Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture – 21
Example Problems: Property of Norms, Problems on Convex Sets

Hello welcome to another module in this massive open online course. So, we are looking at example problems related to matrices and convex sets. Let us continue our discussion.

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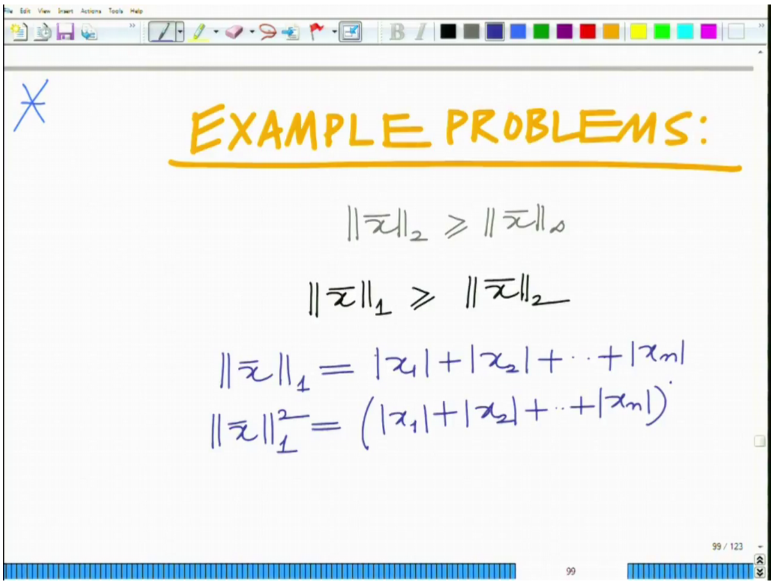

So, you are looking at example problems in particular in the previous module we started looking at the properties of norms, correct in and in particular we have seen that for instance the l 2 norm is greater than the l infinity norm.

Now, in the same way and one can also show that the l 1 norm for a vector is greater than or equal to the l 2 norm ok. And this can be shown simply as follows. If you look at the l 1 norm that is simply for an n dimensional vector, the sum of the absolute values of the magnitudes that is what we have seen that is the l 1 norm ok. And if you look at the square of the l 1 norm the square of this quantity that is simply magnitude x 1 plus magnitude x 2 plus magnitude x n whole square.

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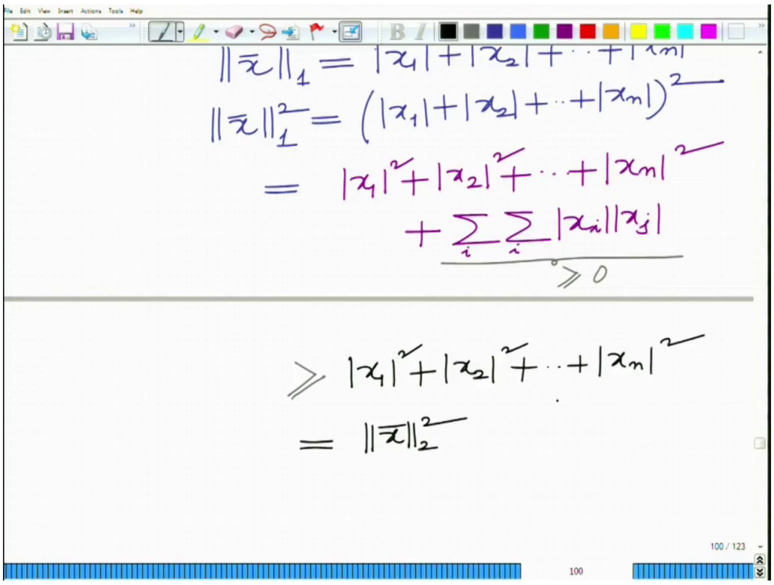

Which you can expand as follows which is basically well it is magnitude x 1 square plus magnitude x 2 square plus so, on magnitude x n square plus summation over all combinations of i comma j the product magnitude x i into magnitude x j.

Now, this quantity this products cross products, the sum of all cross products, this is greater than or equal to 0. Because the magnitudes are positive implies that this is greater than or equal to magnitude x 1 square plus magnitude x 2 square plus magnitude x n square and this is nothing, but the 2 norm square norm x 2 bar square.

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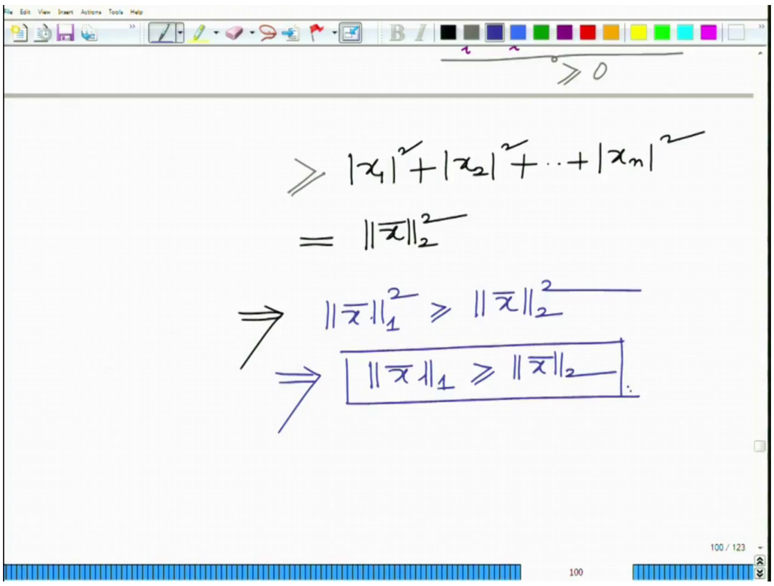

