Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture - 23
Introduction to Convex and Concave Functions

Hello, welcome to another module in this Massive Open Online Course. In this module let us start looking at a new topic and that is of Convex Functions alright.

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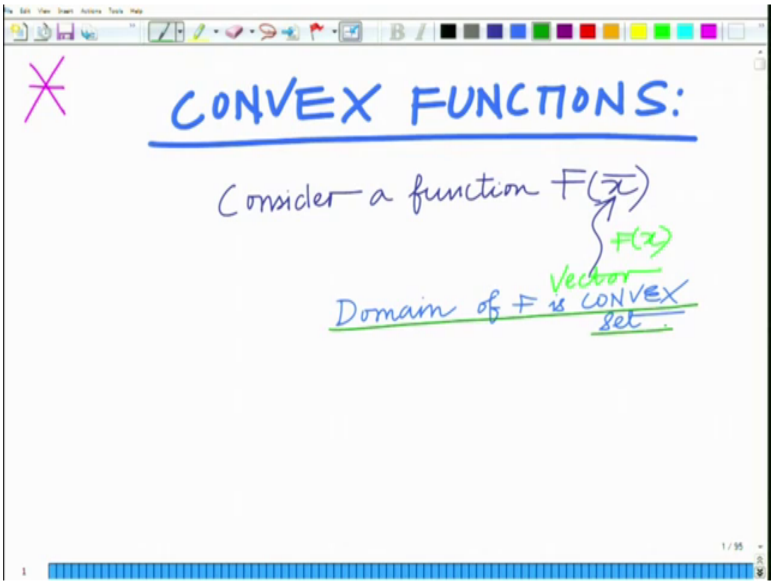

So, want to start looking at another important aspect or building block of the optimization framework that we want to eventually build. And therefore, you want to start looking at convex functions because, these are going main important for part of the optimization problems that we are going to consider. Now what is the convex function? A convex function for instance can be defined as follows consider a function F of x bar.

Now, x bar now x bar this can be simply F of x. So, that can be x bar in general it can be a vector ok. So, you can either you can also have F of x, but in general you can have F of x bar that is a function of a vector. Now, let also the domain of F that is the set over which it is defined domain of F is a convex we already seen what is a convex set. So, the domain of F is a convex set plus F is now F is convex, if the following properties satisfied.

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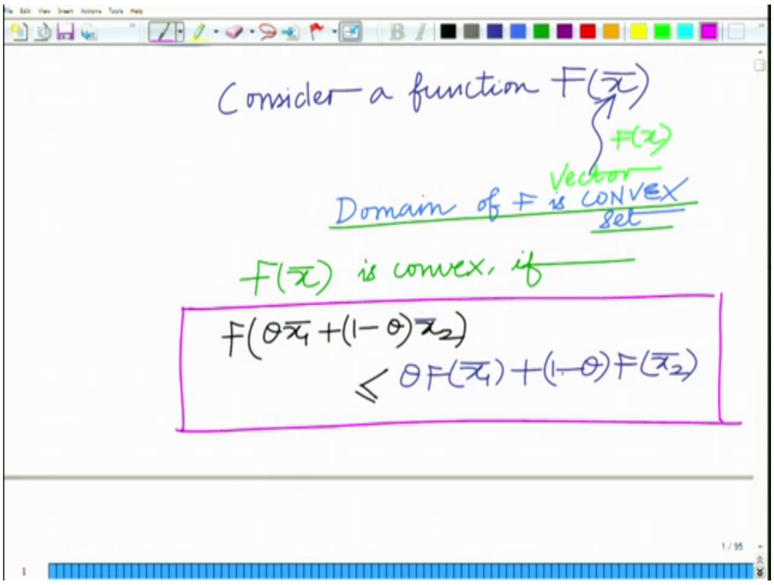

If you have two points x 1 bar and x 2 bar. And you take a convex combination theta times x 1 bar and 1 minus theta time x 2 bar. And evaluate the function at that convex combination that has to be less than or equal to the convex combination of the values of the function itself at x bar x 1 bar and x 2 bar.

So, that would be 1 minus theta F of x 1 bar plus 1 minus theta F of x 2 bar. So, what we must have is that F to be convex it must be the case that F of for two points x 1 bar x 2 bar that has be that belong to the domain of F. F of theta time x 1 bar plus 1 minus theta times x 2 must be less than equal to theta times F of x 1 bar plus 1 minus theta times F of x 2 bar. This can be represented pictorially as follows to better understand this look at the following picture ok.

