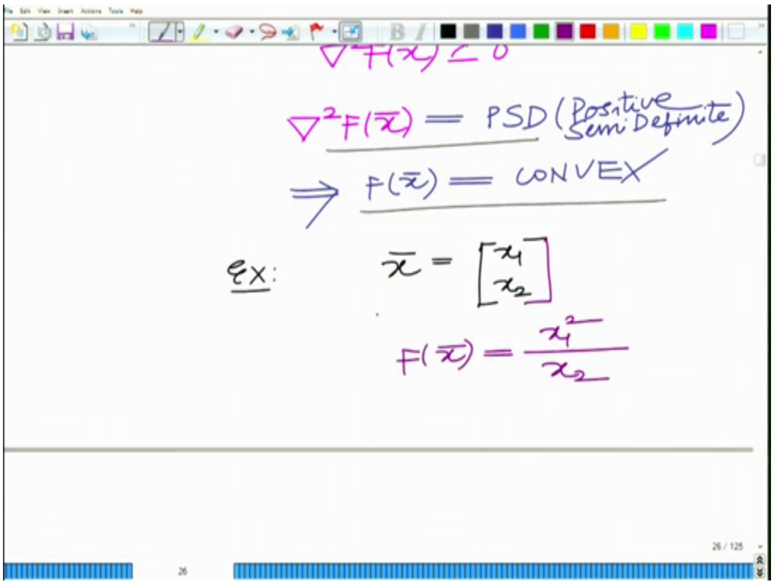


Let us understand that by looking at a simple example. A simple example let us consider F of x bar, I am going to first talk about a simple x. So, first let us start with a two-dimensional vector x bar so, x bar equals x 1 ok. And we have F of x bar equals x 1 square let me write this clearly F of x bar x 1 square divided by x 2.

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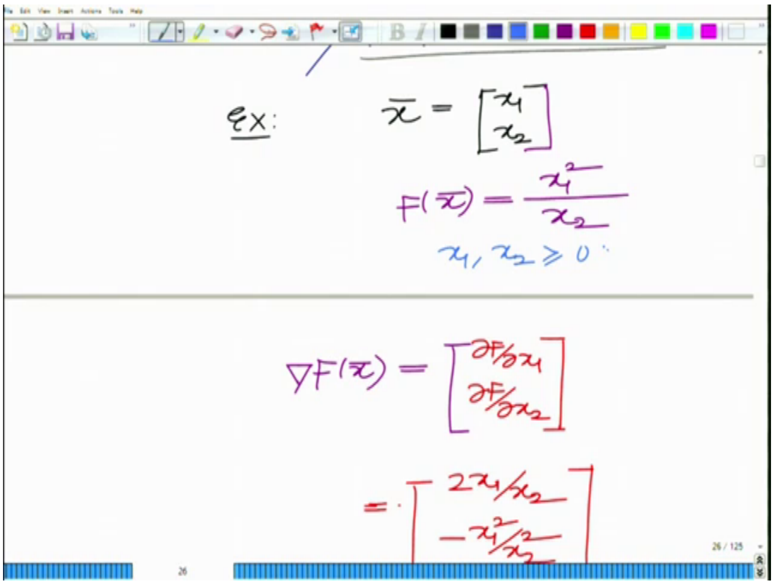


Then first let us consider the gradient with respect to x bar, so that we know is the partial of F with respect to x 1, the partial with respect to x 2 ok. Now, partial derivative with respect to x 1 treat x 2 as constant differentiate with respect to x 1 that gives us 2 x 1 by x 2 partial with respect to x 2, treat x 2 as constant differentiate with respect to I am sorry treat x 1 as constant differentiate with respect to x 2 so that will be minus x 1 square by x 2 square correct. So, you are treating x 1 as a constant and differentiating with respect to x 2 ok.