So, we basically have what we have is that norm x bar the 1 norm square is greater than or equal to the 2 norms square and therefore, this implies both these quantities are positive the square of 1 is greater than the square of the other this means the 1 norm is greater than or equal to the 2 norm.

And we are already seen that the 2 norm is greater than the infinity norm. Therefore, putting all these things together, you have the 1 norm is greater than or equal to the 2 norm is greater than or equal to the infinity norm.

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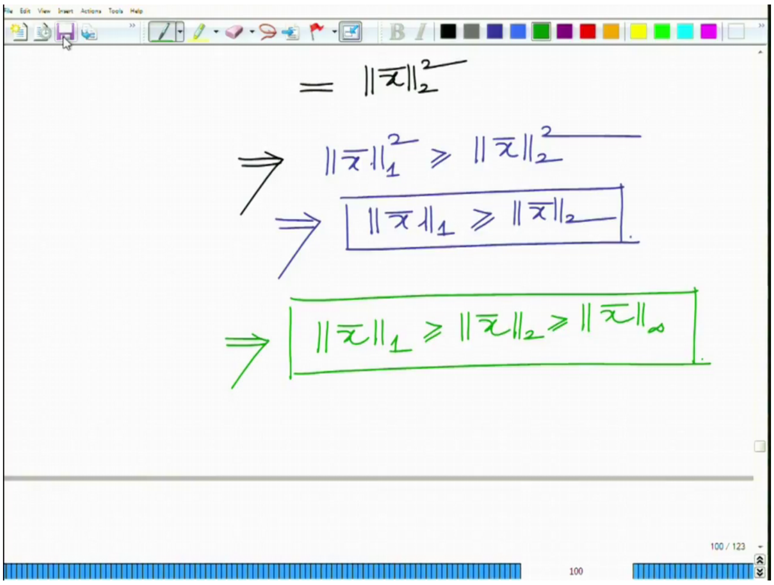

This is the result that we have the 1 norm of a vector is greater than or equal to the 2 norm is greater than or equal to that infinitely norm. So, 1 norm is sum of the magnitude values of the element. The 2 norm is you can think of this as the length the Euclidian the length of the vector in Euclidian space. And the infinity norm is the maximum value of the magnitudes of the different elements of the vector. These are the different norms ok. Let us continue our discussion, now let us move on to look at example problems related to convex sets and their applications alright. So, we have defined convex sets look at their properties. Let us do some example to understand these things better ok.

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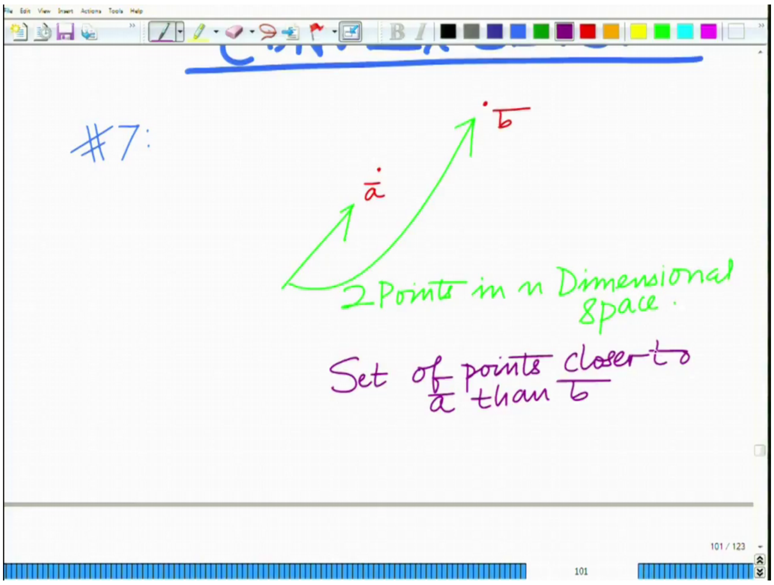

So, what we want to look at next is convex sets, if you want to explore different types of convex sets their definition classification and so, on ok. Now, for instance let us start with the first problem, problem number let us call this as problem number the previous one as problem number we had this, let us call this as problem number. So, we had problem number I think 6. So, let us call this as problem number 5 6. So, let us call this as problem number 7, I think we can start problem number or let us forget this we can start problem number 7 over here.