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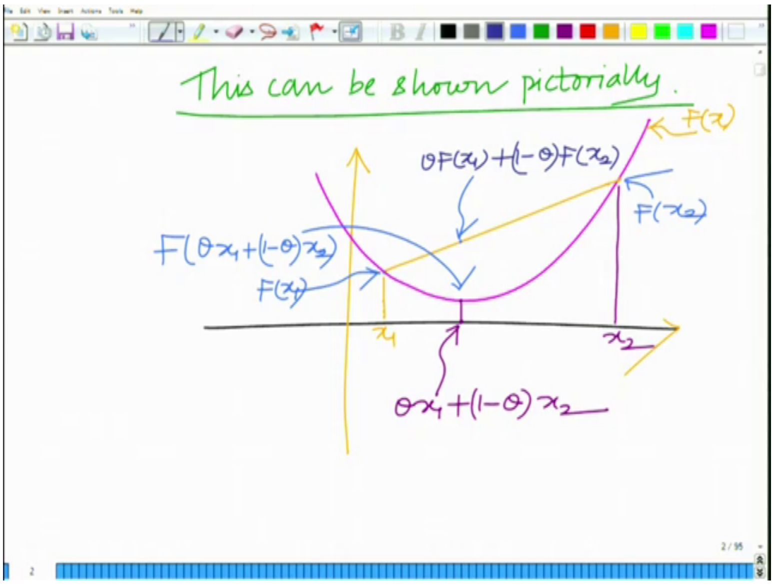

It can be seen pictorially as below I am just going to draw a diagram that will better illustrate this. So, let us say we have function that looks like this. Let us take 2 points and let us join now this is your F of x consider a simple 1 dimensional x let these to be the points let this be your point x 1 correct. So, this is your point x 1 and this is your point x 2.

And let us take a convex combination of x 1 and x 2 that lies along the lines. So, this is your theta times x 1 plus 1 minus theta time x 2 ok. And this value, if you look at this value this is your F of theta time x 1 that is the value of the function at theta times x 1, this is the value of the function.

This is your F of theta times x 1 plus 1 minus theta times x 2. Now on the other hand look at this, this point is F of x 1 that is one end of this chord and this point is F of x 2. And we know now that if we take a convex combination that represents the line segment correct. So, therefore, we now consider the line segment that is joining this points F of x 1 and F of x 2 and theta times F of x 1 plus 1 minus theta times about x 2 is this point theta times F of x 1 plus 1 minus theta times F of x is this point. And you can see therefore, that theta times F of theta times x 1 plus 1 minus theta times x 2 is less than or equal to theta times F of x 1 plus 1 minus theta times F of x 2.

Which in essence is basically, saying that if you think about this what this is essentially saying, that if you take two points on the function, join them by a chord, the chord always lies above the function. So, if you take x 2 points x 1 x 2 join the chord right between F of x 1 and F of x 2 the chord joining F of x 1 F of x 2 always lies above the function between these two points x 1 and x 2 that is a very that is an intuitive definition of convex.

It is for convex function curves upward it looks like a bowl alright, your regular bowl are typically a convex function looks like a bowl that curves upwards although that is not always the case. But typically you when you think of convex functions the prototype of a convex function is a bowl kind of function which curves upwards.

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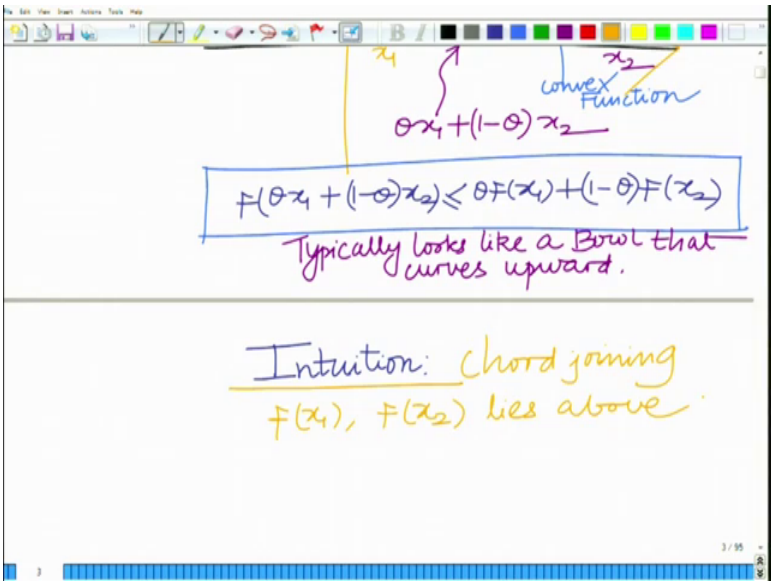

