Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture – 17
Positive Semi Definite Cone And Positive Semi Definite (PSD) Matrices

Hello. Welcome to another module in this massive open online course. So, we are looking at various convex sets and their practical applications relevance to practical applications. So, let us continue our discussion by looking at the set of Positive Semi Definite Matrices which is also known as the Positive Semi Definite Cone.

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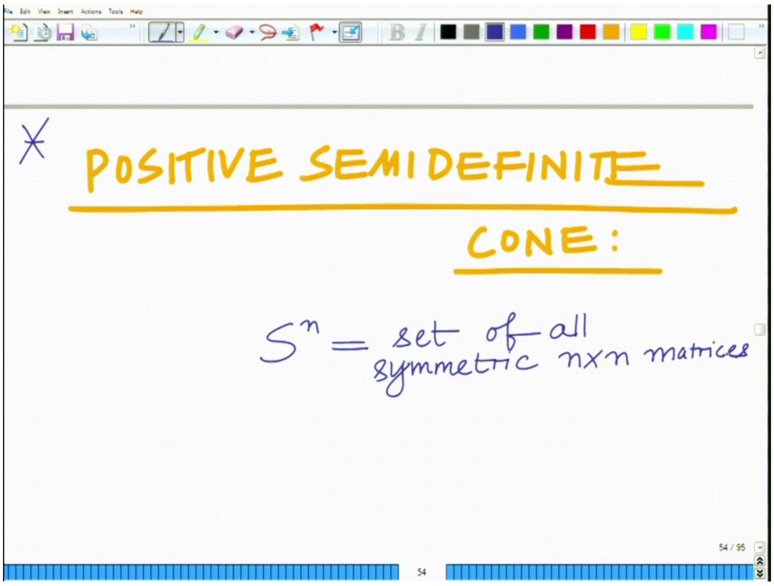

So, what do we want to look at is we want to look at the positive we want to look at the positive semi definite cone. And, well what happens in this, in this case the positive semi definite cone well, let us consider this set of S n. Now, S n equals the set of all symmetric, remember we have defined symmetric matrices before that is if a matrix A equals A transpose for a real matrix it is a symmetric matrix set of all symmetric n cross n matrices.

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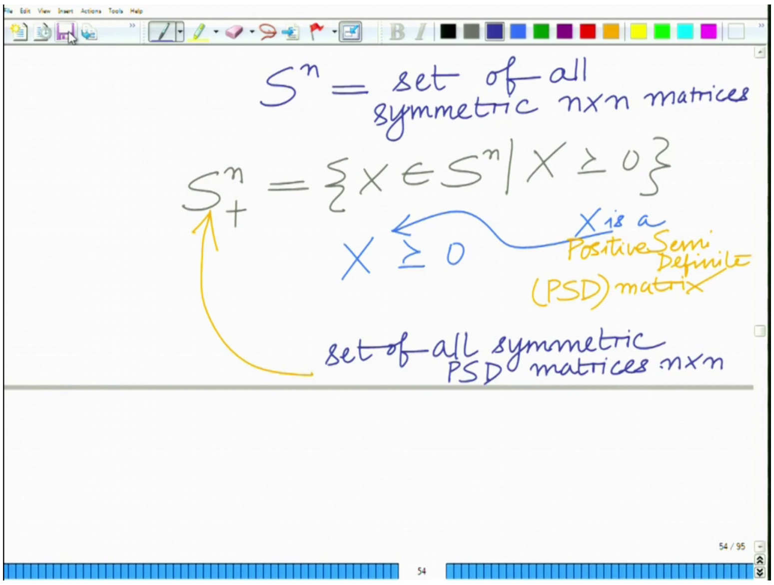

And let us call this matrix S n is not defined another set S n plus as the set of all matrices X element of s n that is a set of symmetric matrices such that X greater than I will describe this what does it mean to say a matrix is greater than equal to 0, where this notation X with this (Refer Time: 02:07) or this curved greater than equal to sign this denotes that X is a positive semi definite matrix X is a positive semi definite or PSD matrix X is a positive semi definite matrix, ok.

And, so, this is the set of all symmetric positive semi definite matrices, of size n cross n, remember. So, s n plus is a set of all symmetric is a set of all symmetric positive semi definite matrices of size n cross n, and remember the definition of a positive semi definite matrix X is that if we take any vector Z bar it must satisfy the property Z bar transpose X into Z bar is greater than or equal to 0, that is the definition of positive semi definite matrix, ok.

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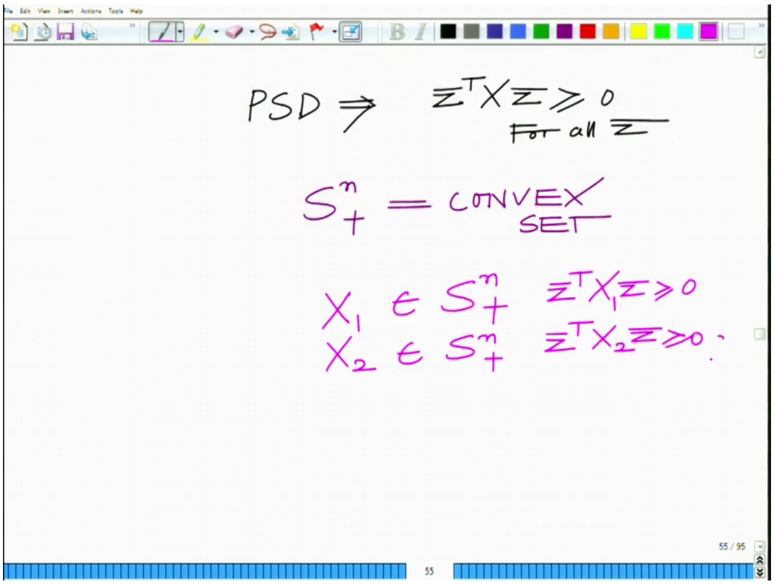