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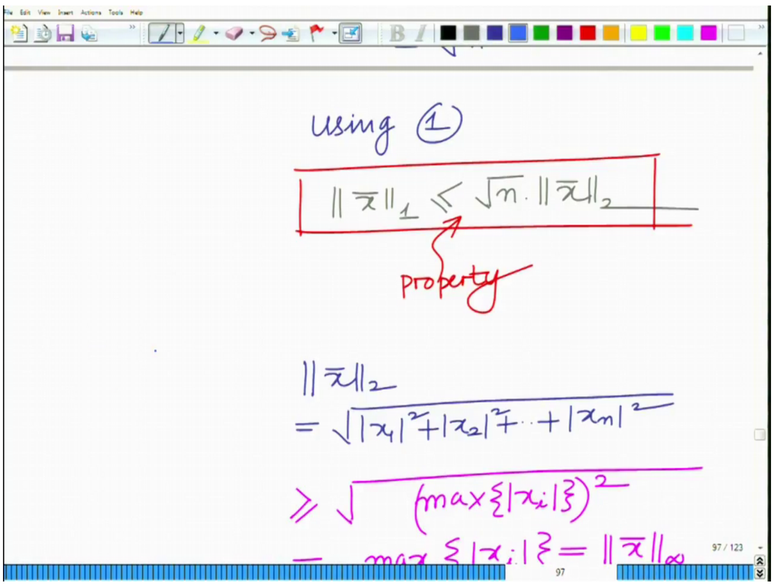

Let us call this as problem number 7 and what we want to show is the set of let us say we can consider two vectors or two points in n dimensional space, denoted by the vectors a bar and the vector b bar and we want to find. So, these are two points in n dimensional space ok, these are all points to points in n dimensional space. Now, we want to find the points set of points closer to a bar set of points that are closer to a bar than b bar.

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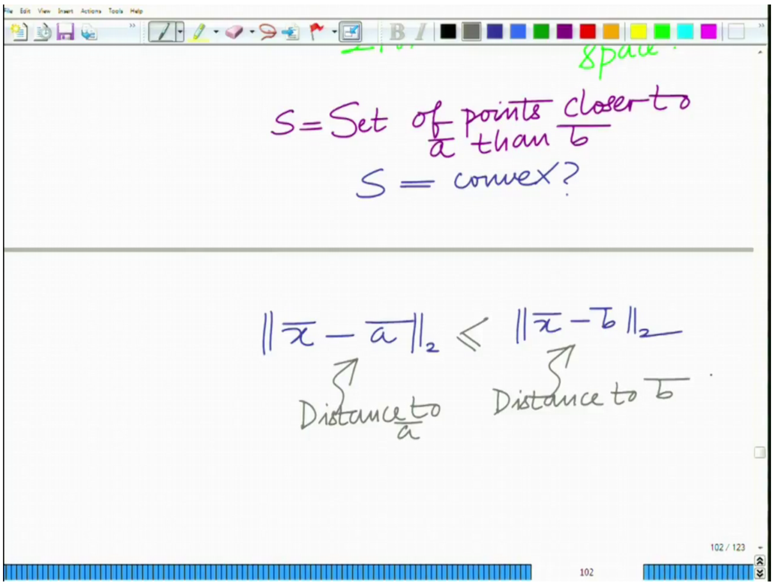

And what we want to see is if we call this set S we want to show, we want to find is S convex is this set S of all points which are closer to a bar that is where our two points in n dimensional space a bar and b bar. It is the set of points x bar which is which are closer to a bar right, than b bar is this set of points convex. That is if you look at the set of points, the distance of any point x bar between a bar we know that distance is the 2 norm, norm of x bar minus a bar. And the distance to b bar is norm the 2 norm x bar minus b bar, we want to see what is x bar belongs to this set if this distance that is this distance to a bar is less than or equal to distance to.

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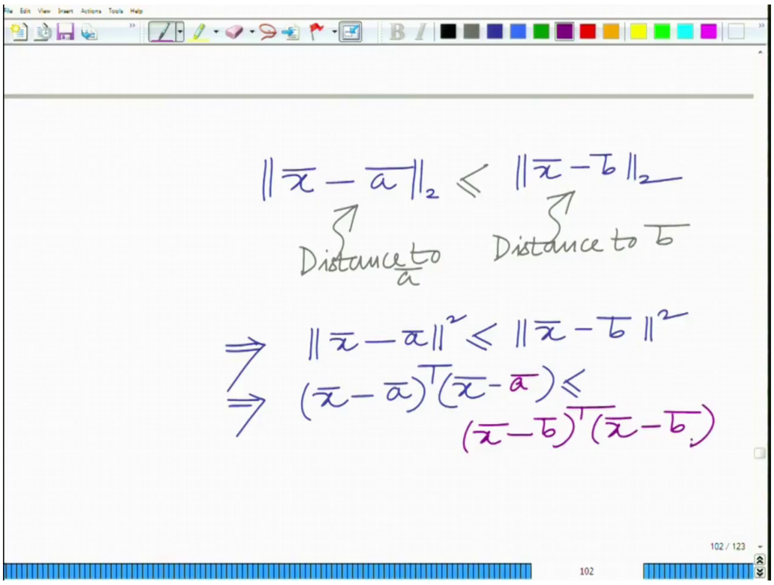

Now, is this set convex well this implies that norm x bar minus a bar whole square less than or equal to norm, x bar minus b bar whole square. Now, remember the norm square of a vector is a vector transpose times itself, this implies norm x bar minus a bar transpose norm x bar minus b bar less than or equal to or norm x bar minus a bar, vector transpose times itself is less than or equal to norm x bar minus b bar transpose norm x bar minus b bar.