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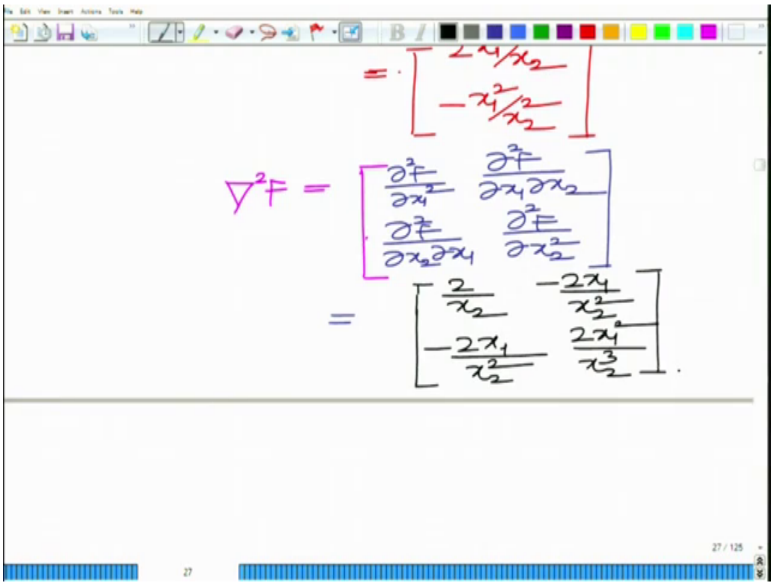

Now, consider the Hessian with respect to x that will be just to refresh your memory that will be dou square F by dou x 1 square dou square F by dou x 1 dou x 2 dou square F by dou x 2 dou x 1. And finally, dou square F by dou x 2 square which you can see is the following thing. This will be well this will be 2 over x 2 this quantity, you can see minus 2 x 1 divided by x 2 square. In fact, dou square F by dou x 1 dou x 2 will be same as dou square F by dou x 2 dou x 1 which will be minus 2 x 1 divided by x 2 square. And finally, dou square F that is the second derivative second order partial derivative with respect to x 2 that is 2 x 1 square divided by x 2 cube.

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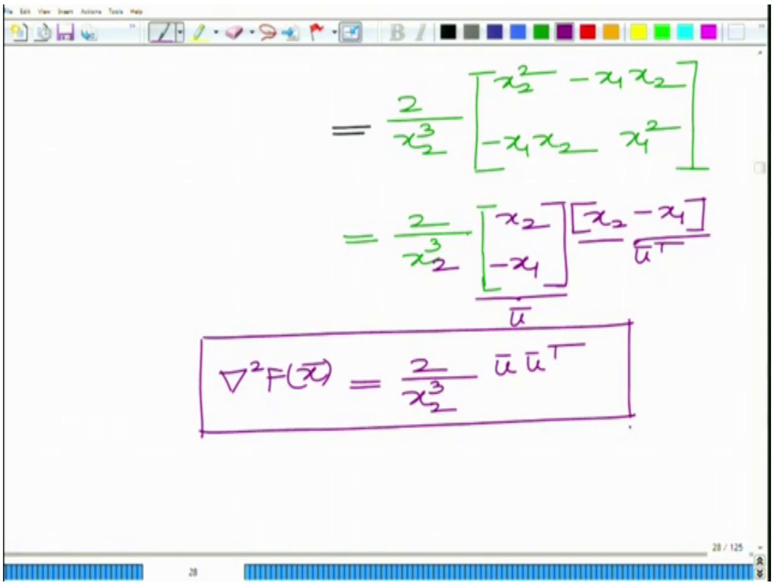

And now if you take 2 over x 2 cube as common, we have x 2 square minus x 1 x 2 minus x 1 x 2 and we will have this one is x 1 square, which you can now write as you can decompose this matrix above as follows. You can write this as you can see this will be x 2 minus x 1 times x 2 minus x 1 and if I call this vector as u bar, this will be u bar transpose. So, this will be 2 by I am sorry 2 by x 2 cube x 2 cube u bar u bar transpose that is the Hessian.

