Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture – 02
Eigenvectors and Eigenvalues of Matrices and their Properties

Hello, welcome to another module in this massive open online course.

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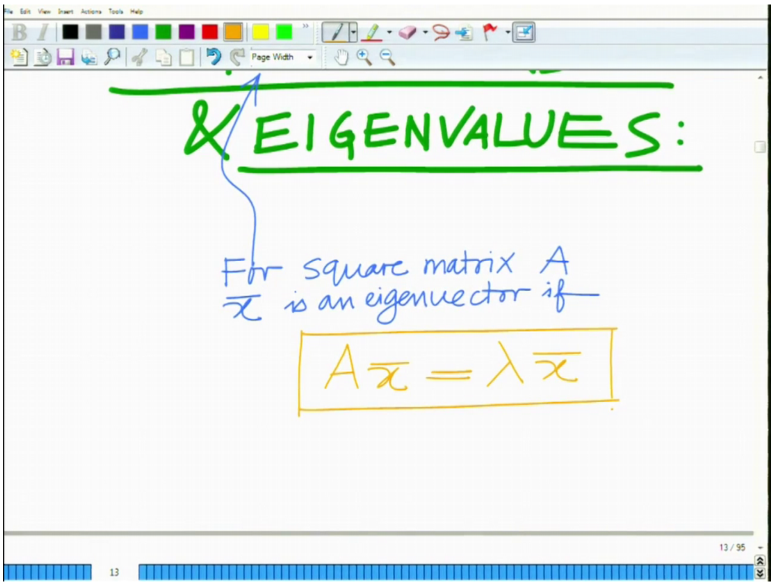

Let us continue our discussion regarding the mathematical preliminaries, for the framework of convex optimization, looking by looking at another very important concept that is of the Eigenvalues, the eigenvectors and eigenvalues of square matrices right.

So, we are going to start talking about the eigen the concept of or rather the concepts of eigenvectors and, eigenvalues. And, now these is the eigenvector the notion of eigenvector and eigenvalue is defined for a square matrix correct.

So, for a square matrix A all right x bar is an eigenvector, x bar is an eigenvector, if we have A x bar that is the product of the matrix A with the vector x bar, A x bar equals a multiple of that is lambda times x bar all right. And this is a fundamental equation for the eigenvector all right and well the this lambda is known as the eigenvalue.

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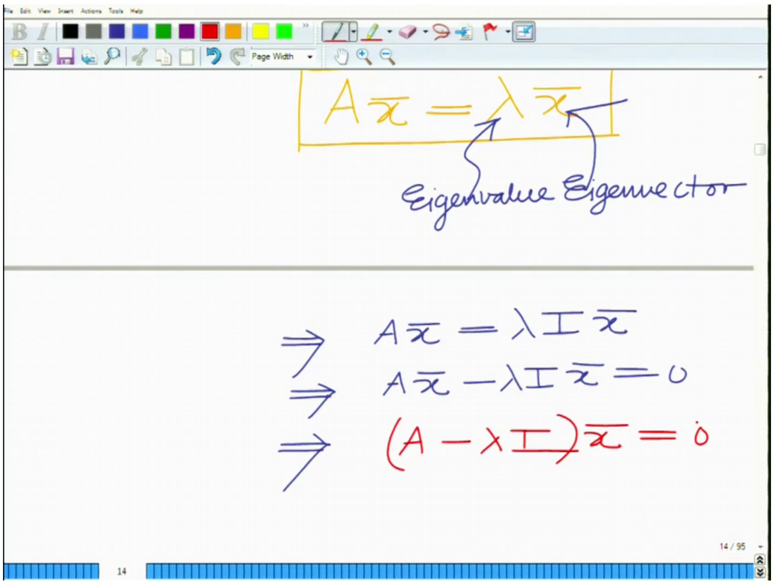

This is known as the eigenvalue and this vector x bar is known as the eigenvector. And, now since A x bar equals lambda x bar that implies A x bar equals lambda, times I can use the identity matrix here, times x bar which implies A x bar minus lambda I times x bar equal to 0, which basically implies that a minus lambda I times x bar equal to 0. Now, this implies what this means is this matrix A minus lambda I this is a singular matrix.

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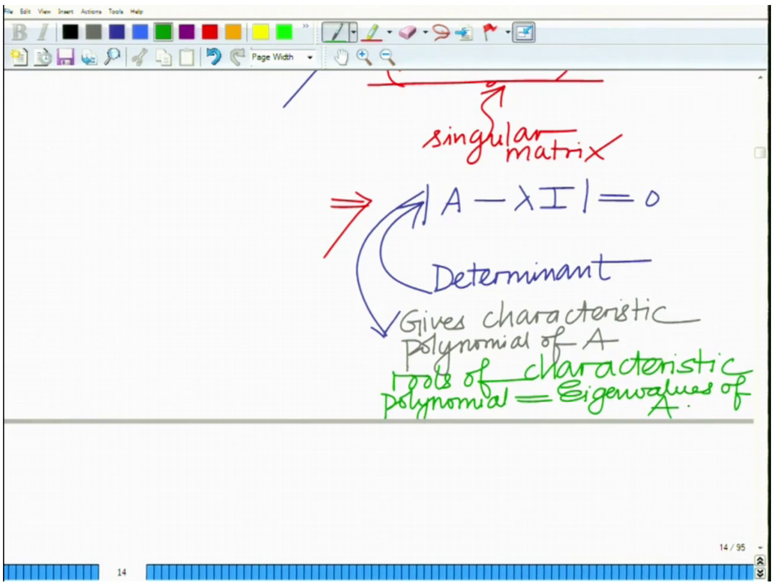

There exists a vector such that a minus lambda I times x bar equal to 0, which implies that the determinant of a minus lambda I equals 0. So, we use this to denote the determinate. So, if lambda is an eigenvalue of A that means, the determinant of A minus lambda equals 0. Now, by evaluating the determinant you can derive an equation that is known as the characteristic equation corresponding to the matrix A, and the roots of that all right, the roots of this equation in lambda give the eigenvalues of the matrix A all right.