So, this basically shows that your. F of theta times x 1 plus 1 minus theta times x 2 less than or equal to theta times F of x 1 plus 1 minus theta times F of x 2 ok. So, this is your definition of convexity and this is what you can see from so, this is your convex function. Typically convex function looks like a bowl that curves upward, it looks like a bowl. And what this means intuitively, the intuition is that the chord joining F of x 1 comma F of x 2 lies above the function.

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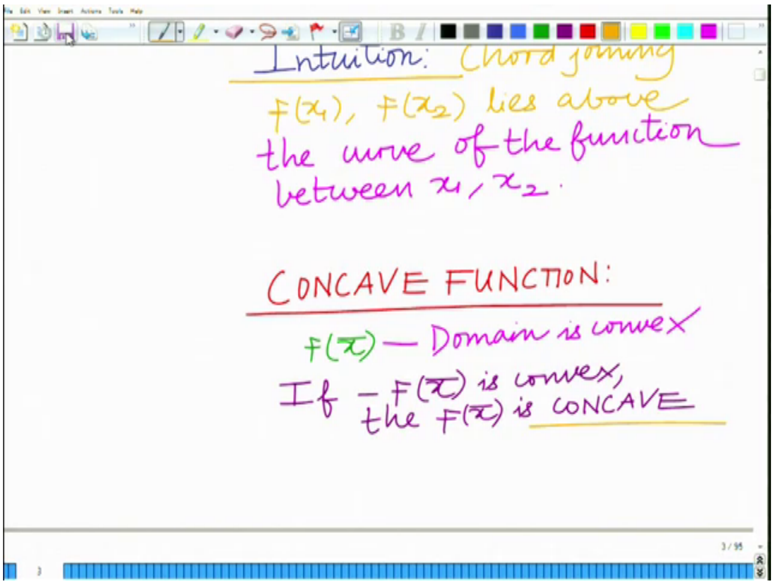

The curve or the plot of the function between x 1 and x 2 alright so, the chord that is joining x 1 and x 2 lies. The chord joining F ox x 1 and F of x 2 lies above the function between x 1 and x 2 ok. And this is a very important class of functions that we will frequently encounter. In fact, most of our optimization problems will be built on convex functions already they felts very important to understand. The definition of a convex function and also the various properties of convex function alright.

On the our hand now, naturally a concave function is 1, which is now for a concave function again it is important to remember that if you look at a function of F x bar. The domain has to still be convex, either concave or convex the domain that is the set over which it is defined is convex. And, if minus F is convex then F of x bar F of x bar is concave alright so, F of x bar is concave if the domain is convex and minus of F is convex.

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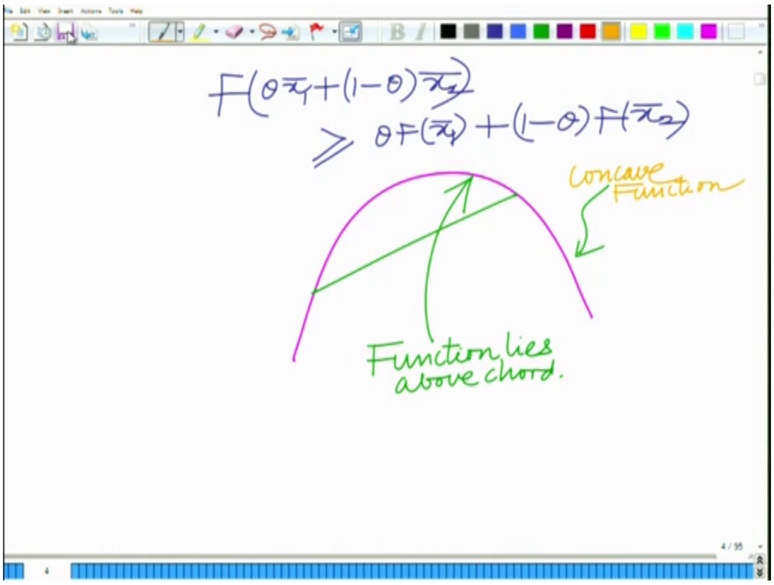

Naturally this implies that air force F of theta times x bar plus 1 minus theta times x bar is greater than equal to theta times F of x bar plus 1 minus I am sorry x 1 bar x 2 bar theta times x 1 bar plus 1 minus theta times x 2 bar F of that is greater than equal to theta times F of x 1 bar plus 1 minus theta times F of x 2 bar. And naturally a concave function curves downwards.