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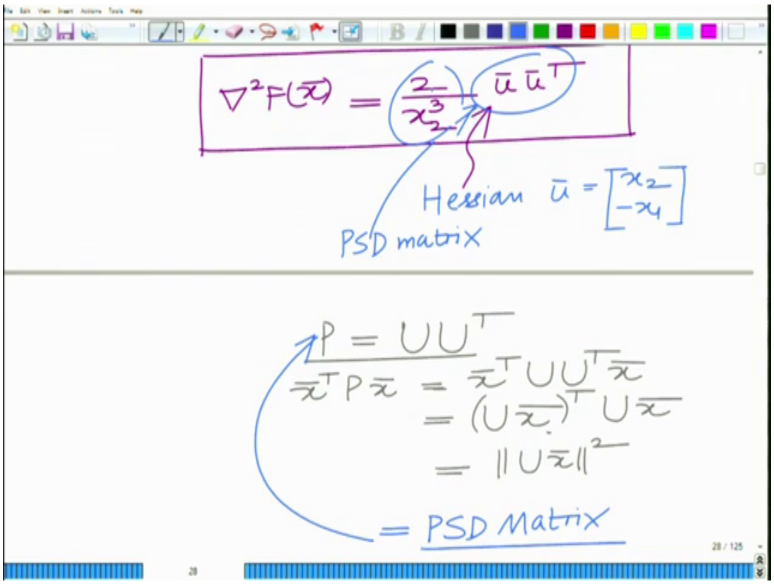


So, this is the Hessian ok, where u bar is the vector x 2 minus u bar is a vector x 2 minus x 1. And now here you can see that if there is any matrix p that can be decomposed as a a transpose or for that matter U U transpose then it is positive semi definite. And this can simply be seen as follows, many of you might already be familiar with it. If you have any matrix p which is basically can be written as a matrix U times U transpose then x bar transpose P x bar equals x bar transpose U U transpose x bar which is U x bar transpose times U x bar which is basically norm of U x bar square.

So, the moment you have any matrix P which is U U transpose then P automatically becomes a this becomes a PSD matrix, P automatically becomes a PSD matrix. Which implies that this 2 by x 2 cube times u bar u bar transpose this is automatically this is a PSD matrix all right. For instance, we can assume that x 2 is greater than or equal to 0. I think one can restrict this domain such that x 1, x 2 greater than equal to 0. So, what we have is this quantity 2 x 2 cube this is greater than equal to 0, this is u bar u bar transpose is the positive semi definite matrix implies.

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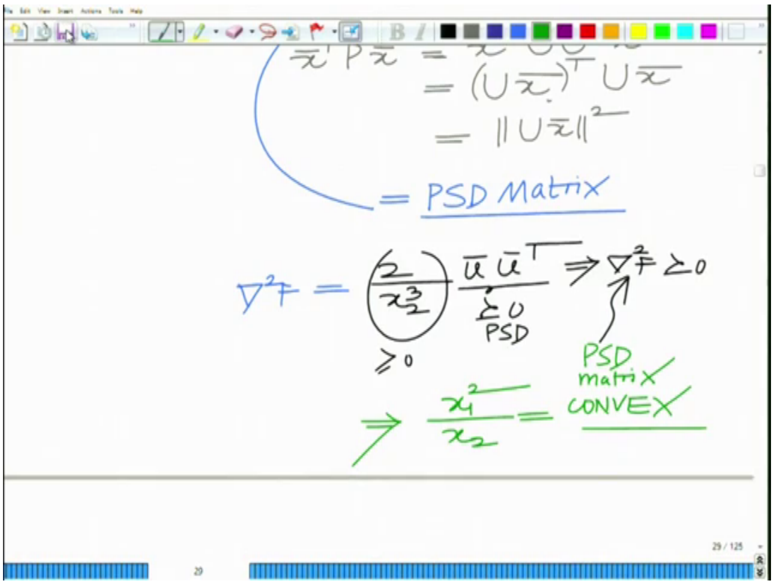


So, this implies let me just write it clearly. This is basically 2 by x 2 cube times u bar u bar transpose. So, this is greater than equal to 0, this is positive semi definite. This quickly or this curved greater than equal to. So, implies delta square F is this is positive semi definite. Implies x 1 square by x 2 a simple function of the vector x bar two-dimensional vector x 1, x 2 this is convex since, the Hessian case. So, we have evaluated the Hessian of this function and demonstrated that the Hessian is a positive semi definite matrix therefore the function is a convex function.