So, this gives so this basically gives the characteristic polynomial correct. So, a minus determinant of a minus lambda I, this gives you the characteristic polynomial, polynomial of a in terms of lambda. The roots and the roots of the characteristic polynomial, these are the roots of the characteristic polynomial are eigenvalues of A for example.

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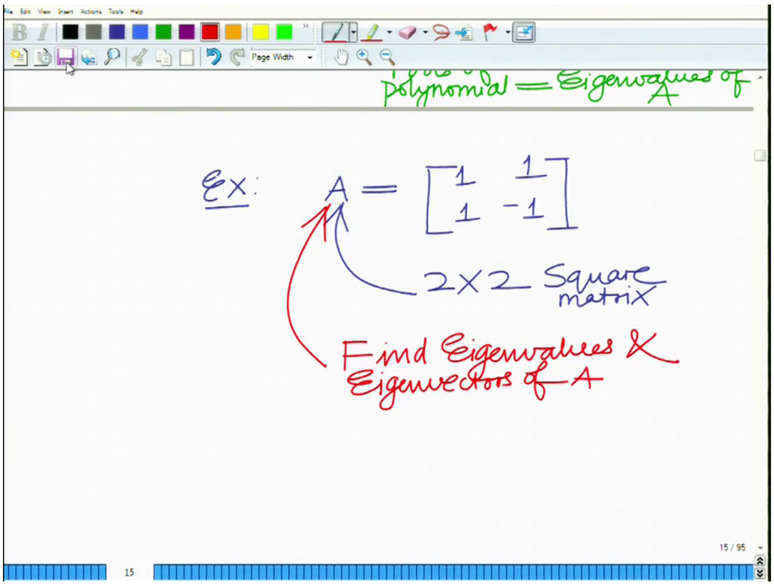

Let us say A is the matrix 1 1 1 minus 1 let us take this simple 2 cross 2 matrix all right. So, this is a square matrix A and now we want to find the eigenvalues ok, for this given matrix A 2 cross 2 matrix, we want to find the eigenvalues and also the corresponding eigenvectors. So, find eigenvalues and the eigenvectors of A.

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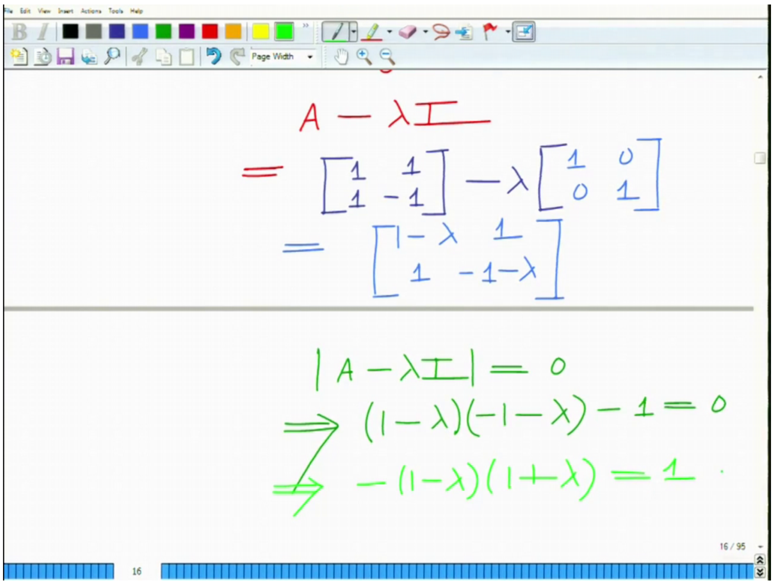

Now, we given a now let us start by considering a minus lambda I, that will be equal to 1 1 1 minus 1 minus lambda times 1 0 0 1, which is equal to 1 minus lambda 1 1 minus 1 minus lambda all right. And, now we have to consider the determinant of this, now determinant of a minus lambda I equals 0, now if you compute the determinant of a minus lambda you will see this is this implies 1 minus lambda into minus 1 minus lambda minus of 1 this is equal to 0, which basically implies that minus of 1 minus lambda into 1 plus lambda equal to 1.

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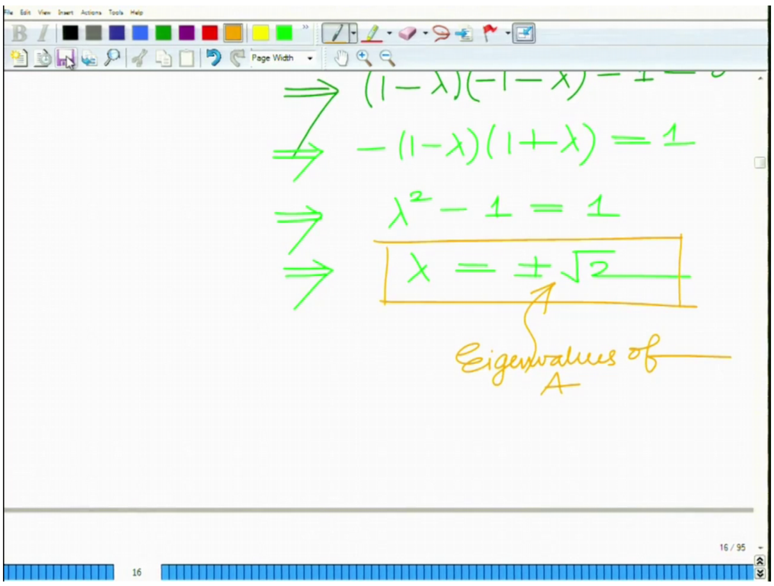

Which implies that lambda square minus 1 equal to 1, which implies that lambda equals plus, or minus square root of 2. So, these are the eigenvalues of A lambda equals plus, or minus square root of 2, these are the eigenvalues of the matrix A. So, we have got 2 eigenvalues that is lambda equals either plus square root of 2, or minus square root of 2. Now, let us find the corresponding eigenvectors of the matrix A corresponding to both these eigenvalues.