So, PSD implies Z bar transpose X into Z bar greater than equal to 0 for all Z bar that is for all Z bar which is an n dimensional vector. Now, we can show and it is not very difficult that this set S n plus this is a convex set this is a very important convex set. The set of all symmetric positive semi definite matrices this is a convex set alright that is S n plus that we have just defined is a convex set and that is not very difficult to see we take any two elements that is we take any two positive semi definite matrix, this is similar to I mean we go back to the fundamental definition of a convex set that is we take two points or in this case two matrices that are positive semi definite.

So, let us say X 1 and X 2; X 1 belongs to S n plus that is X 1 is positive semi definite X 2 belongs to S n plus which implies Z bar transpose X 1 Z bar is greater than equal to 0, Z bar transpose X 2 Z bar is greater than equal to 0.

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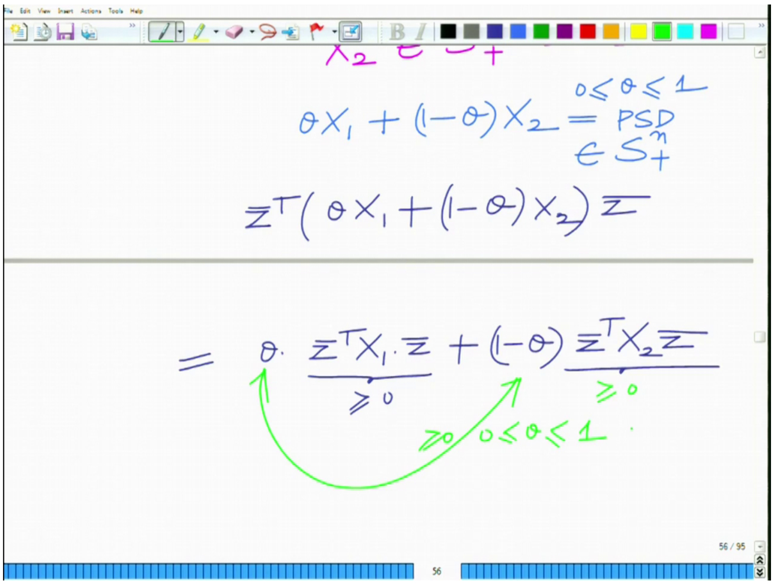

Now, we want to show that if you take a convex combination theta times X 1 plus 1 minus theta times X 2, 0 less than equal to theta less than equal to 1. We want to show this is also PSD or that it belongs to the set S n plus. Very simple, you take Z bar transpose theta X 1 plus 1 minus theta X 2 into Z bar that will be equal to that equals theta Z bar transpose X 1 Z bar plus 1 minus theta Z bar transpose X 2 Z bar.

Now, remember X 1 is positive semi definite. This quantity Z bar transpose X 1 Z bar is greater than equal to 0, X 2 is also positive semi definite. So, Z bar transpose X 2 Z bar this is greater than equal to 0. Now, theta and 1 minus theta both these quantities are greater than equal to 0, because 0 less than equal to theta less than equal to 1. Remember, this is a convex combination, so theta lies between 0 and 1, ok.

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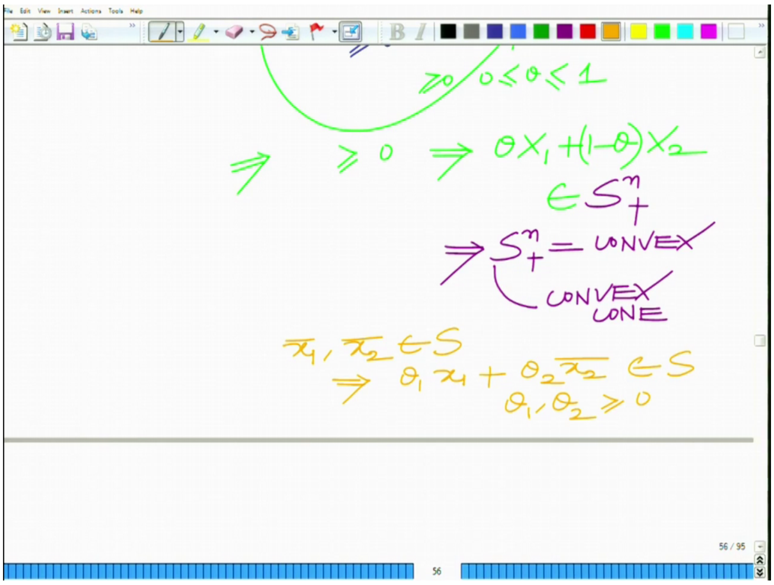

This implies this whole quantity above is greater than equal to 0, which implies that convex combination theta X 1 plus 1 minus theta X 2 this belongs to the set of all n cross n positive semi definite matrices, ok.