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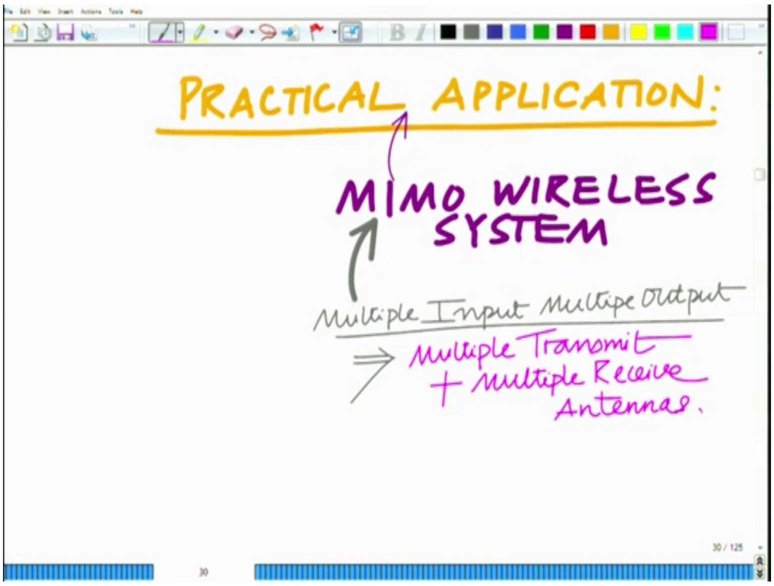

Let us proceed onto look at another example. And we will look at an interesting practical application of this convexity of a function of a vector. So, what I want to look at now is to develop a practical application of this. And in fact, in this practical application what we want to look at we want to look at, I want to look at a MIMO Multiple Input Multiple Output wireless communication system. Consider practical application with the MIMO wireless system, where MIMO some of you might already be familiar the concept of MIMO is basically you have this is stands for multiple input multiple output ok. So, this implies that you have multiple transmit and multiple receive antennas. And having such multiple input multiple output system significantly increases the communication rate or the data rate of a wireless communication system. So, it is considered to be one of the revolutionary technologies in wireless communication.

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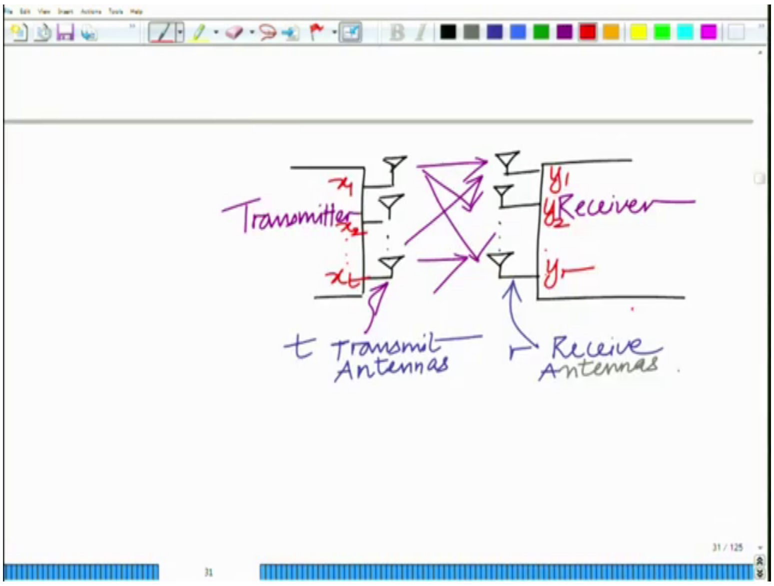


And if you look at this MIMO communication system, you have as I already said you have a system that is you have a transmitter you have a transmitter and you have many possible channels between each transmitter. And so this is your t transmit antennas or let us represent this by small t transmit small t transmit antennas, and this is r receive antennas r receive antennas. And therefore, what we can do is we can transmit the symbols x 1, x 2, x t from the t transmit antennas x 1 x 2 up to x t and we can receive the symbols y 1, y 2 y r.