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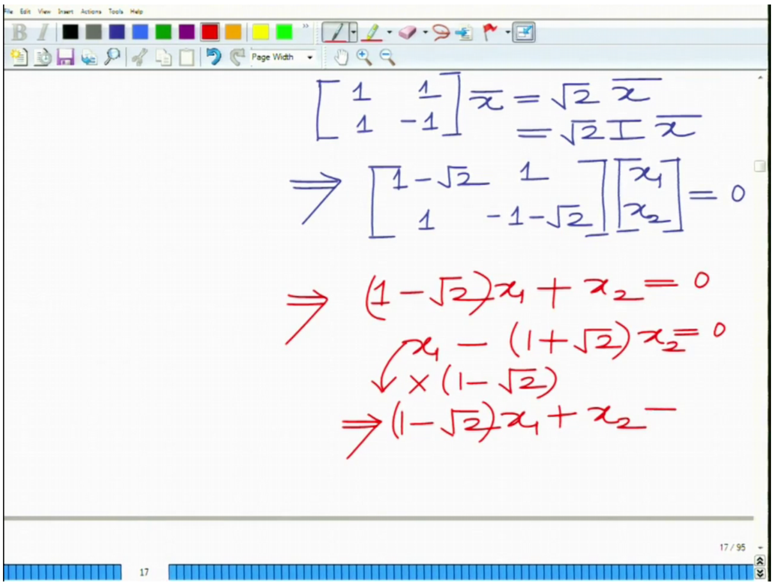

Now, the eigenvector to find the eigenvector ok, what we are going to do is we have 1 1 1 minus 1 A times x bar equals square root of 2 the eigenvalue square root of 2 times x bar this is from the definition of eigenvalue this implies. Now, you can which is also basically square root of 2, now you can insert an identity times x bar.

Which implies now you bring it this on the left. So, this will become a minus square root of 2 times identity which is 1 minus square root of 2 1 1 minus 1 minus square root of 2 into x bar, now I will write x bar as a vector it is a two dimensional vector x 1 x 2 equal to 0. Now, this implies that you get equations in x 1 and x 2 1 minus square root of 2 into x 1 plus x 2 equal to 0 and, x 1 minus 1 plus square root of 2 into x 2 into 0. Now, this equation if you multiply by 1 minus square root of 2, you will realize that you get again the same equation.

So, this will be 1 minus square root of 2 into x 1 minus 1 plus square root of 2 into 1 minus square root of 2 so, this minus of minus 1 so, that is plus 1 plus x 2 equal to 0. So, implies that basically the first and the second equation are identical all right. So, basically you have just one equation and therefore, this is an infinite number of solutions and that is kind of obvious, because if the eigenvector corresponding to eigenvalue is not unique. So, that is if k x is k x bar is an eigenvector, then x bar scaled by any constant k is also an Eigen vector corresponding to the same eigenvalue.

And therefore, correspond to there are infinite number of eigenvectors in that sense. So, this means that these two equations are basically the same.

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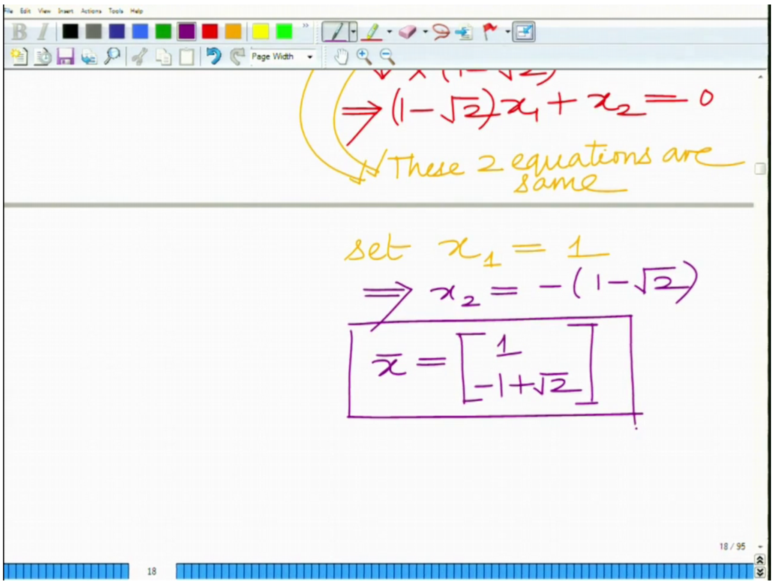

Now, what we will do is to derive a solution you set x 1 to derive any solution, or one such solution you set x 1 equal to 1, this implies x 2 equals minus of 1 minus root 2 and this you can verify is an eigenvector, that is we look at x bar equals x 1 x 2 that is 1 minus 1 minus root 2 that is root 2 minus 1, or not minus 1 plus root 2, or root 2 minus 1 and this you can check. This is an one of the eigenvectors of A.