So, curvature is down so it looks like this and you can clearly see that the chord, if you look at the chord joining any two points the chord lies a function lies above the chord. So, this is function lies above the chord for a concave function, this is your concave ok.

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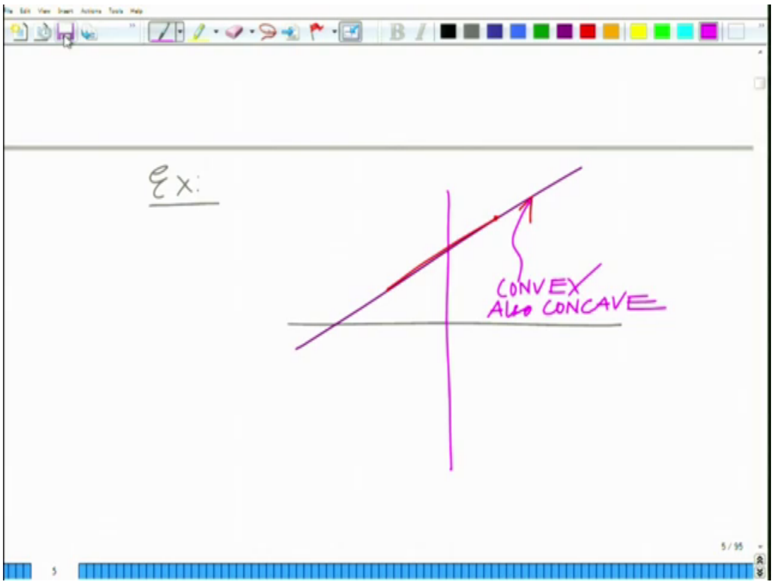

And we cannot look at several examples to understand, this notion of concave and convex functions better. The simplest example is that the most simplest example is that of a straight line, if you look at a straight line you take any two points on the straight line, the chord itself lies on the function the functions of the function.

So, the chord either lies above or below the chord coincides with the functions. So, we can say the straight line is convex as well as concave. So, the straight line is in fact, convex. And it is also concave for that same matter because, of the function coincides with the curve coincide with the chord joining any two points. So, this is convex and this is also concave so, any straight line is convex and also concave ok.

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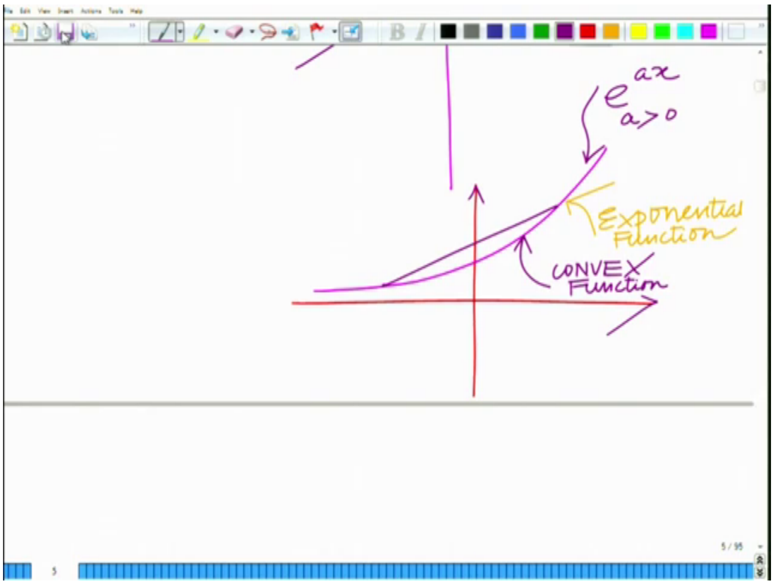

A convex function or what you can also say he is a strictly convex function that is curves upwards. And very important class of such functions is the exponential function if you look at this; this is e to the power of a x, where a greater than 0 ok. So, also known as the exponential this is exponential function. And you can clearly see once again you take any two points, the chord lies above the function.

