Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture – 24
Properties of Convex Functions with examples

Hello welcome to another module in this massive open online course. So, we are looking at complex functions let us continue our discussion.

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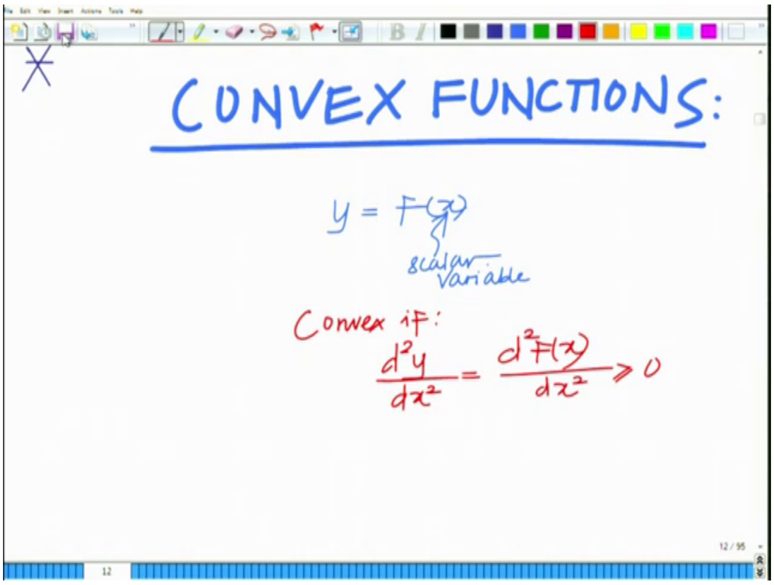

To we want to continue our discussion on convex functions ok. And well now let us look at it test for convexity let us consider first function of a single variable x y equals F of x, this is a scalar variable, we are consider will comes to functions of vectors later, so this is scalar variable. And well this is convex if d square y by d x square equals d square F of x by d x square that is the second derivative is greater than or equal to 0, and remember this is for a function of a scalar there is a function of a of a one dimensional variable x of a single dimensional variable single variable x all right.

So, the second derivative is greater than or equal to 0, then the function is convex ok..

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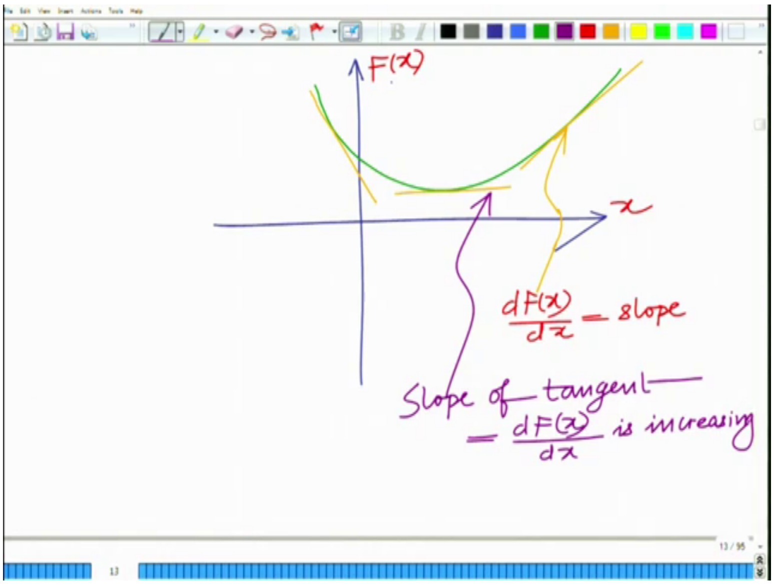

And this can be understood as follows for instance if you plot a convex function, now what you will see is if you look at the derivative of the function at different points ok. If you look at the derivative which is nothing, but the slope derivative is slope.

So, this is your x and this is your F of x. And if you can look the slope of the function right which is by the slope of the slope of the derivative of the function, which is given by the slope of the tangent at each point you can see that the deri the slope of the tangent or the derivative is increasing for a convex function, always slope of tangent equals d F x by dx is increasing.

Which means we know the test for an increasing function is if its derivative increase that is the derivative is increasing all right. So, its function is increasing if its derivative is greater than or equal to 0, all right a function is monotonically increasing with derivative is greater than equal to 0.

Now, here the slope is constantly increase monotonically increasing which means that the derivative of the slope must be greater than or equal to 0.