So, if we take any convex combination theta times X 1 plus 1 minus theta times X 2 0 less than equal to theta less than equal to 1, the convex combination also belongs to the set of positive semi definite matrices and therefore, S n plus is a convex set, ok. So, implies; so, this implies S n plus equals a convex set. And in fact, this is a convex cone it is not just a convex set, you can show that this is a convex cone. The definition of a convex cone is something that very simple if X 1 bar similar to the convex set X 1 bar comma X 2 bar belong to S, if this implies theta 1 X 1 bar plus theta 2, X 2 bar also belong to S for any theta 1 comma theta 2 both greater than equal to 0.

Remember, there is no restriction of theta 1 plus theta 2 equal to 1 that restriction is there both in convex and in affine, right. So, if you relax that restriction if this holds true for any theta 1 theta 2 greater than or equal to 0 such a set is known as a convex cone. So, if you look at this definition any convex cone is also called a convex set because if theta X bar theta X theta times X 1 bar plus 1 minus theta times X 2 bar belongs to S, correct, that is if theta 1 X 1 bar plus 1 minus theta 1 plus theta 2, X 2 bar belongs to S.

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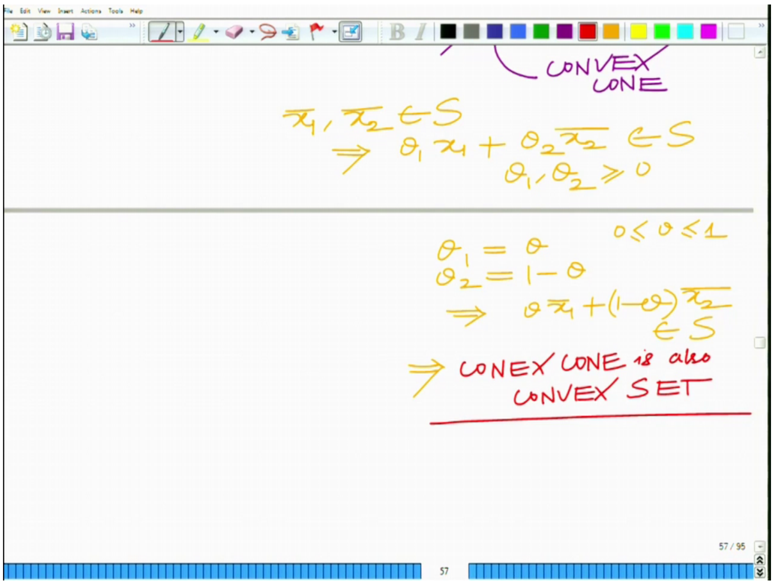

Then, I can simply set theta 1 equals theta, theta 2 equals 1 minus theta and for 0 less than equal to theta less than equal to 1, now theta 1 and theta 2 are both greater than equal to 0. This implies that theta X 1 bar theta X 1 bar plus 1 minus theta X 2 bar also belongs to S, which implies it is a convex sets. So, what this means is any convex cone is also a convex set. So, convex set is a special is a subclass of convex cones, ok.

So, implies the convex set is also a convex cone I am sorry a convex cone is also a convex set, but not the other way around not every convex set is a convex cone, alright. So, every; so, the set of positive semi definite matrices is a convex cone it is known as a convex cone for this particular reason not just a convex set, but it is a convex cone.

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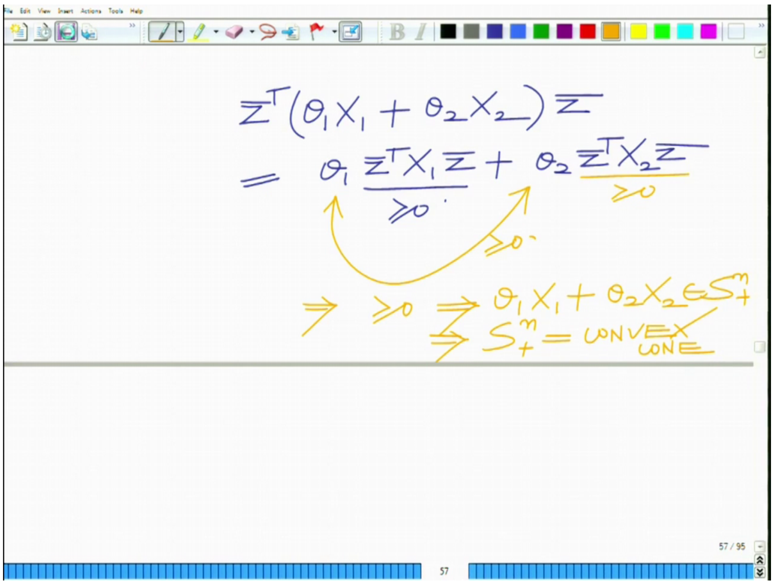

Because, if you take any theta 1 it is not difficult to see any theta 1 X 1 plus theta 2 X 2, where X 1 and X 2 are both positive semi definite you perform Z bar transpose Z bar you have theta 1 Z bar transpose X 1 into Z bar plus theta 2 Z bar transpose X 2 into Z bar. Now, again this is greater than or equal to 0 this is greater or equal to 0, theta 1 theta 2 greater than equal to 0 by assumption implies this is greater than equal to 0, implies again this theta 1 times X 1 plus theta 2 times X 2 belongs to the set S n plus implies S n plus is a cone, implies the set S n plus that is the set of all n cross n symmetric positive semi definite matrices is a cone, ok.

And, therefore, anyway for our purposes right it is important to remember that the set of positive semi definite matrices is a convex cone more importantly it is a convex set, ok. That is, we take the convex combination of any two positive semi definite matrices it is also in turn a positive semi definite matrix.