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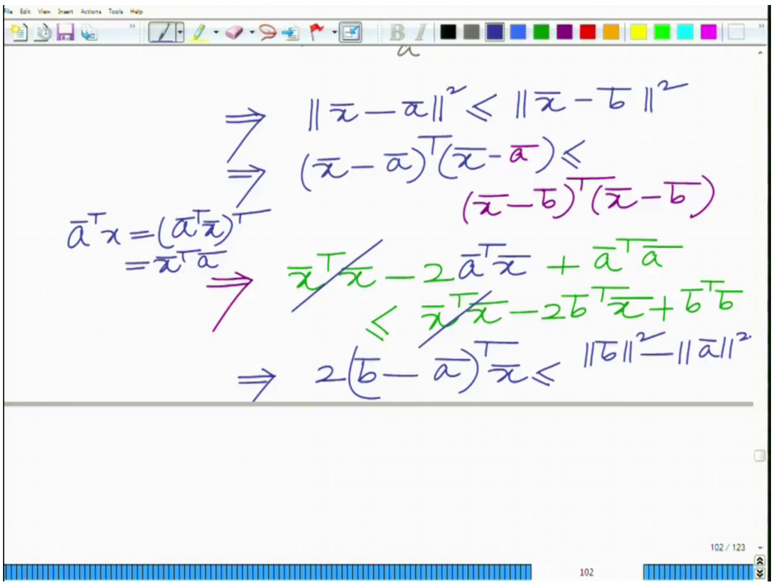

This implies that if you split this or if you expand this will be x bar transpose x bar minus a bar transpose x bar x bar transpose a bar. Now, since they are scalar a bar transpose x bar and x bar transpose a bar are the same. So, this will be minus 2 x bar transpose a bar, plus a bar transpose a bar less than or equal to again the same thing similar x bar transpose a bar minus 2. I can also write it as b bar transpose x bar x bar transpose b bar or b bar transpose x bar both are same plus b bar transpose b bar.

And now you can see x bar transpose x bar x bar transpose x bar cancels this implies that bring the b bar over the other side this implies, let us write this as a bar transpose x bar. Both of them are same x bar transpose a bar note that a bar transpose x bar equals, because it is a number a bar transpose x bar transpose which is equal to x bar transpose a bar because it is a number, it is a scalar quantity.

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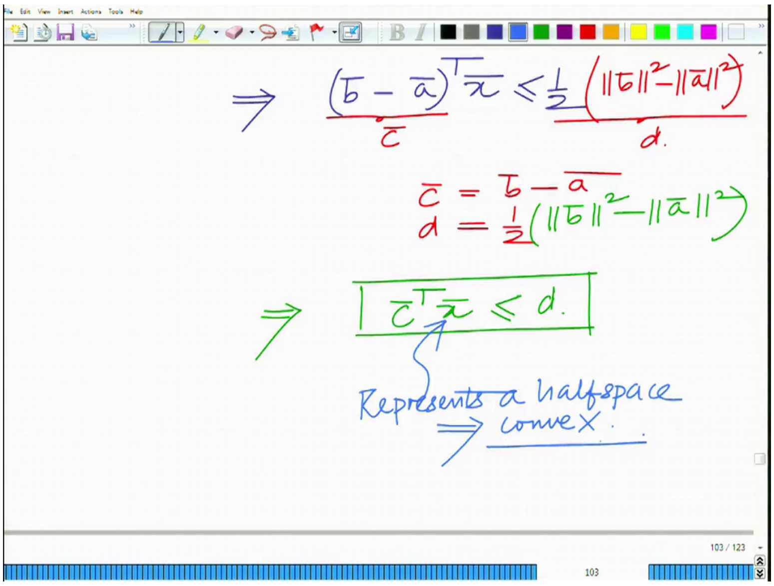

So, this implies b bar minus a bar transpose times x bar less than or equal to well b bar transpose b bar, that is your norm b bar square or there is a two factor of 2 here minus norm a bar square which implies that b bar minus a bar transpose x bar less than or equal to half norm b bar square minus norm a bar square. And therefore, if you call this as if you call this vector as c bar b bar minus a bar, if you denoted by c bar and if you denote this quantity by d that is your c bar equals b bar minus a bar and this constant d equals half, norm b bar square minus norm a bar square.

We have an equation of the form c bar transpose x bar less than or equal to well d and, if you look at this nothing, but a this is nothing, but a equation of the half space alright c bar transpose x bar into equals d bar that is a hyper plane. And c bar transpose x bar less than equal to d some constant d that is the half space.

So, this set of all points x bar which are closer to a bar, then b bar represents half space and therefore, the set is convex that completes the proof. So, if you look at this set it represents a half space, implies this is convex.