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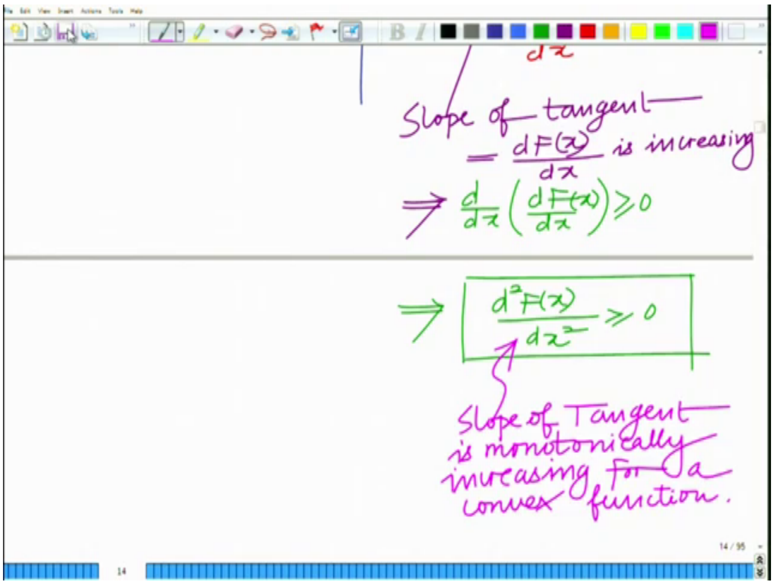

Which implies this is the slope is constantly increasing, this implies that the derivative of the slope d by dx of d x d F x by dx must be greater than or equal to 0 for a convex functions, and which gives us the result the d square F of x by dx square greater than or equal to 0. This is basically nothing, but the condition that the slope or the derivative is constantly increase slope of tangent is monotonically increasing for a convex function, is monotonically increasing for a convex function.

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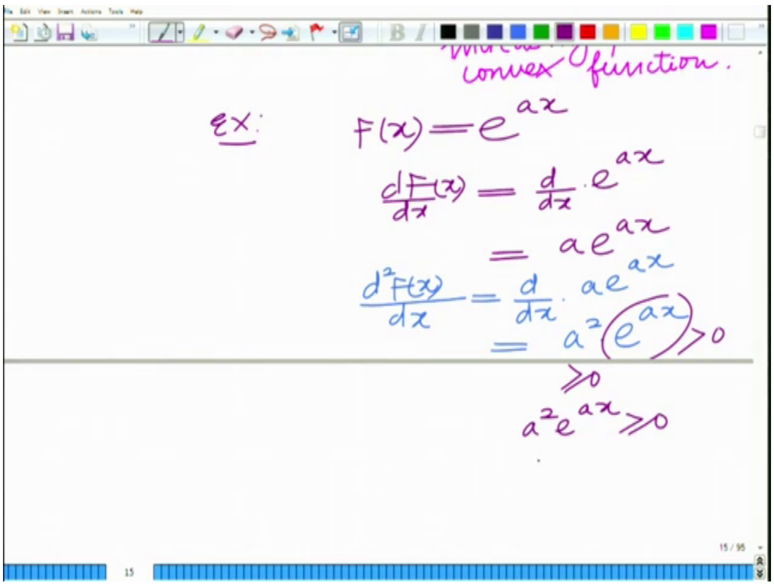

Let us take an example, consider F of x equals e raise to the power of F e raise to the power of ax for any a can be less than 0 or equal to 0 greater than 0, then d F x by dx equals d over dx of e raise to ax, this we know the derivative of the exponential this is a e raise to a. Now, if you take the second derivative F x by dx equals d square or derivative of the derivative that is d over d x of a e raise to ax, which is a square e raise to ax.

Now, we know e raise to ax exponential this is always greater than in fact, this is always greater than 0, a square is greater than equals to 0 implies this is greater than or equal to a square e raise to ax is greater than or equal to 0 all right. Because the a square is always non negative all right a square is always greater than equal to 0.

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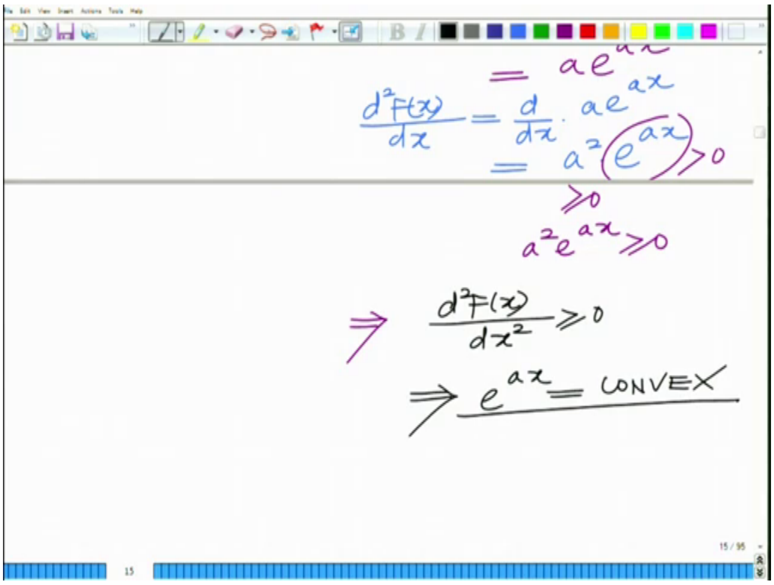

So, this implies for any value of a d square F x by dx square is greater than equals to 0 implies power e ax equals convex and it is always convex irrespective of a, a can be either negative be its negative it is a decreasing exponential, if it is possible is an increasing exponential and both are convex. So, e raise ax is always convex and the derivative tests that is the second order derivative also conforms that all right.