And therefore, this is of course, we are going see later a rigorous test to establish convexity and concavity, but right now based on our intuition you can clearly see this is a convex function and this exponential function there is an increasing exponential which is convex function; it is an important class of convex functions. And in fact, one can also consider a decreasing exponential that e raise to the power of a x that arises for a less than 0. And we will also we will also see you can also quickly see just by visual inspection that that is also a convex function.

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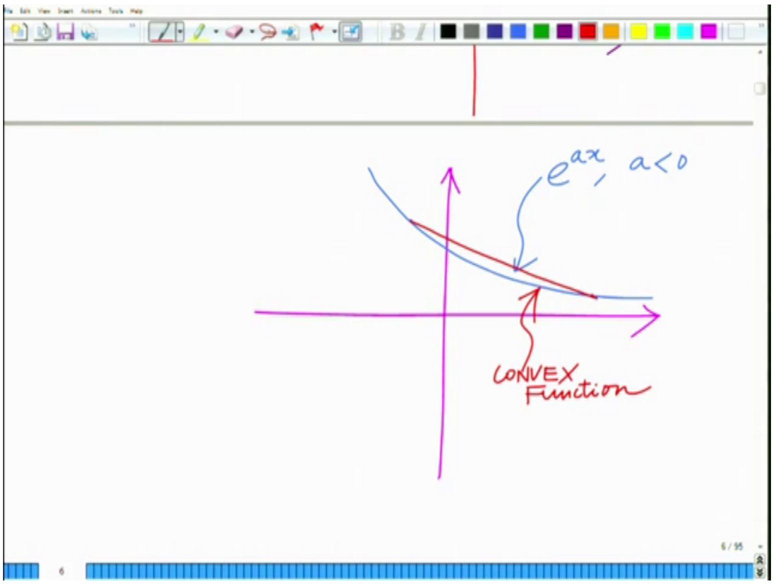

So, what happens? If it is not very difficult to see if you plot e raise to the power of this is your e raise to the power of a x a less than 0, also an exponential. And you can again see that take two points join the two points the chord ok. You take any two points and you join the two points, the chord lies above the function and therefore, this is a convex function.

And of course, if a equals 0 this becomes e raise to a e raise to the power of 0 which is 1 which is again a convex function. So, one can say e raise to the a x in general for any real value of a is a convex function.

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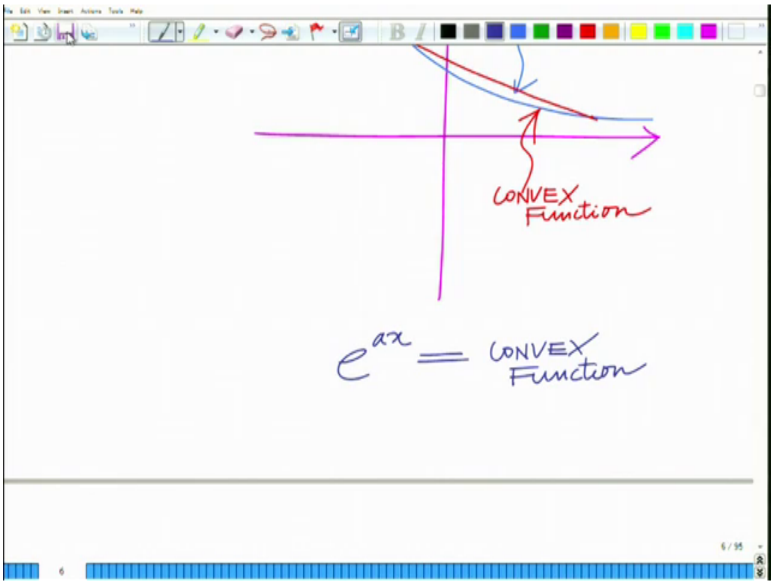

So, one can conclude that e raise to x e raise to x for any value of a the exponential function for any value of a is a convex function. If a is less than 0, it is decreasing if a is greater than 0 it is increasing ok, but in any case it is a convex function.