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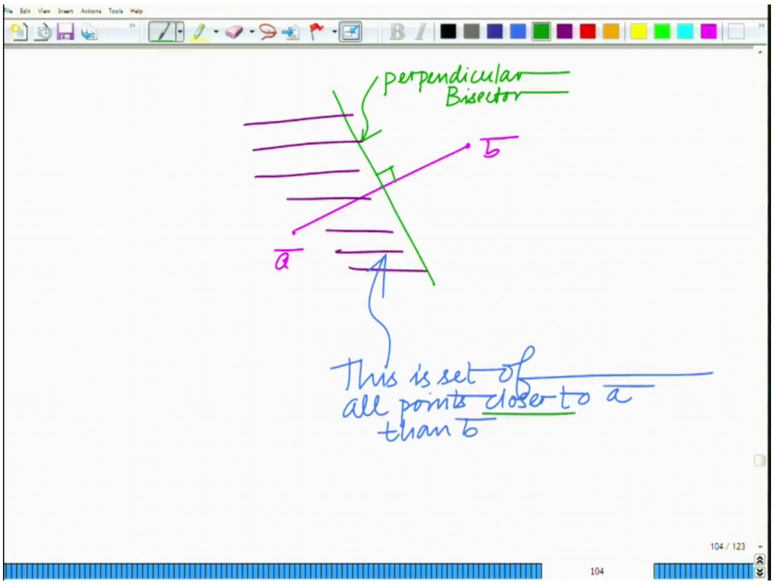

And interestingly if you look at that if you are wondering what that set is, if you look at these points a bar and b bar and the line joining this points a bar b bar. And if you look at the perpendicular bisector of this ok, if you look at the perpendicular bisector of the line joining a bar b bar. All the points that are closer to a bar than b bar lie on basically one side of this perpendicular bisector and this is the half space that you are talking about essentially this is the set of all points closer to a bar, this is a set of all points that are closer to a bar than b bar.

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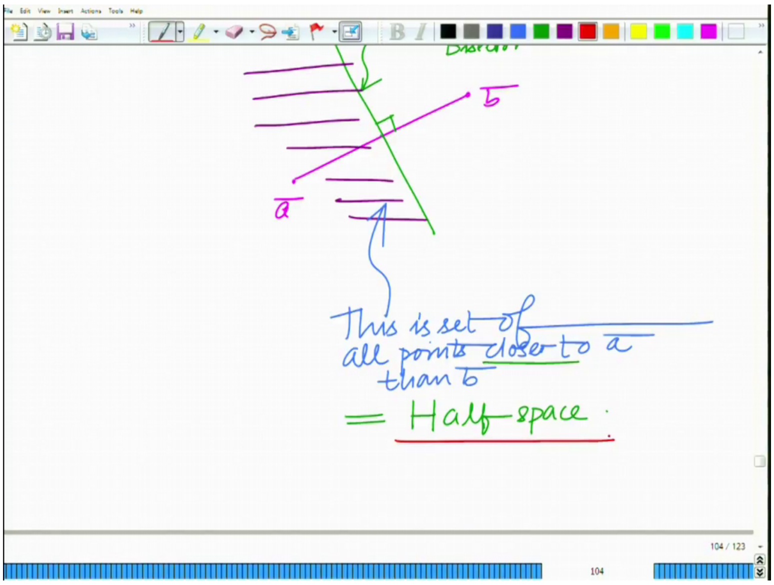

And therefore, this is equal to this is indeed you can see clearly this is a half space is a set of all points. It is we look at the perpendicular bisector of the line joining a bar and b bar. This is the high half space that lies towards a bar this is a half space that is the perpendicular bisector, divides a bar divides it into two half spaces the half space, that includes a bar that is the set of all points which are closer to a bar than b bar, which is indeed the half space and therefore, it is indeed convex ok

So, that basically complete the proof of course, we have proved it analytically and that should leave no question and this is basically an insight into the convex set ok.

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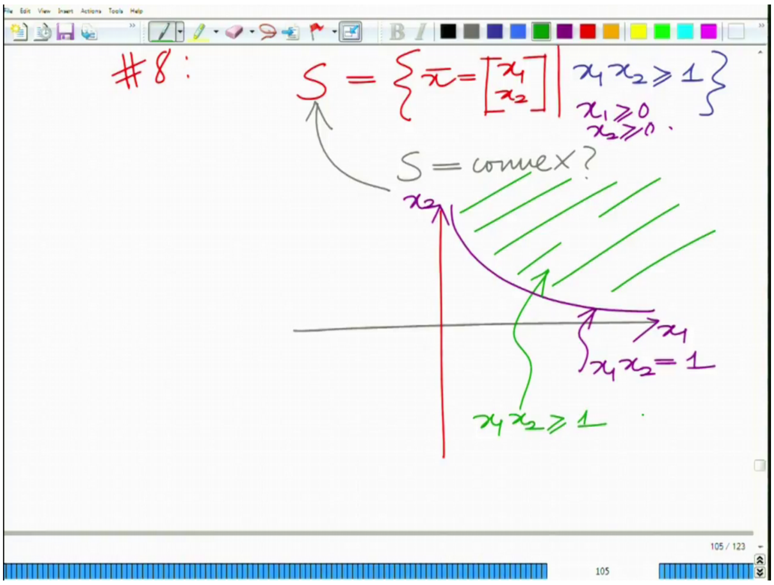