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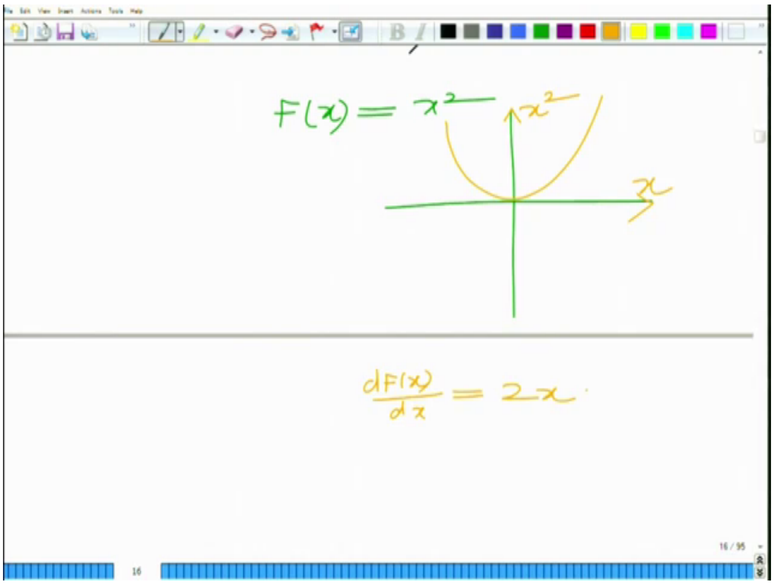

Let us look at another function that we have seen yesterday. That is F of x equals x square, you can recall that something like this correct this is a F of x equals to your x and this is your x square. And you can see that..

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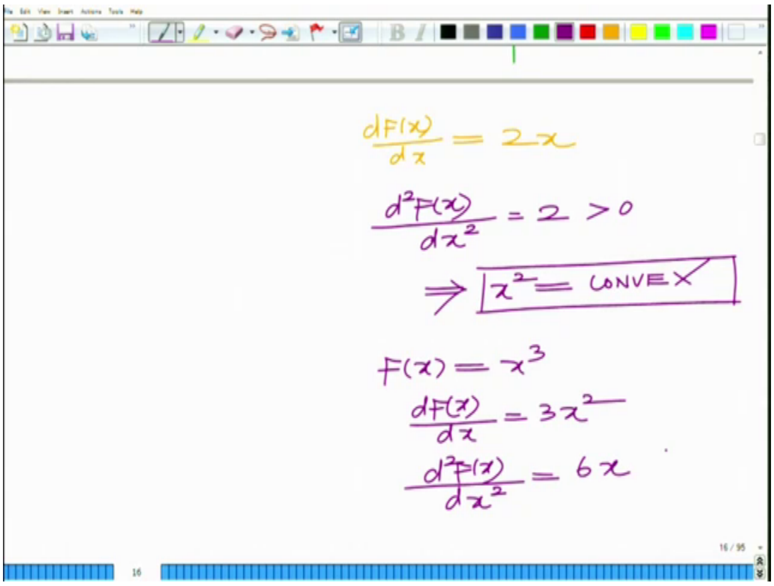

If you take this is very straightforward d F x by dx equals 2 x and d square F x by dx square equals to which is greater than 0 implies x square equals convex x square is the convex function.

On the other hand if you take F of x equals to x cube, you will notice that d F x over dx equals 3 x square and d square F of x over dx square, second order derivative 6 x greater than 0 greater than is equal to 0 only x is greater than equal to 0.

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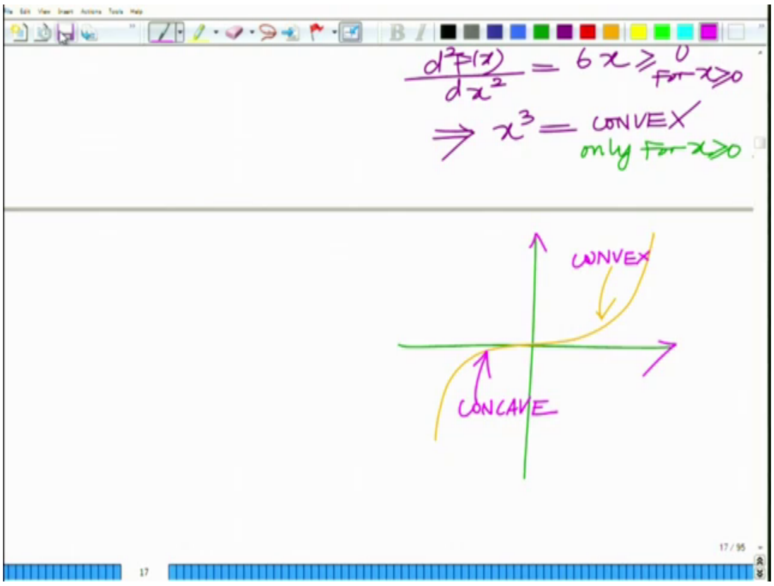

Now, this is than equal to 0 for x greater than equal to 0 implies, x cube is convex only if only for x greater than equal to 0. In fact, what is that we had seen in the previous module, if you plot x cube just to refresh your memory it look something like this all right at so, this part is convex and less than 0 it is concave all right.