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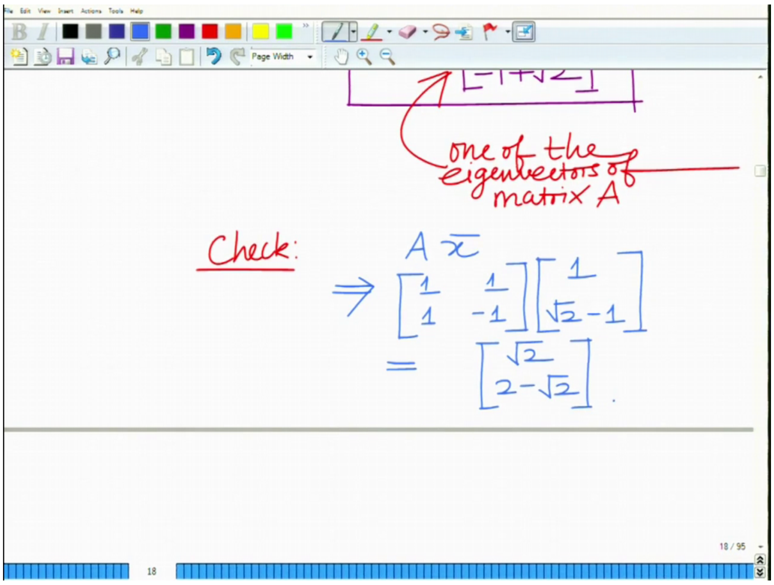

One of the Eigen vectors of matrix and you can check this ok, let us do that if you look at A times x bar that is equal to 1 1 1 minus 1 times x bar that is 1 root 2 minus 1. So, this will be 1 into 1 plus root 2 minus 1 so, this will be root 2 1 into 1 minus root 2 minus 1.

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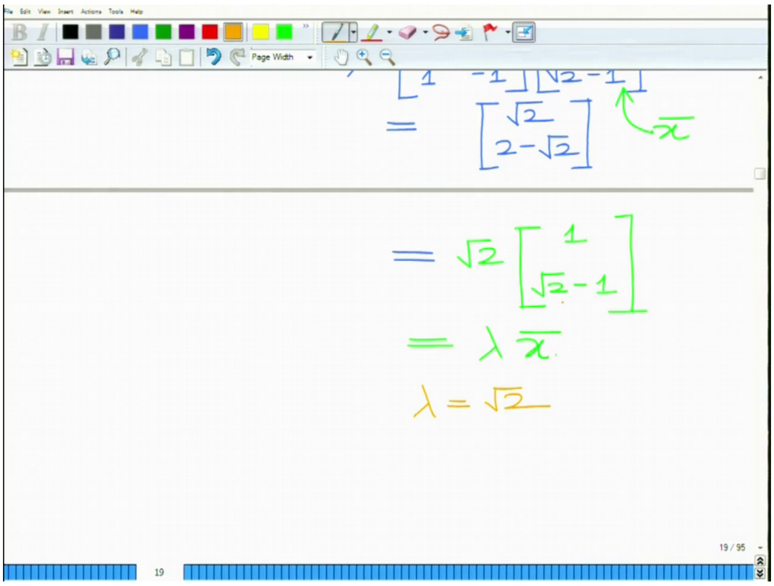

So, this will be 2 minus root 2, which if you pull out this constant square root of 2, this will be well 1 square root of 2 minus 1. And which is basically nothing, but if you call this as your x bar this is nothing, but lambda times x bar where what is lambda, lambda equal square root of 2.

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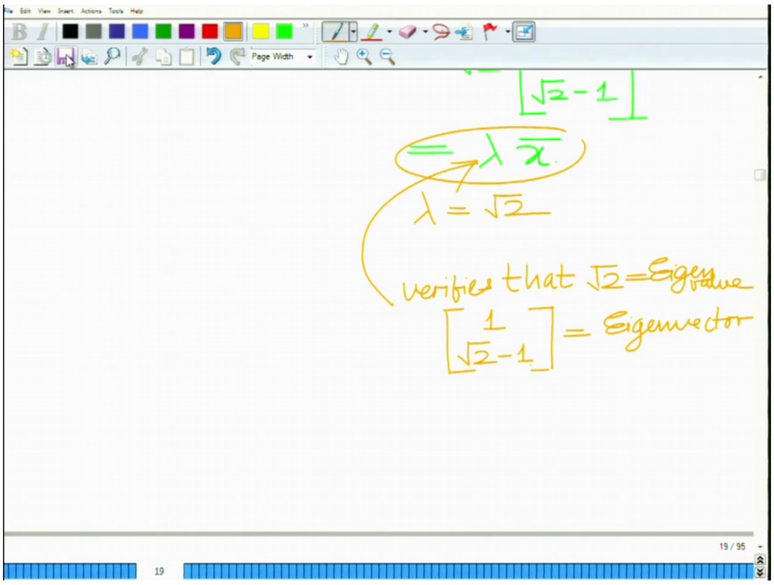

And we already seen x bar is 1 square root of 2 minus 1 so, this verifies that verifies that square root of 2 equals the eigenvalue and the fact that 1 comma, it verifies both right and that 1 comma square root of 2 minus 1 equals an eigenvector. So, this verifies basically both the facts that square root of 2 is the eigenvalue of this matrix A and square root of 1 square root of 2 minus 1 is the eigenvector.