Let us look at another problem let us look at another interesting set. So, we have S equals the set of vectors x bar. So, set of two dimensional vectors x bar equals x 1 x 2 such that x 1 x 2 greater than or equal to 1. And we want to ask the question again is S convex is the set x convex. Now, if you look at this two dimensional plane ok, if you plot x 1 equals x 2 that will be this hyperbola, correct x 1 equals x 2 or x 1 x 2 equals 1. And of course, we are considering the positive let us also maintain that let us also impose this additional constraint x 1 greater than equal to 0 x 2 greater than or equal to 0 ok.

And now if you look at this now if you look at this so, this is your x 1 this is your x 2 and x 1 x 2 greater than or equal to 1, that includes this set ok. So, this is a set of all points such that x 1 x 2 greater equal to 1 and you can see visually see that this is convex.

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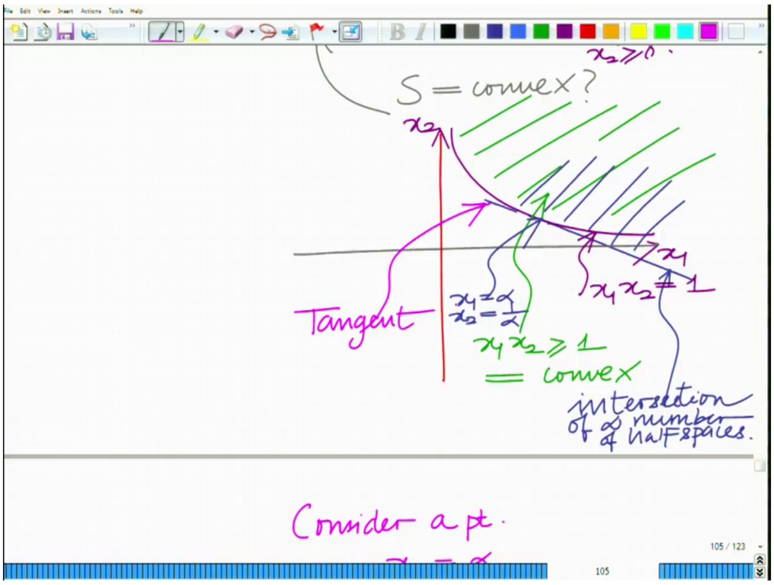

We are going to take an alternative approach what we are going to show is that this half space this convex set can be represented as a intersection of a infinite number of infinite number of half spaces. And since it is the intersection of half spaces it is essentially a polyhedral, it is well it is an infinite number of half spaces its essentially a convex set, because is the intersection of several convex sets that half spaces basically.

And you can clearly see that if you draw that tangent and, if you look at the corresponding half space ok. And if you draw all tangent all such tangents the infinite number of tangents, you can represent this as the intersection of an infinite number of such half spaces. Now, let us look at any point well where x 1 equal to alpha implies x 2 x 1 x 2 equal to 1 so, x 2 equals 1 minus alpha.

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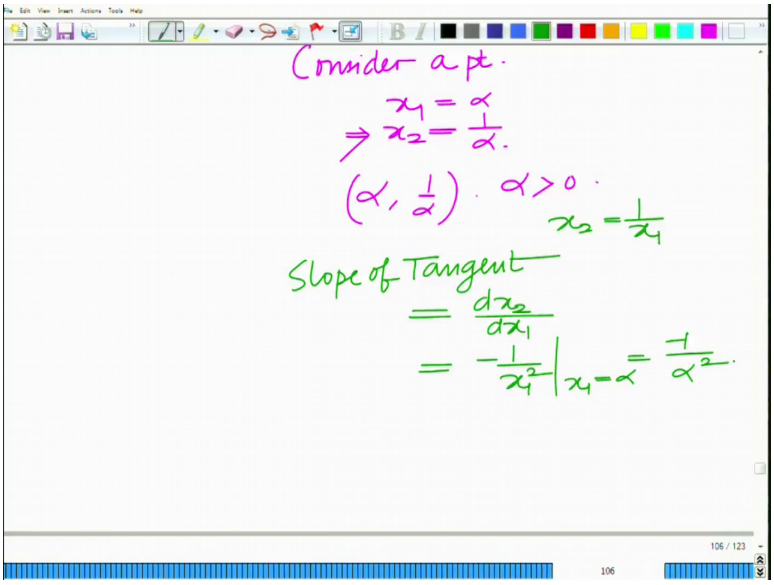

So, we are considering any point consider any point so, let us start with this consider a point x 1 equals alpha this implies x 2 equals 1 over alpha since x 1 into x 2 equals 1. So, we consider this point alpha comma 1 over alpha with alpha greater than 0. And now if you look at this quantity d y b y d x d x 2 by d x 1,

That is if you look at the tangent to the curve, if look at the slope of the tangent. So, this is your tangent and the slope of the tangent to the curve equals d x 2 by d x 1 that is at that point the tangent has the same slope as the curve. So, d x 2 by d x 1 which is remember x 2 equals 1 over x 1 which implies d x 2 over d x 1 is minus derivative of x 2 that is 1 over x 1 with respect to x 1 is minus 1 over x whole square.