Now, how about concave functions, let us look at we have to test for the convexity of the negative of that function correct.

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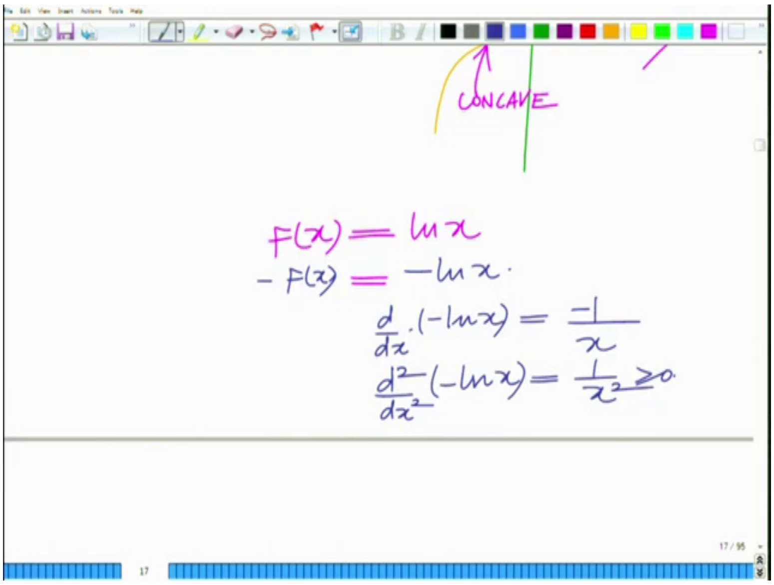

So, let us consider again or classic examples for concave functions that is the natural logarithm of x, now minus F of x equals minus ln of x. So, to show first differentiate minus the natural logarithm of x this is minus 1 over x you will considering minus of x. Since to demonstrate concavity we have to demonstrate the convexity of minus of F x ok.

Now, take the second derivative d square over d x square of minus ln of x equals 1 over x square, which is greater than equal to 0 implies minus ln x minus ln x equals contacts what we had seen in the previous module implies ln x equals the natural logarithm of x is convex.

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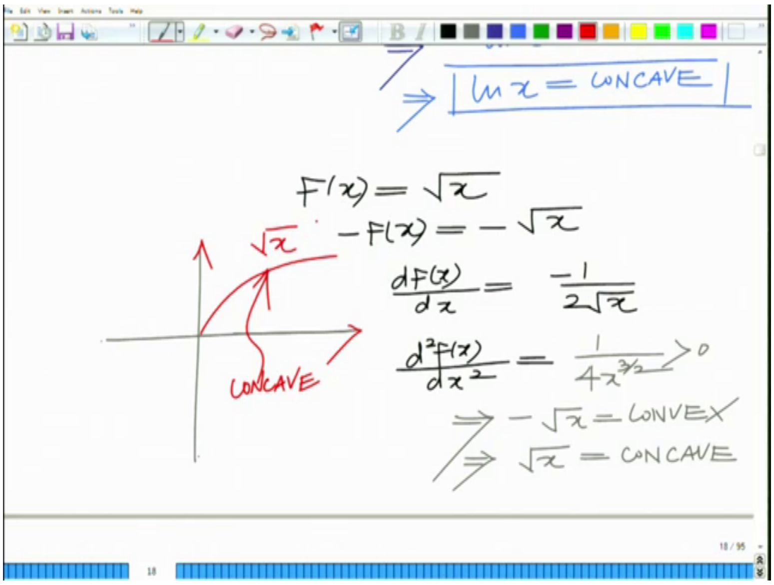

Again square root of x F of x equals square root of x minus of of x equals minus square root of x d F x by dx equals well d F x by dx equals well this is minus over twice square root of x. And d square F x by dx square this is equal to 1 over 4 x to the 3 by 2 which is greater than 0 implies well minus square root of x equals x convex implies square root of x equals concave ok. And that is what we had seen yesterday, if you plot square root of x it looks like this and this concave and this is concave. So, this is square root of x all right.

Let us now come to the norm again norm cannot use the second derivative test. Let us look at another interesting function in fact, let us look at a practical applications of this is an interesting aspect and we are also seen this before.