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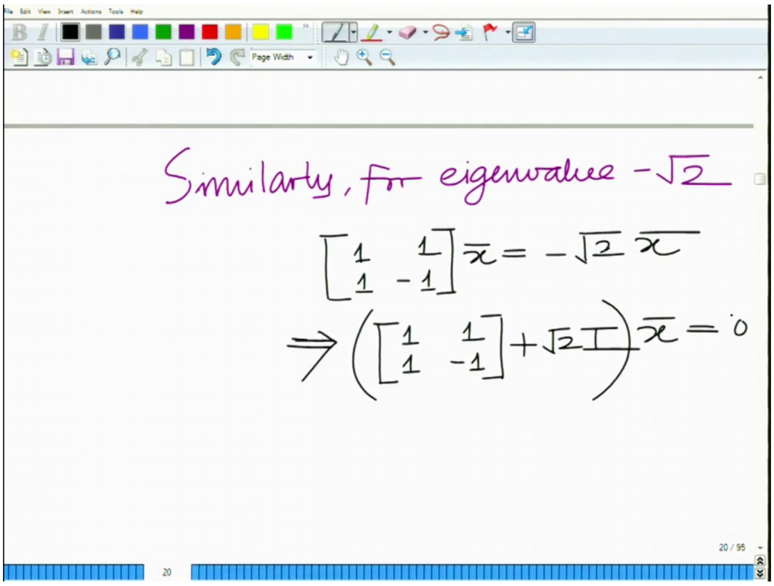

Now, similarly one can find the other eigenvector all right corresponding to eigenvalue minus square root of 2. Similarly for eigenvalue minus square root of 2, we have 1 1 1 minus 1 of x bar equals minus square root of 2 x bar, this implies 1 1 1 minus 1 plus square root of 2 times, identity into x bar equals 0.

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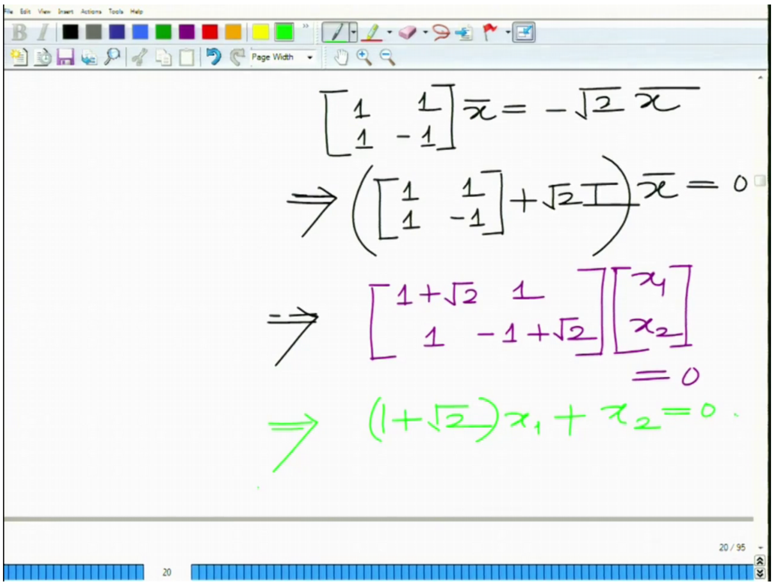

This implies that basically if you look at this implies 1 plus square root of 2, 1 1 minus 1 plus square root of 2 into the vector x bar, into the vector x bar that is x 1 x 2, this is equal to 0 and this implies basically 1 plus square root of 2 times x 1 plus x 2 equal to 0 and, you can see both the equations will reduce to the same thing.

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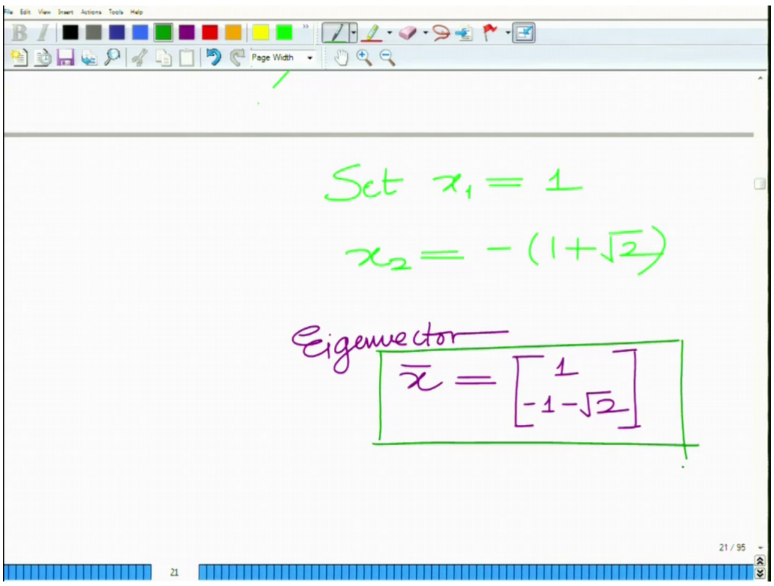

And now once again what we will do is we will set x 1 equals 1, that implies x 2 equals minus of 1 plus square root of 2, or minus 1 minus root 2. And therefore, the eigenvector x bar equals 1 minus 1 minus root 2. So, this is the other eigenvector corresponding to eigenvalue minus square root of 2.