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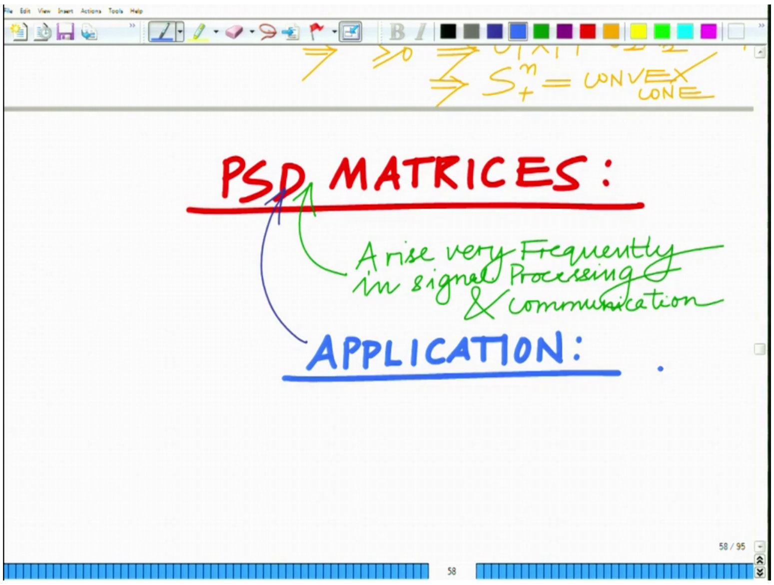

Now, PSD Positive Semi Definite matrices are very important in signal processing and communication as well. So, if you look at PSD matrices so, the set of all positive semi definite matrices these have a lot of applications these have arise very frequently these arise very frequently in signal processing and communication and for instance a simple application can be demonstrated as follows.

Let us consider a simple application of this concept of positive semi definite matrix. For instance let us consider a signal vector let us consider a discrete signal vector given as follows.

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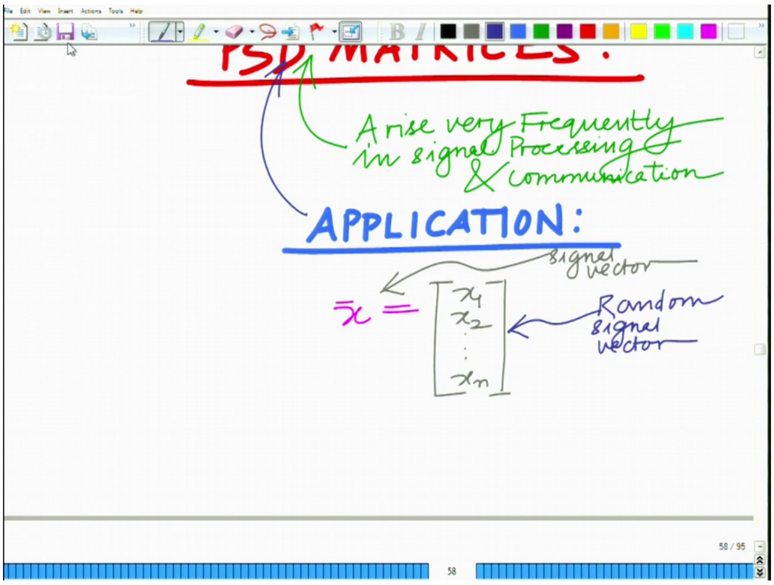

Given as x bar equals this is a vector of samples or you can say also symbols x 1, x 2, x n of size n, ok. So, this is a signal vector can arise in any scenario, ok. This is a; and let us further consider that this to be a random signal vector. Let us say this is a random signal n dimensional random signal vector with average value that is we look at this mean.

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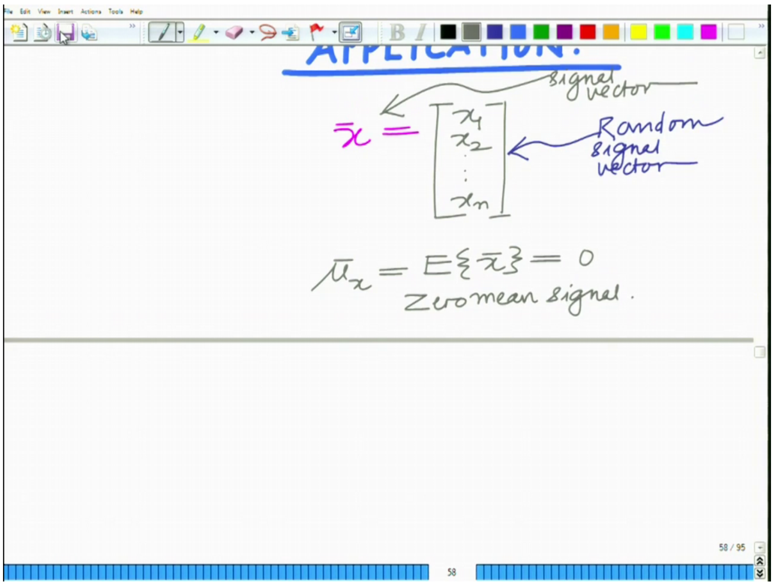

That is if you look at the mean of x mu bar of x is the expected value of x bar is equal to 0 which means this is a zero mean signal, ok, not important, but just for let us say convenience of analysis we are setting this to be also a zero mean signal. Now, this can arise in several scenarios. For instance, again let us go back to our multi antenna system.