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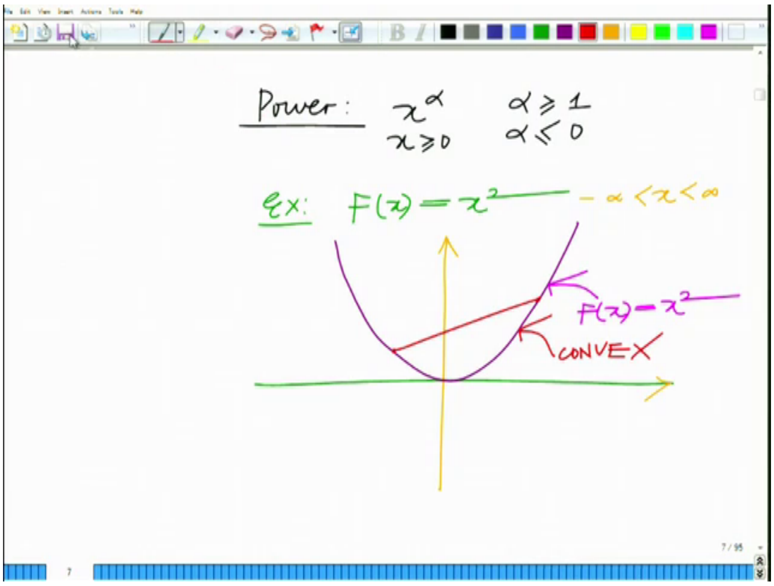

Let us look at the power x to the power of alpha. If alpha is greater than or equal to 1. This is defined for x greater than equal to 0 alpha greater than equal to 1 or alpha less than or equal to 0. In both cases you can see it is convex for instance. If you look at F of x equals x square; you can see you are well familiar that this is a classic bowl shape function. In fact, F of x equal to x square you can define it for entire minus infinity less than x less than infinity and it looks something like this correct.

And at x equal to 0, it is 0 and this is your classic example of a convex function. This is F of x equals x square and you can clearly see if you join any two points by chord it lies above the function. So, this is a classic example of a convex function. In fact, it looks like a perfect bowl this is your F of x equals to x square.

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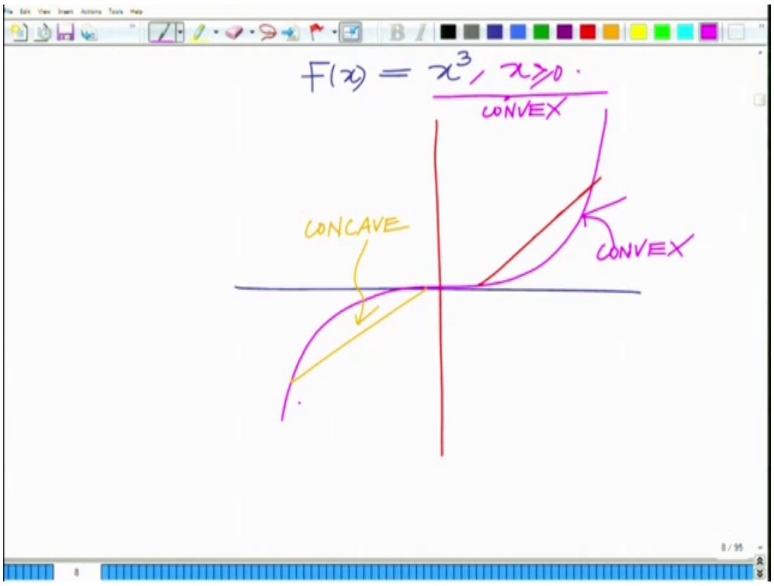

And now on the other hand if you look at F of x equal to x cube, now, you might be under the misconception that F of x equal to x to the power of alpha. For alpha greater than equal to 1 is always convex, but if you look at F of x equal to x cube for x less than 0 this is negative; for x less than 0 this is negative. And this is negative and x greater than 0 it is positive, but you can see here x greater than 0 the chord lies above the function.

But for x less than 0 the chord lies below the functions. So, the x less than 0, it is concave and for x greater than 0 it is convex. So, this is only convex for x greater than or equal to 0 F of x cube. So, that is why it is important to consider the suitable domain. So, this is now, convex F of x equal to x cube for x greater than equal to 0 is convex in this region x less than equal to 0 it is concave. Further one can show that well we can check this later.