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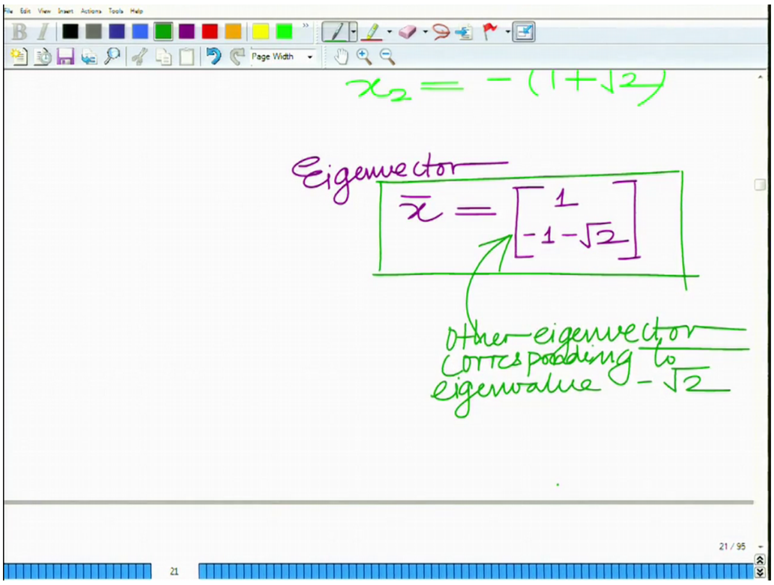

This is the other eigenvector, corresponding to the other eigenvalue minus square root of 2 all right. So, that is a brief introduction to the concept of eigenvectors and eigenvalues of the matrix. Let us look at another important concept that is the concept of symmetric and the Hermitian symmetric matrices, these are Hermitian matrices all right.

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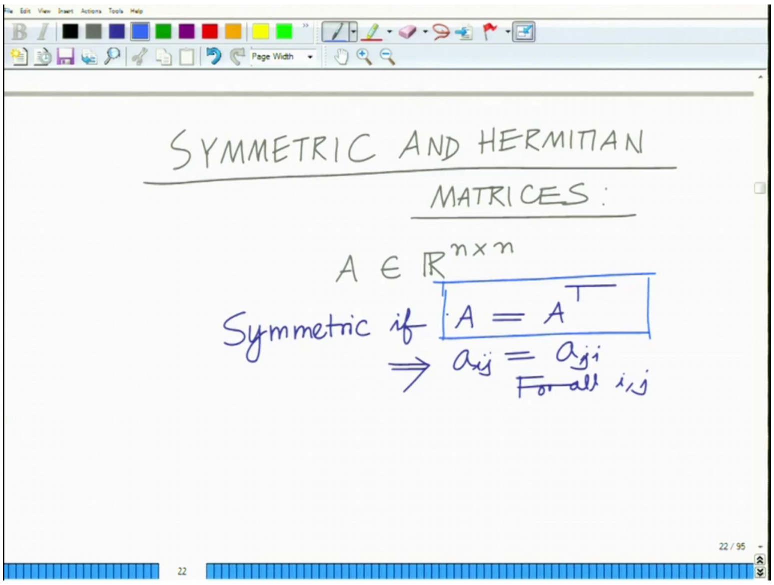

So, what we want to look at now is basically the notion of what is what are known as symmetric and Hermitian symmetric and Hermitian matrices. So, let us say A is a real matrix n cross n real matrix, we say symmetric A is symmetric, if A equals A transpose that is this is symmetric equals transpose a transpose that implies take any element a i j, that is equal to a j i for all pairs i comma j. So, basically for a symmetric matrix we must have that equals A transpose. And naturally that implies this must be a square matrix all right, because only this is the symmetry is only preserved of the matrix is a square matrix.

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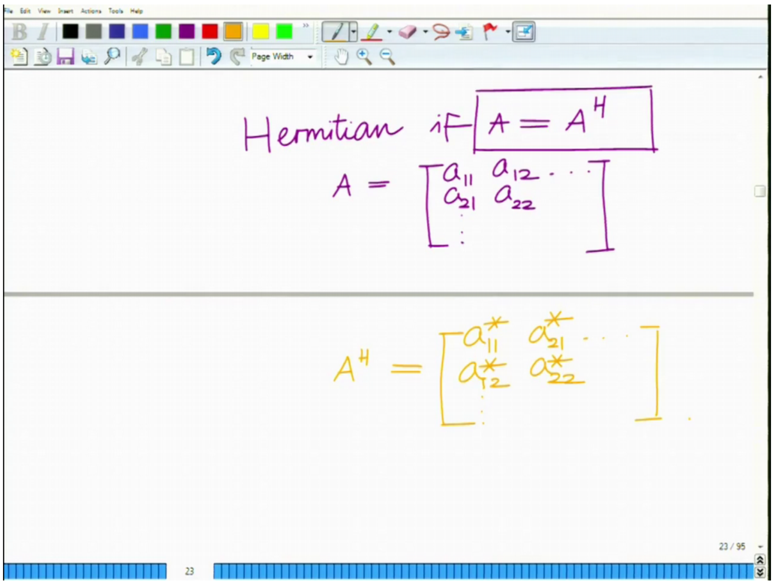

And it is Hermitian or a Hermitian symmetric matrix Hermitian, if A equals A Hermitian and what is a Hermitian? Now, let us say A is our matrix a 1 1, a 1 2, a 2 1, a 2 2 and so on. For the Hermitian matrix what you have to do is you have to take the transpose and you have to take the complex conjugate of each element. So, this will be a 1 1 conjugate, since you are taking the transpose this becomes a 1 a 2 1 and the conjugate of that a 2 1 becomes a 1 2 and the conjugate, this is a 2 2 and the conjugate.