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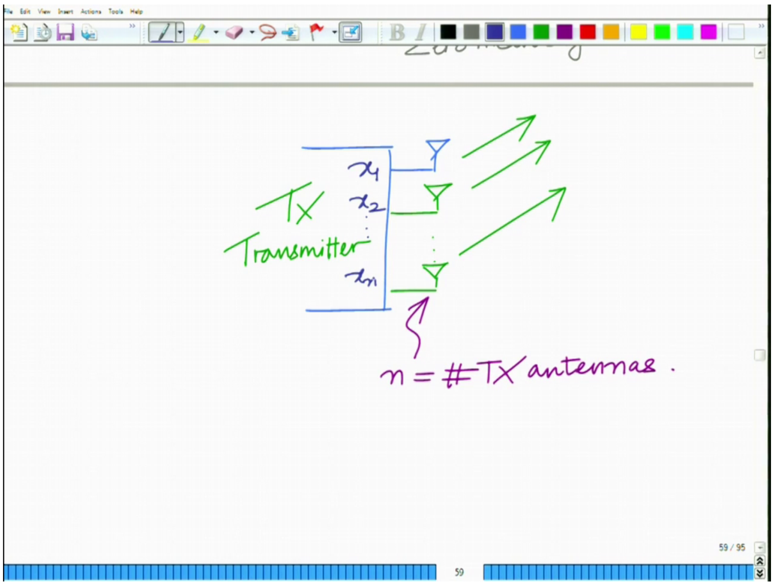

Which you must now be very familiar with if you look at your multiple antenna system, let us say we have in this case a set of multiple transmit antenna which are transmitting the signal, ok. So this is your transmitter and let us say in the wireless communication system your transmitting symbols x 1, x 2. So, let us say you have 1 or let us say you have n transmit antennas. So, number of transmit antennas equals n number of transmit antennas equals n and then let us say the transmits symbols are given by x 1, x 2, x n, ok.

So, we have n symbols x 1, x 2, x n; x 1 is transmitted from the first transmitter antenna x 2 is transmitted from the second transmitter antenna so on, x n is transmitted from the nth transmitter antenna. So, therefore, x bar also denotes the transmit vector or the vector of transmit signals that is we have x bar equals x 1, x 2, x n this is the vector of transmit symbols which is also known as the transmit vector, ok.

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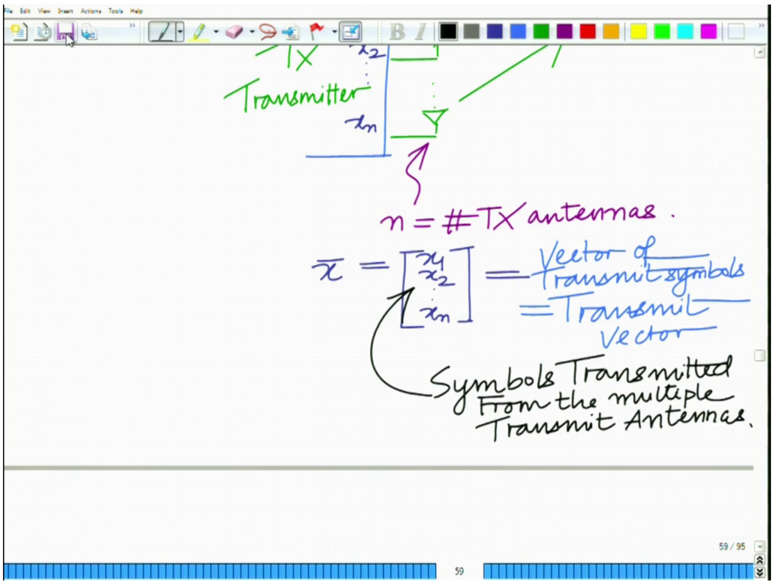

So, you have x bar which is equal to x 1, x 2, x n this is your vector of transmitted symbols from the multiple transmit antennas this can also be called as the transmit, vector of transmit symbols from the multiple transmit antennas. So, these are the symbols transmitted from the n transmit antennas. These are the symbols that are transmitted these are the symbols that are transmitted from the multiple transmit antennas.

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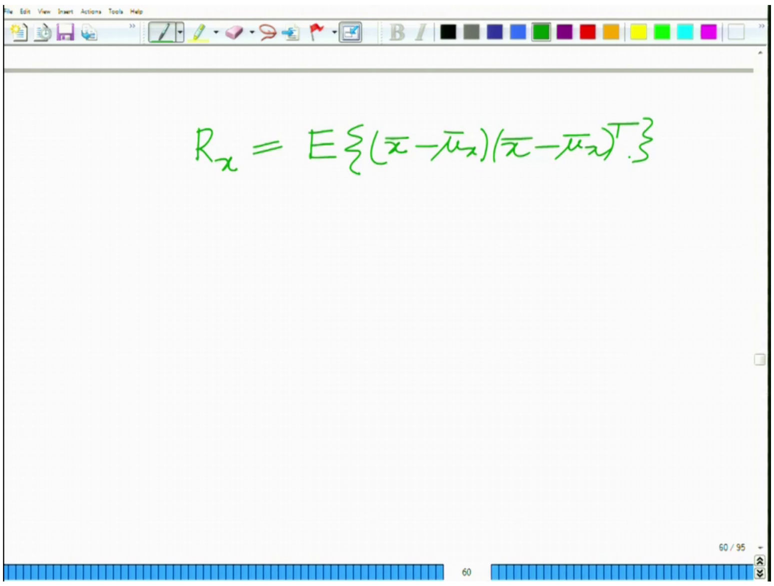

So, the transmit covariance is given as that is if we denote the transmit covariance by this matrix or we can also call this is as a covariance matrix of the transmit vector x bar that is expected value of x bar minus mu bar x into x bar minus mu bar x transpose, this is a expression for the covariance matrix.