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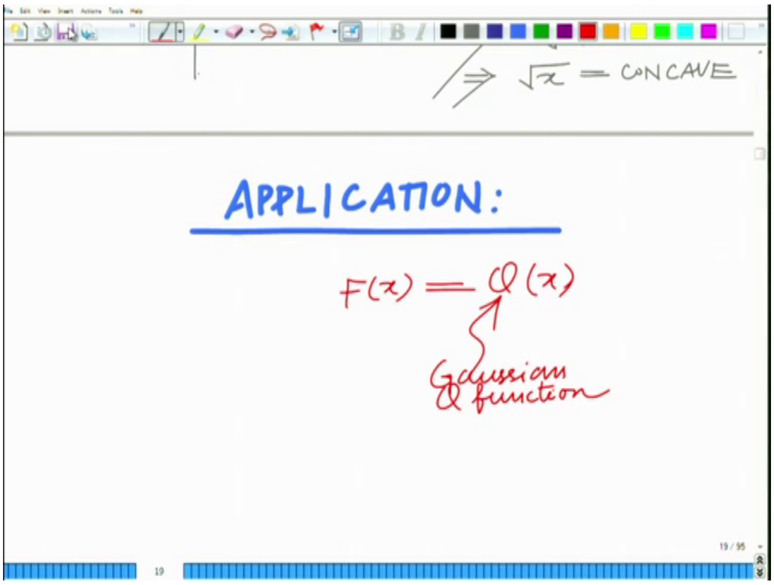

So, let us come to an application, let us look at y of x I am sorry, let us look at F of x equals Q x. So, your Q x is the Gaussian Q function the Gaussian Q function remember this is the CCDF, complementary cumulative distribution function of the standard normal random variable, there is a Gaussian random variable with mean 0 and variance 1, this is also the tail probability of the standard Gaussian random variable ok.

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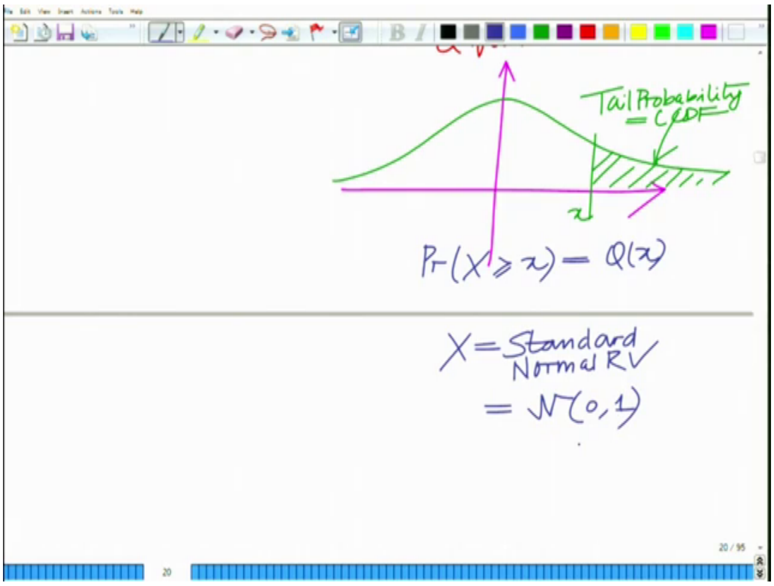

You can recall this is given as follows, if you have a PDF of the standard normal, random variable mean equal 0 variance equals one this is the CCDF that is the probability that x is greater than or equal to x ok.

So, this is the tail probability ok. Also the CCDF the complementary cumulative distribution function, which is basically the probability that takes is greater than or equal to X this is your Q x where x equals standard normal random variable, denoted by N of 0 1 that is Gaussian random variable mean equal to 0 variance equal to 1.

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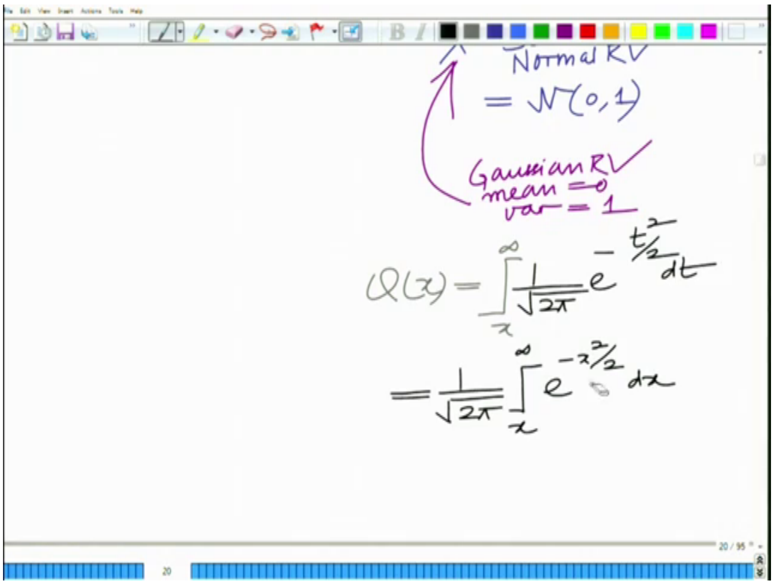

Gaussian random variable with mean equals to 0 and variance equals to unit that is basically the CCDF of that, random variable complementary cumulative distribution function of that random variable, is basically the q function.