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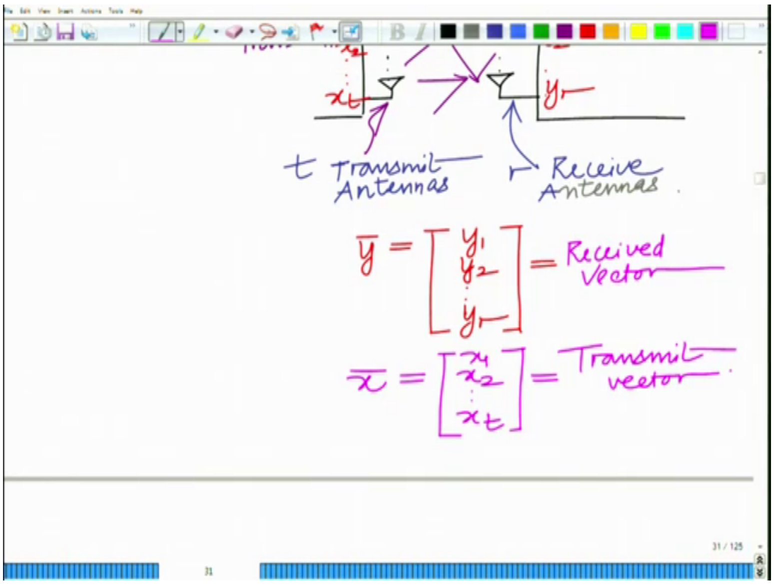


So, we can write this as y bar equals the received vector corresponding to the symbols y 1 y 2 up to y r this is the received vector or you can say this is the vector of received symbols. And for that matter if you look at x bar that is x 1, x 2, x t, because you are transmitting t symbols one from each transmit antenna. So, this is your transmit this is your transmit vector.

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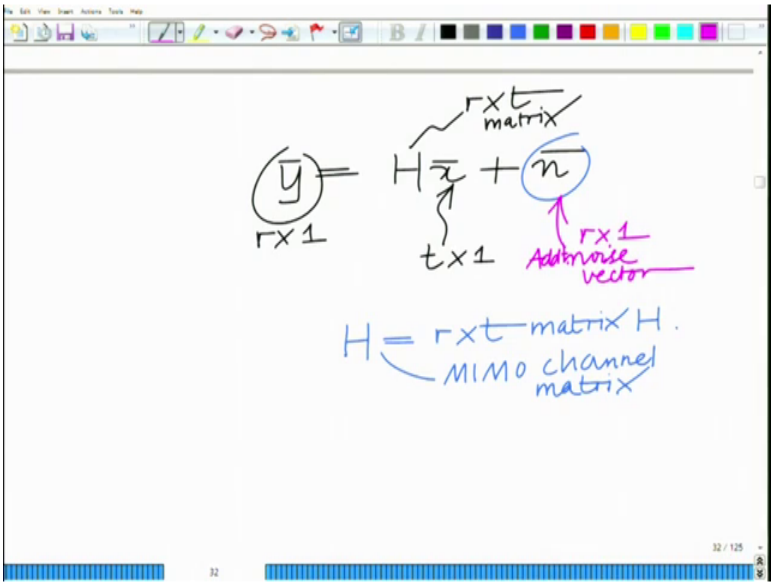


And our model for this MIMO system is therefore given as y bar equals H x bar plus n bar where we know this y bar this is an r cross 1 receive vector, this is a t cross 1 transmit vector which implies that this must be an r cross t matrix H. And this is known as the MIMO channel matrix. So, this r cross t matrix H which where r is the number of receive antennas t is the number of transmit antennas this is your r cross t matrices. This is also the MIMO channel matrix multiple input multiple output channel matrix. Of course, this n bar, this is an r dimensional noise vector. You can see this is an r cross 1 noise vector. So, this is the noise vector. This is also you can say additive noise vector; this is an additive noise vector because just adding to H x bar ok.

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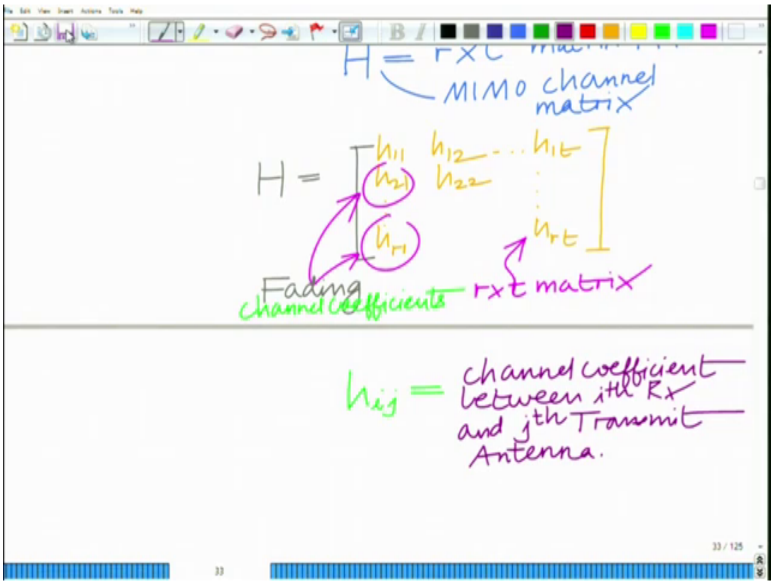