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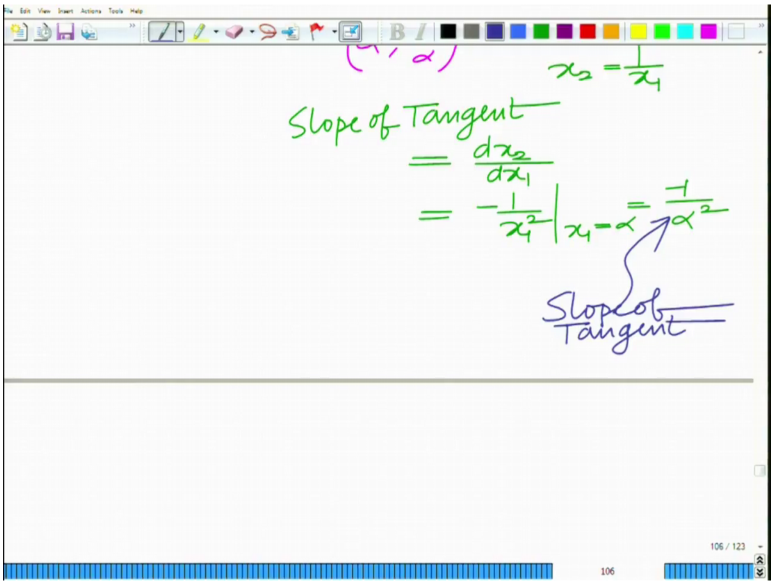

And at this point evaluated at x 1 equals alpha, so, this will be minus 1 over alpha 1 square. And so, this minus 1 over alpha square what is this is basically the slope of the tangent, why is this the slope of the tangent at that point alpha 1 over alpha that point x 1 comma x 2. If you look at the derivative correct, if you look at the derivative of the curve that itself gives the slope of the tangent ok.

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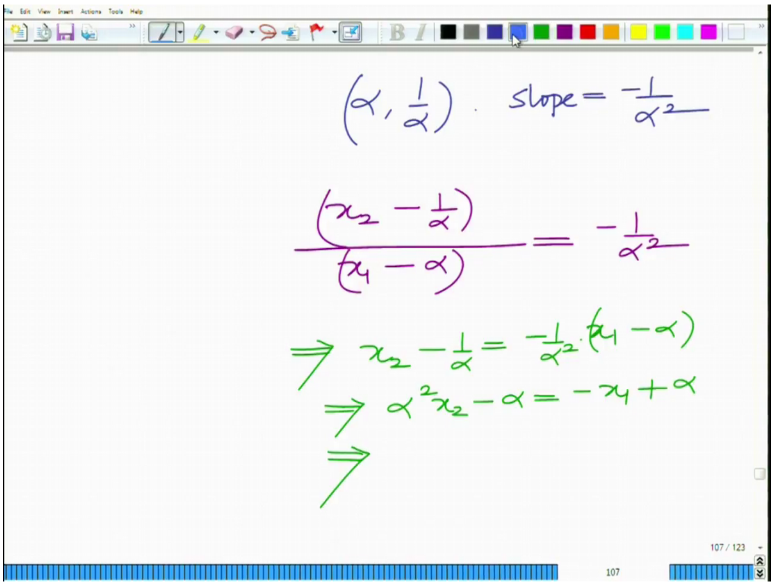

And therefore, we are in the slope and we have the point the point is 1 over 1 over alpha and the slope equals minus 1 over alpha square. So, the tangent will be point through alpha 1 over alpha with slope minus 1 over Alpha Square. So, that will be given as well x 2 minus x 2 minus 1 over alpha divided by x 1 minus alpha the slope is nothing, but difference of the y coordinate by difference of the x coordinate.

So, x 2 minus 1 over alpha divided by x 1 over alpha divided by x 1 over alpha equals minus 1 over alpha square, this implies x 2 minus 1 over alpha equals minus 1 over alpha square x 1 minus alpha this implies alpha square x 2 alpha square x 2 minus alpha equals well minus x 1 plus alpha this implies, if you can see this implies alpha square x 2 plus x 1 equals 2 alpha.

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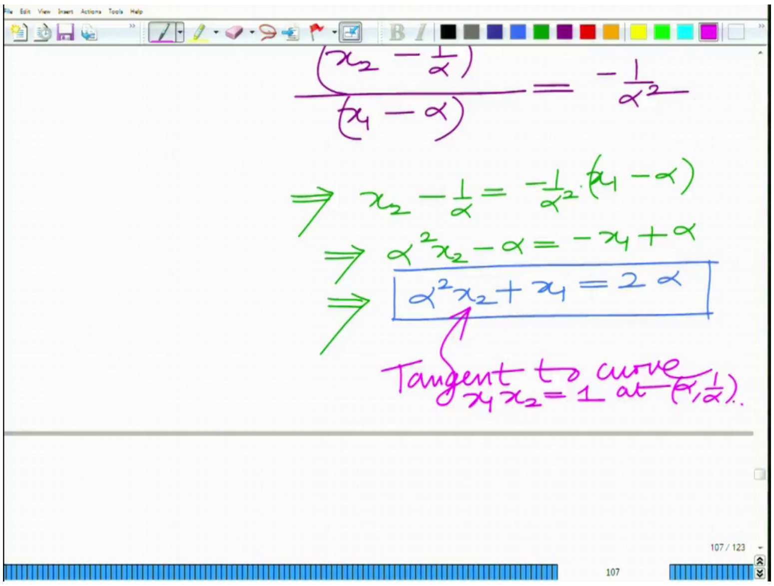