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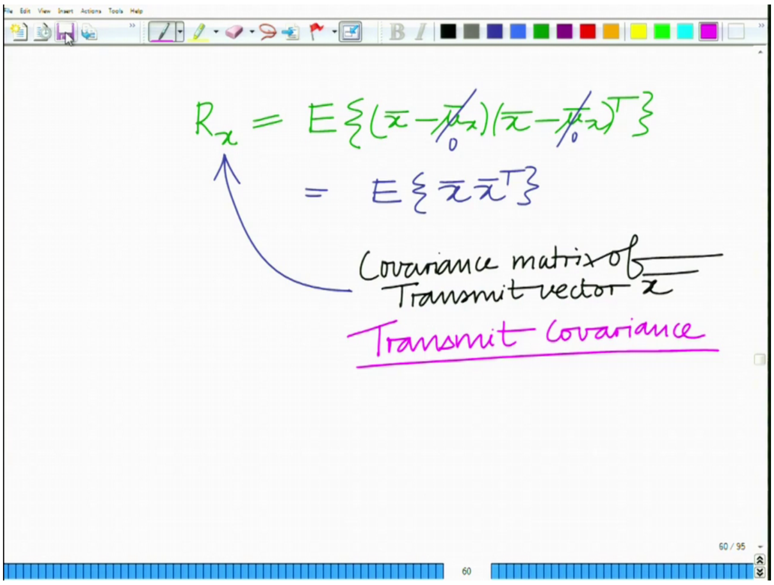

Now, in this case we already seen mu bar x that is the mean is 0. So, this is basically 0, 0. So, this is your expected value of x bar, x bar transpose and this is termed as this is denoted as I already said this R x this is also termed as the covariance matrix of x bar. This is the covariance matrix of transmit of the transmit vector x bar or they simply also known in practice and frequently in literature or research as simply the transmit covariance, ok. So, this is also simply known as the transmit covariance which is expected value of x bar x bar transpose.

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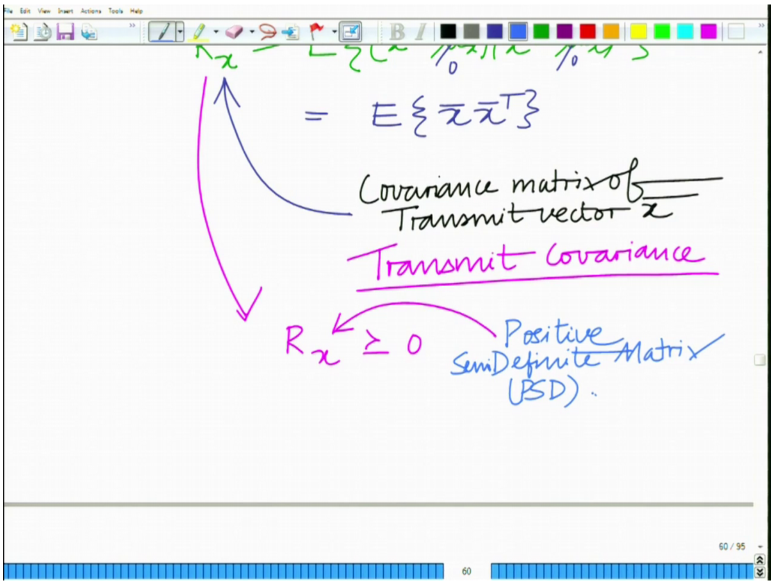

And, we can show that any such covariance matrix that is R x is positive semi definite it can be shown very easily. In fact, we will do just that that this transmit covariance matrix which arise frequently this is a positive semi definite matrix. So, this is a PSD matrix, ok.

So, this is a very important property of which arises very frequently that is positive semi definiteness and one of the most important types of matrices that we are going to see are basically the covariance matrices of these random vectors. And, every such covariance matrix for instance the transmit covariance matrix which is the covariance of the transmitted vector is a positive semi definite matrix this can be shown simply as follows.

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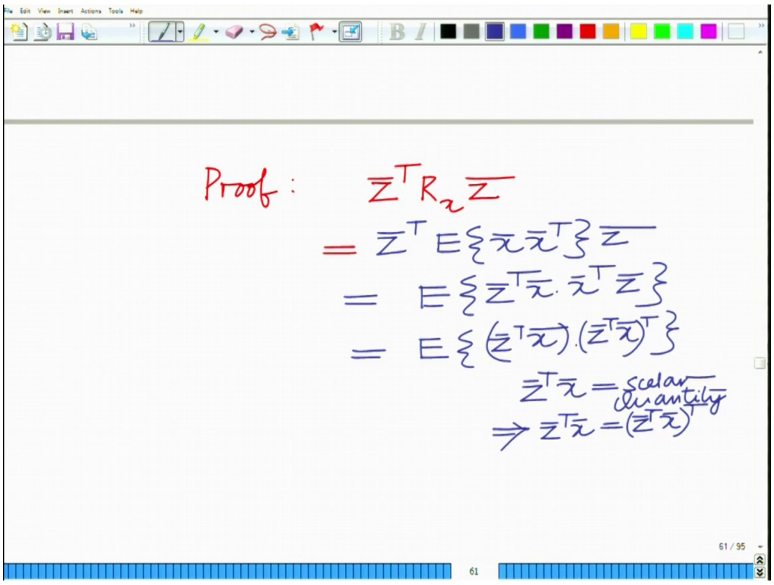

So, we want to show that R x is the transmit covariance is a positive semi definite matrix. So, we perform Z bar transpose R x Z bar we have to show this is greater than equal to 0. So, this is equal to Z bar transpose expected value of x bar x bar transpose in into Z bar which is now taking the Z bar inside the expected value. So, we have Z bar transpose x bar times x bar transpose Z bar which is basically expected value of you can see this is Z bar transpose x bar times Z bar transpose x bar transpose.