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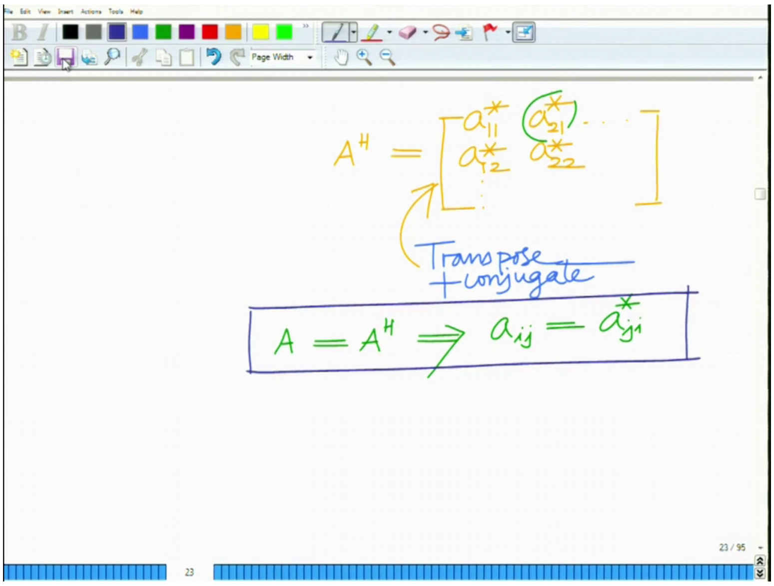

So, basically for the Hermitian you take transpose plus conjugate of each element. Now, A equal to A Hermitian that is it is Hermitian matrix implies that a i j equals a j i conjugate that is this is Hermitian symmetric, if a 1 2 equals a 2 1 conjugate and so on. So, that is that matrix is known as a Hermitian matrix.

So, this basically is a Hermitian matrix. Now, there are several interesting properties of this Hermitian and symmetric matrices one of the most interesting properties is that the eigenvalues of both symmetric and Hermitian matrices are real all right.

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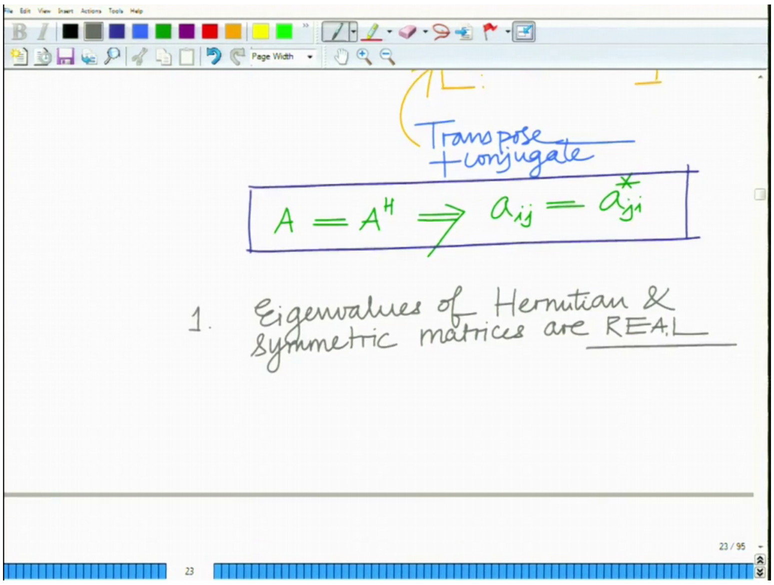

So, the first property is that the eigenvalues of Hermitian and, symmetric matrices are real; these are real quantities all right.

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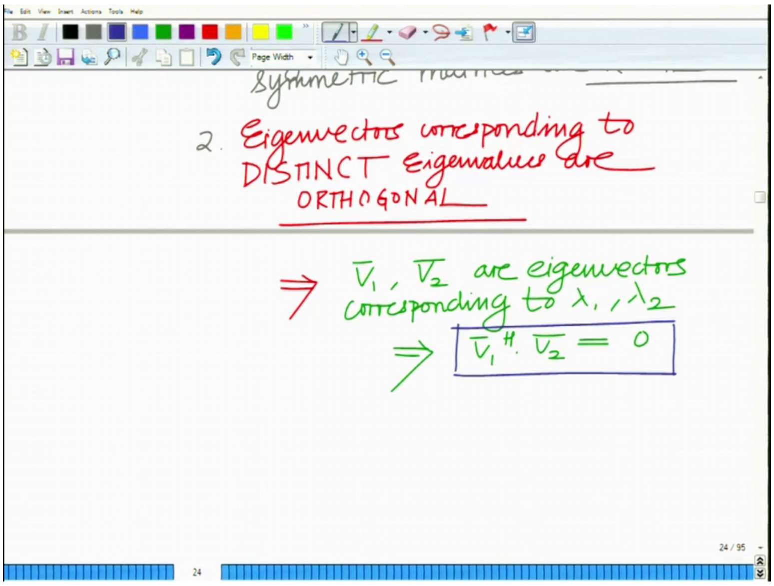

And second properties is another interesting property, Eigen vectors corresponding to distinct eigenvalues that is different eigenvalues not the same Eigen value, but distinct eigenvalues are orthogonal and this is an important property. This implies that if V 1 bar comma V 2 bar are the eigenvectors corresponding to distinct eigenvalues lambda 1, comma lambda 2 this implies for a symmetric matrix, V 1 bar Hermitian V 2 bar equal to

0. This is the meaning of vectors being orthogonal that is two vectors are orthogonal, if that is x 1 bar x 2 bar are two vectors, they are real vectors x 1 bar transpose x 2 bar is 0, they are complex vectors x 1 bar Hermitian x 2 bar equal equals 0, then the vectors are said to be orthogonal.