And remember the expression for the Q function, we have also seen that or you must also be familiar with that that is basically 1 over square root of 2 pi integral of the Gaussian probability density function, or let me write it integral of x to infinity since is the probability that its greater than equals to x integral of x to infinity e power minus x square d by 2 dx, which if you take the constant outside you can also write this as integral x to infinity e raise to x square I am sorry e raise to we can use a different variable of integration e raise to minus t square by 2 dt ok.

Now this Q function is a very interesting and deserve it frequently arises in communication and signal processing also, because this q function represents the bit error rate might have seen the expression Q of 2 raise Q of square root of 2 E b or n not which is an also write as Q of under root of SNR or two times SNR depending on how you define it.

So, this denotes the bit error rate this is the bitter the rate of the wireless computer rate of additive white Gaussian noise channel BESK, bit error rate of BPSK binary phase shift key over an a additive white Gaussian noise channel. This is the bit error rate over an AWGN channel and in fact, it has a lot of applications arises quite frequently as I said in communications as well as single processing all right. So, now, want you want to show is the this Q function which is a lot of practical applications that this is convex.

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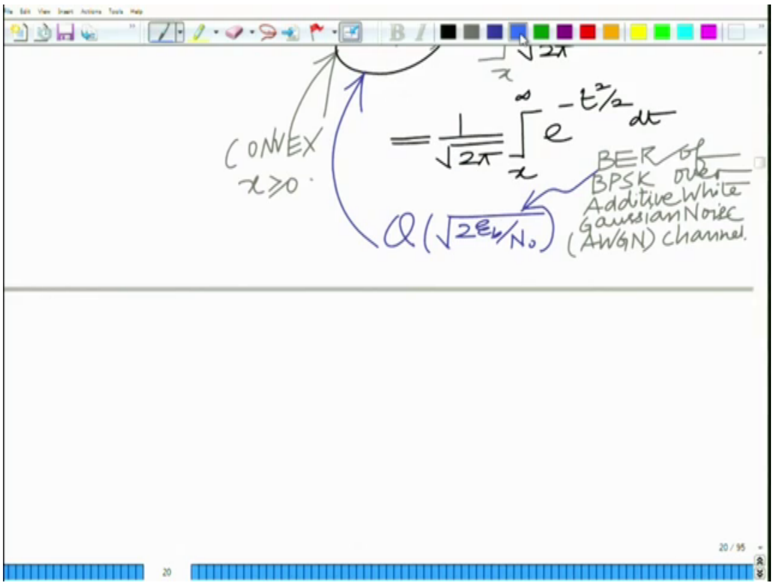

We want to demonstrate that this is convex and in fact, that is convex for x greater than or equal to 0, the slight qualification its not convex over the entire x convex only for x greater than equal to 0.

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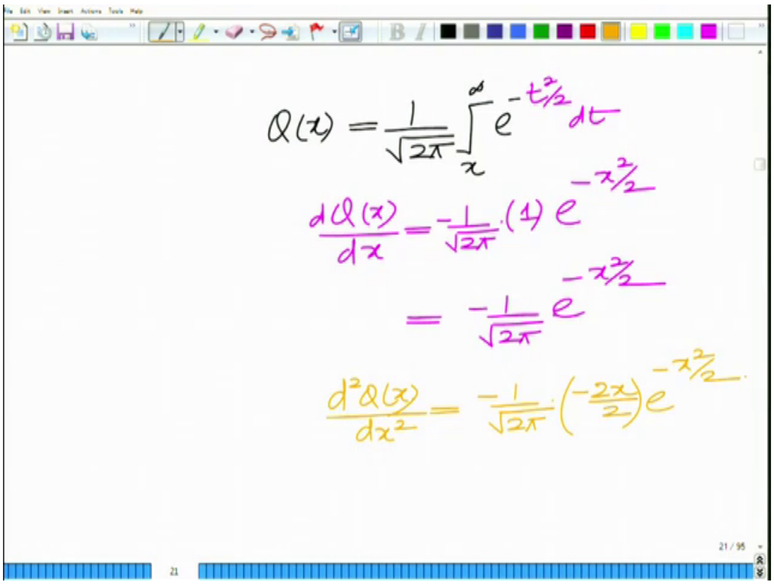

In fact, we start with the definition q of x equals 12 1 over square root of 2 pie, integral x to infinity e raise to minus x square by 2 dx you take e raise to minus t square by 2 dt apologize. You found the first derivative of this that is 1 over square root of 2 pi derivative of the top limit which is 0, because is a constant minus the bottom limit correct. So, we have minus derivative of the bottom limit is derivative of x which is times 1, times the integral evaluated at the bottom limit that is e raise to minus x square by 2 that is it. So, the derivative of Q of x phase minus 1 square root of 2 pi e raise to minus x square over 2.