Now, you can see this Z bar transpose x bar is a scalar quantity ok, transpose of a vector times another vector so, this is a scalar quantity. This implies Z bar transpose x bar equals Z bar transpose x bar transpose, because for a scalar quantity the transpose that is when the quantity simply a number real number right the transpose of the quantity is itself.

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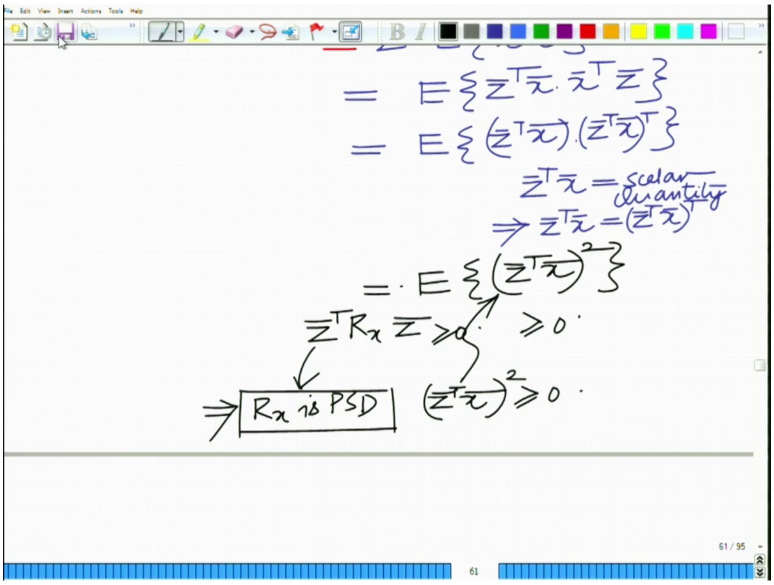

Therefore, this is simply equal to expected value of Z bar transpose x bar into itself which is basically Z bar transpose x bar square and this is the expected average value of the positive quantity so, this is greater than equal to 0. So, this because Z bar transpose x bar square is always greater than equal to 0. So, we take it is mean the expected value that is also going to be always greater than equal to 0 which means basically your Z bar transpose R x Z bar is always greater than equal to 0, which implies that R x is a that is the covariance matrix. In fact, for that matter any covariance matrix is positive semi definite, ok.

So, that completes basically the proof. This shows that the covariance matrix, if I not just the transmit covariance, but the receive covariance or any covariance matrix of a random vector is a positive semi definite matrix and covariance matrix has an important role to play.

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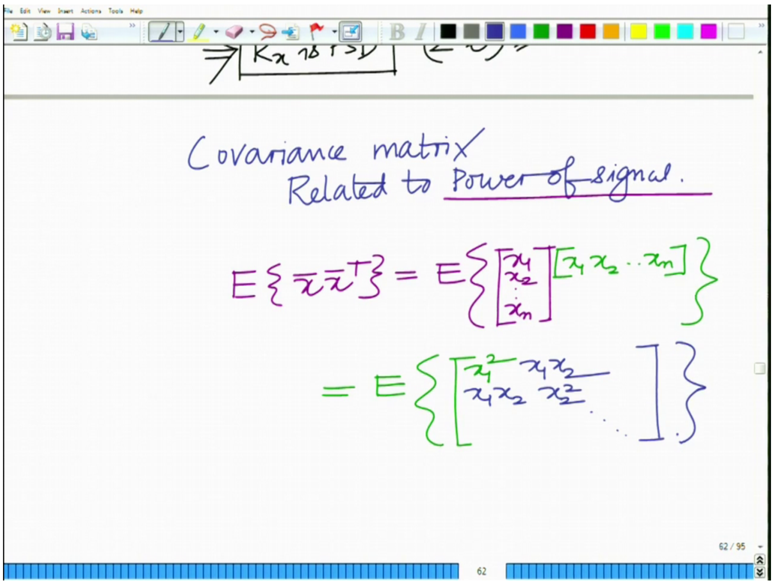

In fact, the covariance matrix is related to the transmit power of the signal. It is a very important property that is if you look at the covariance matrix, just expand it a little bits to give you a better idea. So, this is if you write this as expected value of x 1, x 2 up to x n times x 1, x 2 up to x n this is equal to expected value of x 1 square x 1, x 2, x 1, x 2, x 2 square and so on.

And, now if you look at the trace of this, now trace implies the sum of the diagonal elements, correct.