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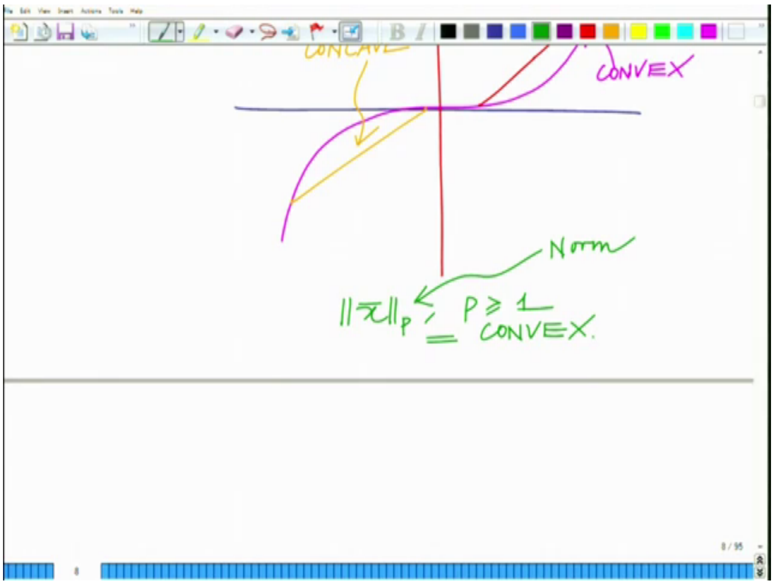

That is any norm x bar base p, p greater than or equal to 1. This is this is of course, of a function of a vector, this is convex field keep this for later, but this is important.

That is the norm, p norm for p greater than or equal to one this is convex ok. So, we have seen several examples of convex functions very straight forward mostly one dimensional function mostly functions of single variable. So, we have seen that the exponential function is convex the straight line of course, is convex and concave exponential function for all values of a is convex, x square is convex over the entire real line, x cube is convex only if x is greater than or equal to 0. Let us look at the examples of concave functions.

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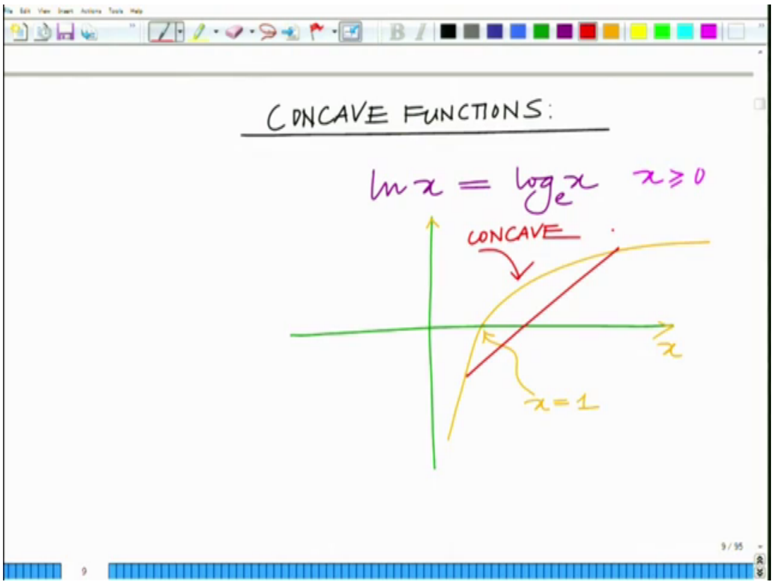

The classical example of concave function; concave function the classic example is l n x always you can use log of x the natural logarithm is also known as a natural logarithm log x to the base e. And if you draw this it is only defined for of course, x greater than equal to 0, if you draw this at x equal to 1 it becomes 0. And at x equal to 0 tends to minus infinity and as x tends to infinity tends to infinity and if you look at any two points join the chord the function always lies about the chord this is concave.

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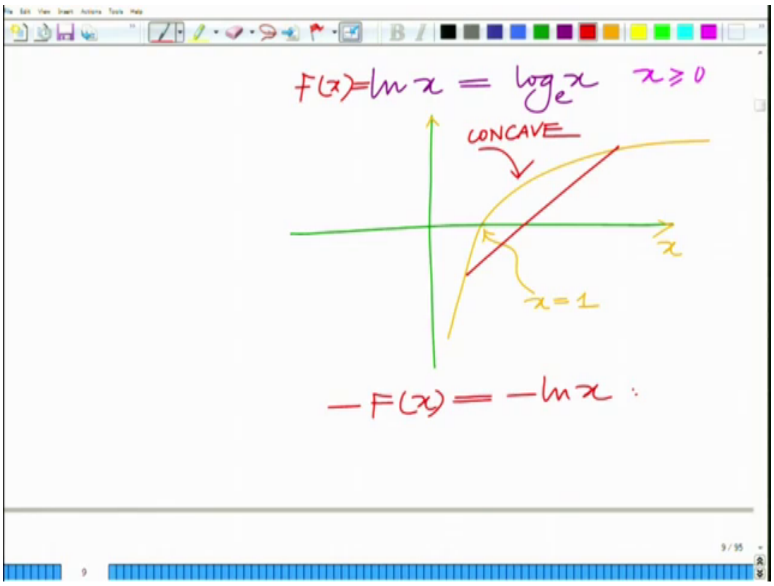

In fact, the right way to do it is you have to look at F of minus of F of x. So, if you look at minus of F of x so, this is your F of x. Minus of F of x because as for a definition remember minus of F of x, if minus of F of x is convex then F of x is con concave.