Now, which means if you take the second derivative of this d square Q x by dx square that will be minus square root of 1 1 over minus 1 over square root of 2 pi minus, 2 x over 2 e raise to minus x square over 2.

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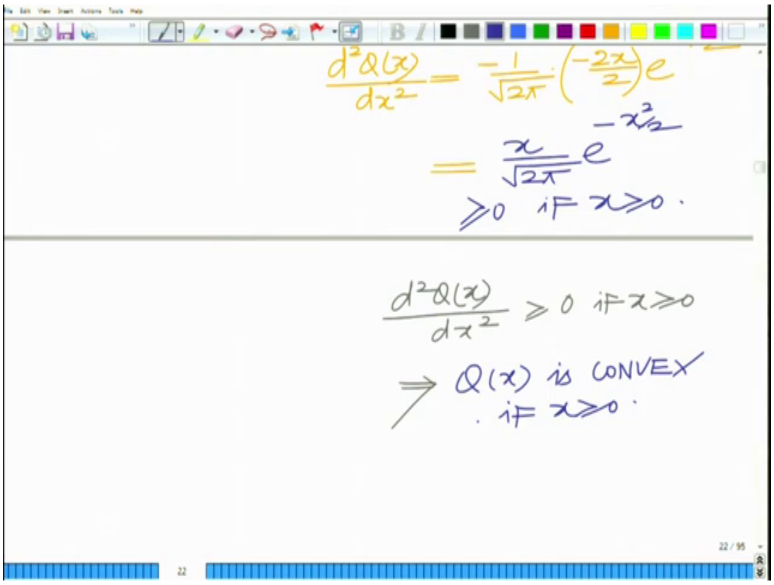

Which is equal to if you look at this which is equal to x over square root of 2 pi e raise to minus x square by 2. And now you can see is the, this is always greater than equal to 0 e raise to minus x square by 2, x is a course greater than equal to 0. So, when x is greater than equal to 0 this is greater than equal to 0 which means greater than equal to 0, if x is greater than equal to 0.

So, we have the second derivative d square Q x by d square greater than equal to 0, if x greater than equal to 0 implies that the Q function of x is convex, if x is greater than or equal to 0 and this is in fact, very interesting property.

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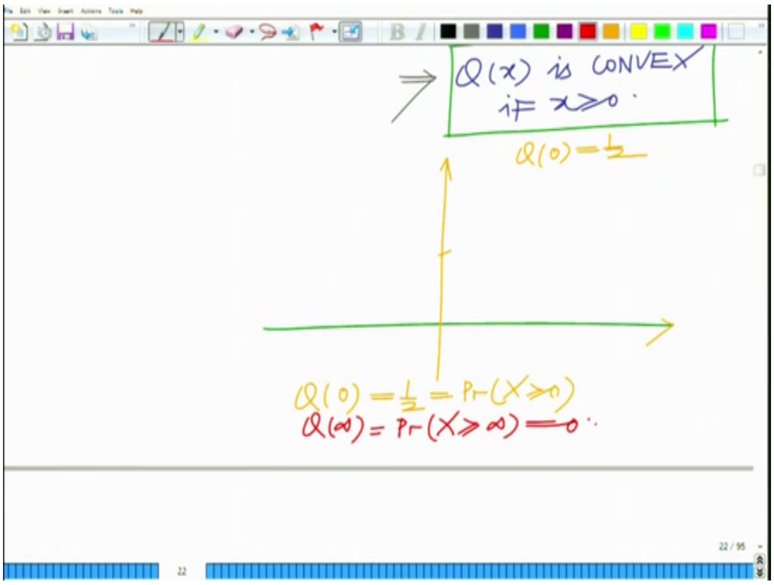

That will again use in fact, quite some in the future, if you look at a plot of the Q function the CCDF decreasing constantly 0, we have Q of 0 equals half, because this is the CCDF the probability that X is greater than 0 is the point of symmetry correct Q of 0 is the probability, that X is greater than equal to 0 which is half which is equal to the probability that the random variable X is less than equal to 0 for this, standard Gaussian random variable with mean 0 and variance 1 ok.

So, Q of 0 equals half equals probability x greater than equal to 0 Q of infinity equals probability x greater than equal to infinity equal 0 at infinity it is 0.