And if you look at this channel matrix H that has the following structure that has h 1 1 h 1 2 h 2 1 h 1 I am sorry h 2 1 h 2 1 h 1 2 h r 1 h 1 t this is h 2 2, and this is finally, h r t. So, this is an r cross t matrix, r equals number of rows which is the number of receive antennas t equals number of columns that is the number of transmit antennas. And if you look at h i j, so each of these quantities correct, each of these quantities these are basically your fading channel coefficients ok. So, these are fading wireless channel because the wireless channel is the fading channel. So, these are fading these are fading channel coefficients in particular if you look at this quantity h i j this is the channel coefficient between the ith receive antenna and jth transmit antenna. This is the channel coefficient between the ith transmit between the ith transmit, ith receive I am sorry ith receive and jth transmit antenna ok.

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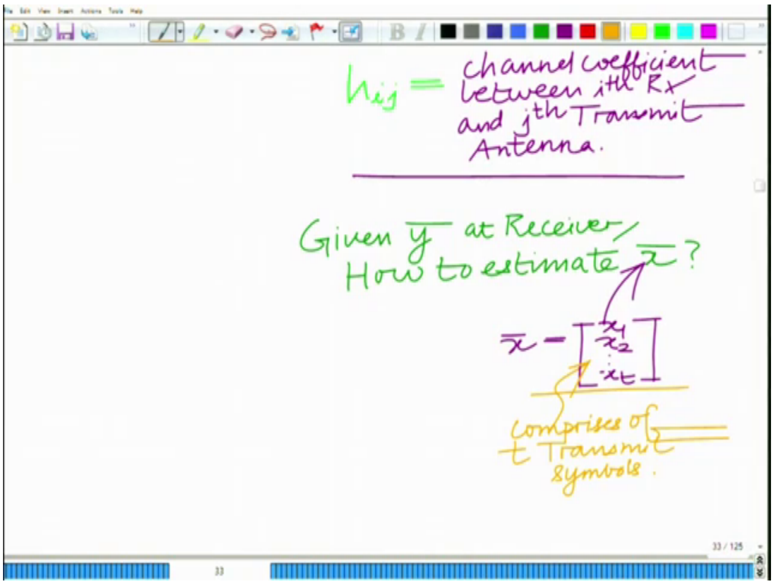


And now our challenge is the problem that we want to address is basically which we will continue in the subsequent module is that remember at the receiver we have this vector y bar correct which we know at the receiver all right. Now, with this received vector y bar one has to decode or one has to estimate the vector x bar that has been transmitted alright that is the problem of communication. In a communication system you transmit some symbols these symbols have to be recovered at the receiver correct.

So, in a MIMO system you have this received vector y bar, but you do not know at the receiver you do not know that vector x bar that has been transmitted which comprises of the t transmit symbols x 1 x 2 x t correct. So, one has to estimate this transmitted vector x bar and that forms the problem or the problem of MIMO receiver design. So, one has to design a suitable algorithm or a technique for the MIMO reserve receiver which recovers this transmitted vector or the transmit vector x bar.

So, what we want to look at is this problem that is given y bar, given the vector y bar how to estimate where remember x bar comprises of your vector of t transmits symbols. So, this vector comprises of the t transmit symbols ok. So, how do you detect this vector x bar that comprises of the t transmit symbols, how do you recover the t transmit symbols from the given vector y bar at the receiver all right, so that forms the problem of MIMO receiver design all right, which we will look at in subsequent module. The problem of MIMO receiver design and its relation to convex optimization, the problem of MIMO receiver design and its relation to convex optima convex optimization is something that we want to explore in the subsequent module all right.

Thank you very much.