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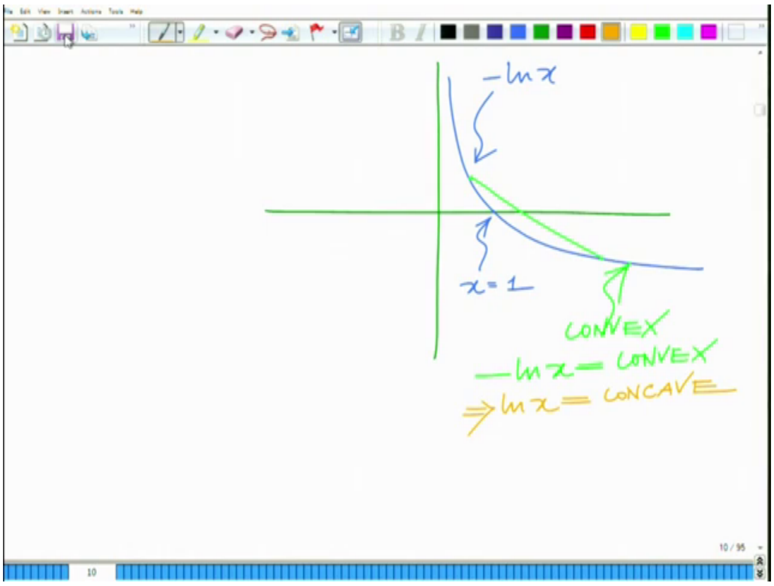

And if you plot minus of F of x then that looks like this; this is 0 at x equal to 1 as x tends to 0, x tends to infinity as x tends to infinity tends to minus infinity. So, this is minus of the natural logarithm of x chord lies above the function so this is convex correct. So, minus natural logarithm is convex and this implies that the natural logarithm of x equals is a concave function. That is the technically the correct way to demonstrate concavity.

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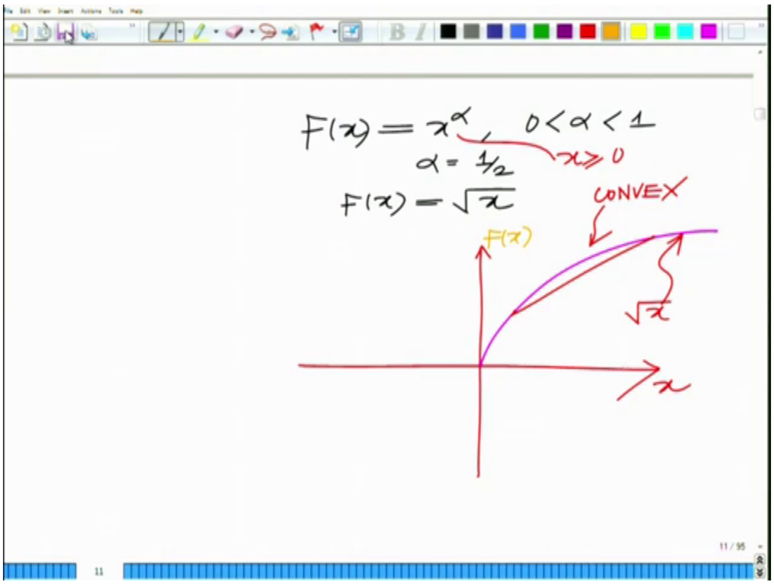

Another important class of concave functions is F of x equals x power alpha for alpha for 0 less than alpha less than 1 ok. And in fact, for example, let us say alpha equal to half, we have F of x equals square root of x and naturally this is defined only for x greater than or equal to 0 correct. So, we will restrict the domain here x greater than equal to 0. And if you plot square root of x for x greater than equal to 0, it looks something like this and you can once again see the function lies above the chord so, this is square root of x correct.

Square root of x and this is convex, this is your function F of x equals square root of x for x greater than equal to 0 it is convex alright. So, in this module we have seen a very important definition that of convex and concave functions; I urge you to go through this once again and understand it thoroughly because we are going to invoke this and use this property very frequently.

Thank you very much.