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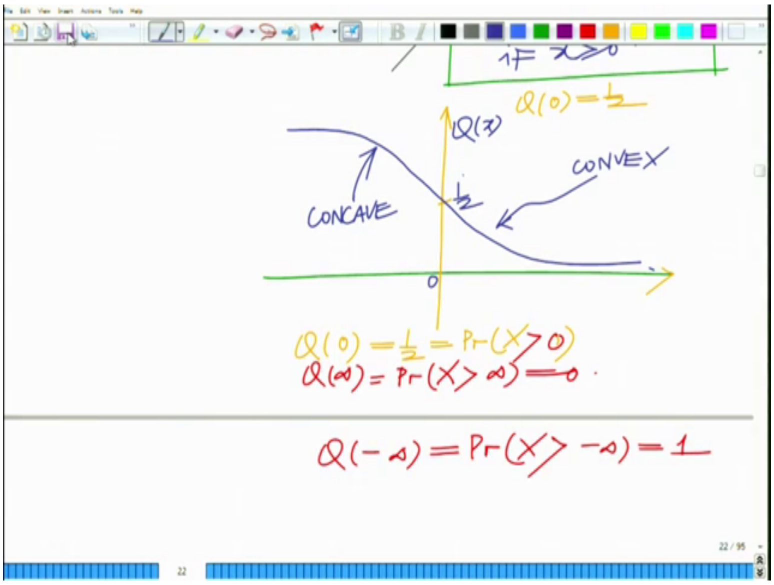

Q of minus infinity equals probability X greater than equal to minus infinity. Since it has to be greater than equal to minus infinity I am sorry X has greater than equal to yeah greater than in fact, we can just replace this by greater than equal to greater than equal to greater equal does not matter ok.

So, this is X greater than minus infinity is 1 therefore, starts at 1 and it decreases and therefore, if you look at this portion this is convex and this is half at x equal to 0 this is half, Q of x it is half this is convex and this is this portion which is less than 0 this is concave ok. So, this is your Q function it is convex for X so, q function is basically convex for X greater than or equal to 0 ok. So, that is what we have seen all right.

and finally, coming now to the norm the norm, it is a straightforward to show that it is convex the 2 norm so, well first we note the triangle. So, first consider the norm.

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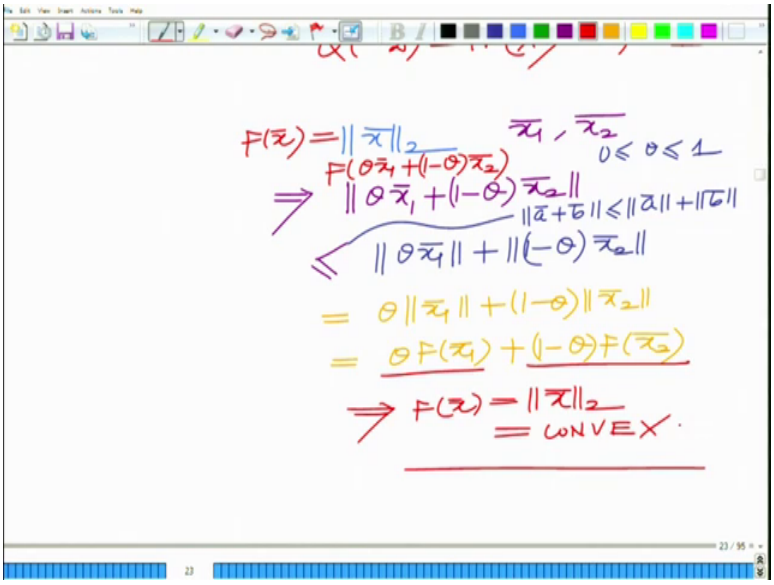

Now, what we want to do is we want to consider any two vectors x 1 bar x 2 bar we want to find the norm of theta x 1 bar plus 1 minus theta x 2 bar ok. The norm or the value of the function at the convex combination now using triangle inequality we know that this is less than. Now, observed that of course, for any convex combinations 0 is less than equal to theta less than equal.

So, norm of theta x less than equal to norm theta x 1 bar, from the triangle inequality ok. We are using the property here is there is a bar plus b bar is less than equal to norm a bar plus norm b bar, now theta in 1 minus theta are greater than equal to 0, because 0 less than equal to theta less than equal to 1. So, this is simply norm of theta times x 1 bar is theta times norm of x 1 bar plus 1 minus theta norm of x 2 bar, which is basically your theta times F of x 1 bar plus 1 minus theta times F of x 2 bar, where we have F of x bar equals norm of x bar the 2 norm. So, all these are basically the two norm and this is your f of theta 1 bar plus 1 minus theta times x 2 bar.

So, what we have shown is F of theta times x 1 bar plus 1 minus theta times x 2 bar is less than or equal to theta times F of x 1 bar plus 1 minus theta times F of x 2 bar implies F of x bar equals norm, the 2 norm this is convex ok. So, this is sort of a straight forward way to show that the 2 norm is convex the l 2 norm is convex. However, this might be a slightly converge some way to show it especially for especially for any function of a vector, correct or a function of more than one variables that is the function of a vector all right.

So, we have to one has device a general test, to demonstrate similar to the test that we are shown earlier the second order derivative test for a scalar for a function of a scalar variable single variable x 1 has to derive or one has to arrive at a similar test for a function of two vector all right, which are something that we are going to develop or look at in a subsequent modules.