So, this is the tangent to this curve x 1 x 2 equal to 1 at alpha come 1 over alpha tangent ok.

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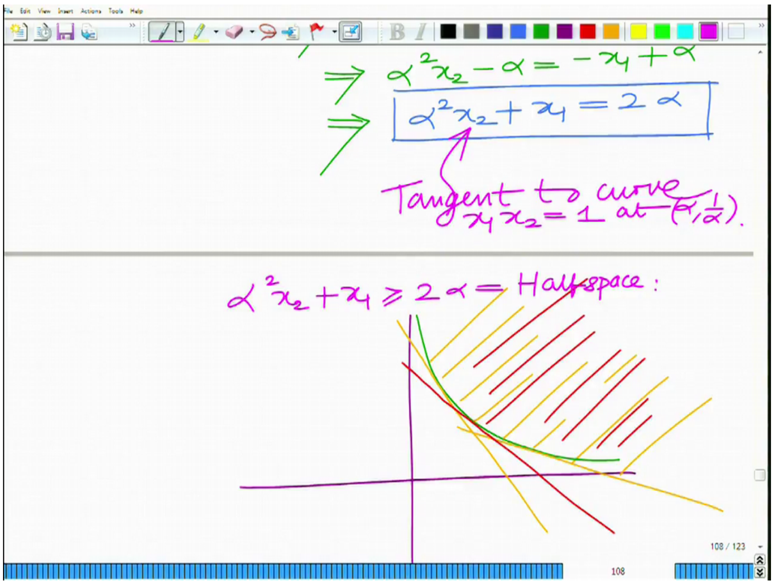

So, basically what we have is you have you have this curve x 1 x 2 equal to 1 and you have this particular tangent you have this half space. Now, you take all such tangents and you take the intersection of these half spaces and what you will see, you will end up getting this original convex set.

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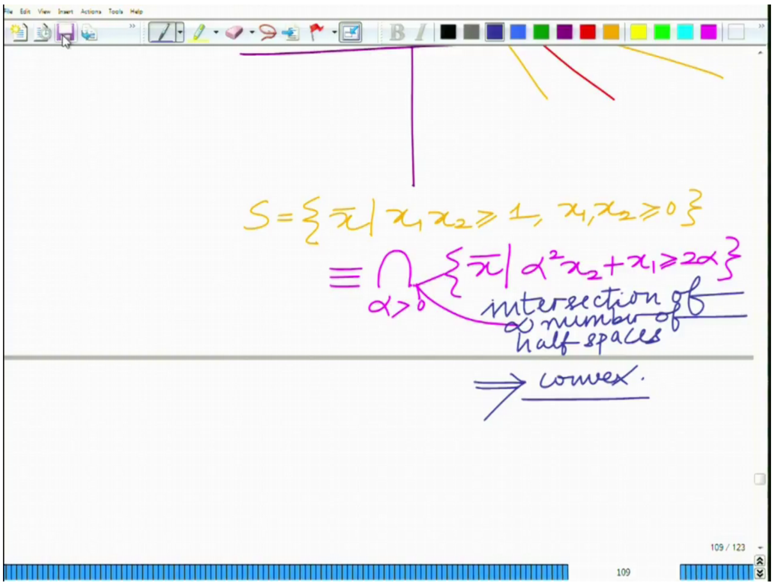

So, set x 1 x 2 so, if you look at the set x 1 that is x bar such that x 1 x 2 greater than or equal to 1 comma x 1 greater than or equal to 0 x 2 greater than or equal to 0. If you look at this set, if you look at this set this set is equivalent to the set, that is this it is equal to the intersection of all the sets intersection over what intersection over any point alpha greater than 0 such that x bar such that alpha square x 2 plus x 1 greater than or equal to 2 alpha.

So this is alpha square x 2 plus x 1 equal to 2 alpha, this is the tangent which implies alpha square x 2 plus x 1 greater than equal to 12 alpha this is the half space ok. So, this convex set x 1 x 2 x 1 x 2 greater than equal to 1. This intersection of all these half spaces the intersection of an infinite number of half spaces implies, this is convex. So, the set x 1 x 2 greater than equal to 1, we represented that as the intersection of an infinite number of half spaces and therefore, indeed it is convex let. So, these are interesting applications interesting problems which demonstrate which show you how to demonstrate the convexity of set and how to visualize the different properties or different aspects of a convex set alright. So, we will stop here and continue in the subsequent modules.

Thank you.