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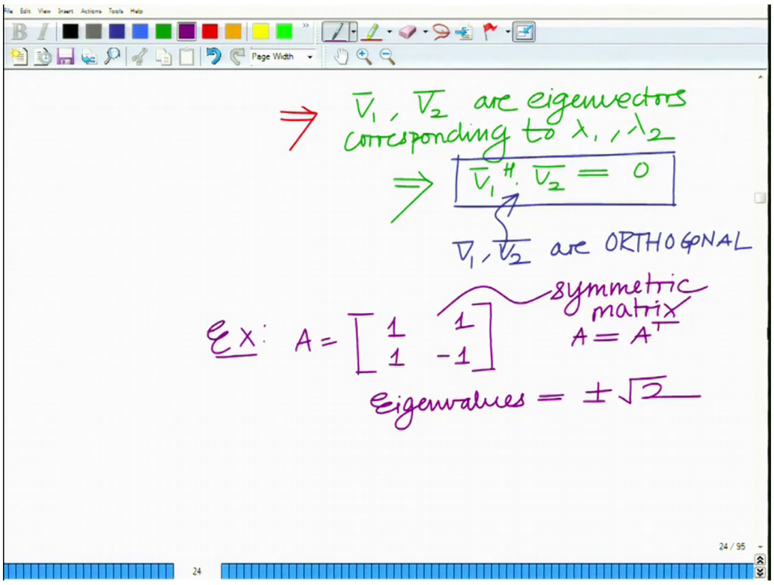

So, this is orthogonality of vectors, this is an important property in general orthogonality of vectors is also a very important property. Now, let us go back to our earlier example to illustrate this fact for instance, you have probably realized that if you look at our previous matrix that is 1 1 1 minus 1, this is a symmetric matrix you can see, this is a symmetric matrix, we have A equals A transpose. And the eigenvalues are natural if you look at the eigenvalues equals plus, or minus square root of 2 and these are real quantities ok, we already seen this all right.

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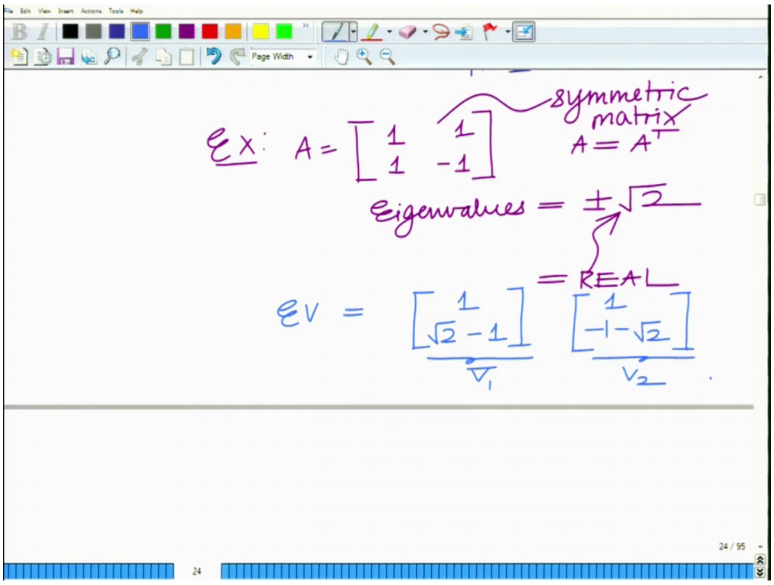

So, you can see that the eigenvalues of this symmetric matrix are real as given by the property. And, now let us look at the eigenvectors and we will show that the eigenvectors are orthogonal the eigenvectors are 1 root 2 minus 1. Let us call this as your V 1 bar and the other eigenvector is 1 minus 1 minus root 2. Now, since these vectors are real we can simply take the transpose so, this is V 2 bar n because transpose or Hermitian will give the same thing for real vectors.

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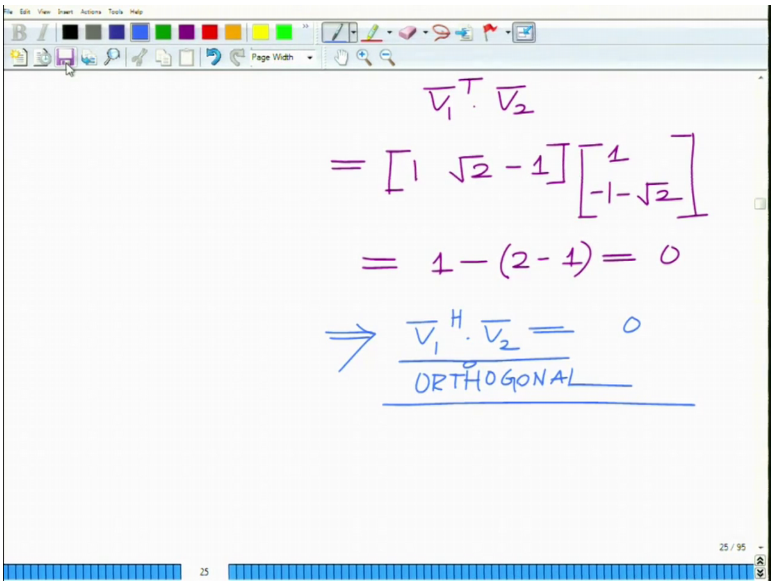

Now, if you look at V 1 bar into V 2 bar we will bar transpose into V 2 bar that gives 1 square root of 2 minus 1 times 1 minus 1 minus root 2, which is equal to you can clearly see 1 minus root 2 plus 1 into root 2 minus 1 that is 1 minus 2 minus 1, which is basically 0. So, this basically is showing you that V 1 bar Hermitian V 2 bar equal to 0 implies these vectors are orthogonal all right.

That is a very interesting property and that is arising, because the matrix is symmetric all right. So, in this module we have looked at various important and very interesting and also very important concepts of eigenvalues, eigenvectors and symmetric matrices. And these are very important, because these are going to be used frequently in our discussion and the development of the framework of optimization for various applications.

Thank you.