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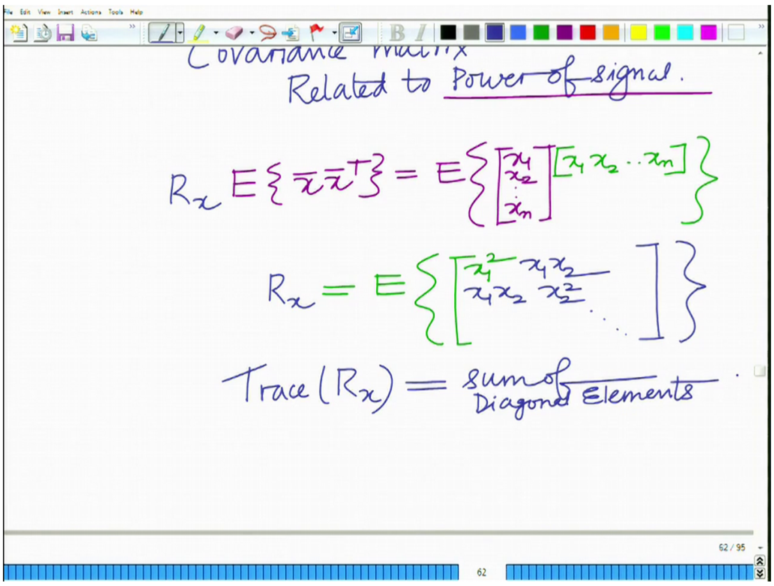

And, so if you look at the trace so, this is your basically your R x that is your covariance matrix, ok. So, if you look at the trace of R x that is equal to the remember trace of a square matrix is the sum of the diagonal elements.

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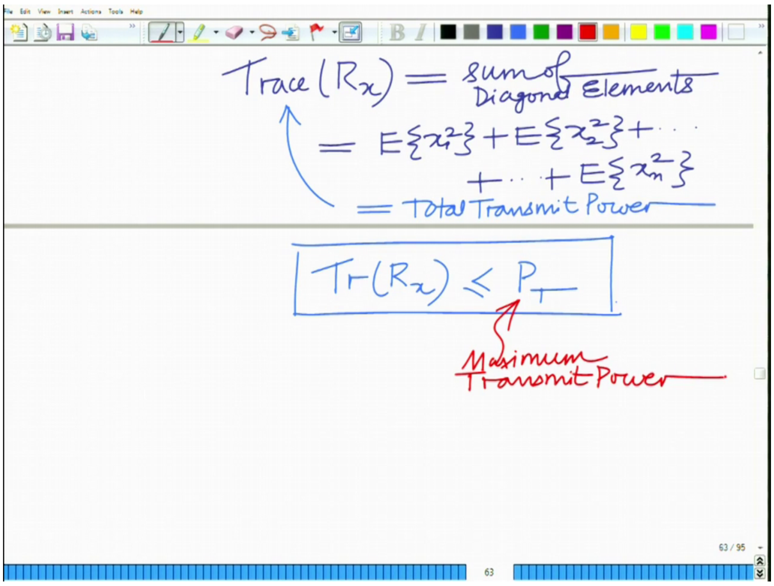

That will be equal to expected value of the diagonal elements are x 1 square plus expected value of x 2 square plus so on plus expected value of x n square and this is nothing, but the power of each symbol expected value of x i square is the power of symbol x i square. So, this is basically total transmit power total.

So, the trace of the covariance matrix is nothing, but the total transmit power and that has to be less than or equal to the maximum transmit power at the transmitter. Therefore, we will have this constrained in a practical communication system. We will have the constraint that is if you look at the trace of R x that is less than or equal to P T. Let us denote the maximum transmit power by P T. So, this is an alternative way of writing the transmit power constraint this is the maximum transmit power.

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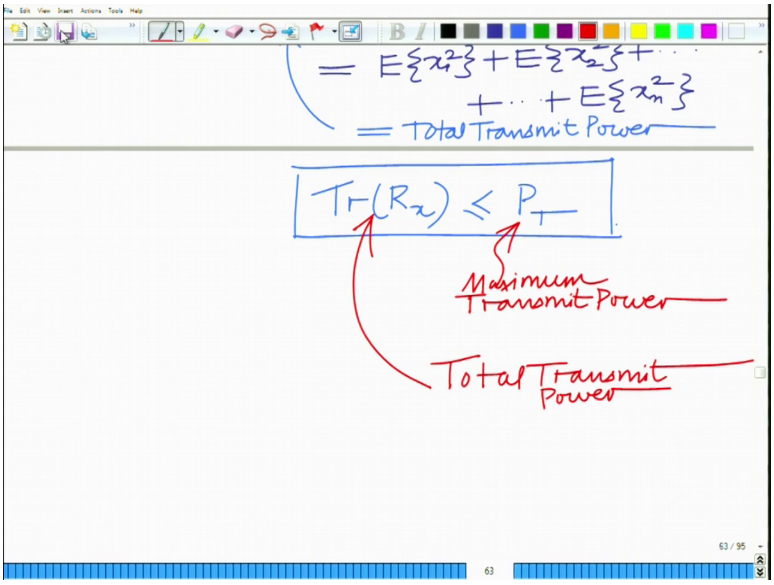

And, this is basically your trace of R x so, this is the total transmit power. So, we have that total transmit power is less than or equal to the total transmit power is less than or equal to the maximum possible transmit power. So, this covariance matrix, alright prominent role to play in wireless communications in fact, we will frequently encounter this notion of transmit covariance, receive covariance matrices and so on or the interference covariance matrix and so on and all has to do with the power of a particular signal, alright. It gives an indication of what is basically the power of the signal which is the indeed a random vector, ok, alright.

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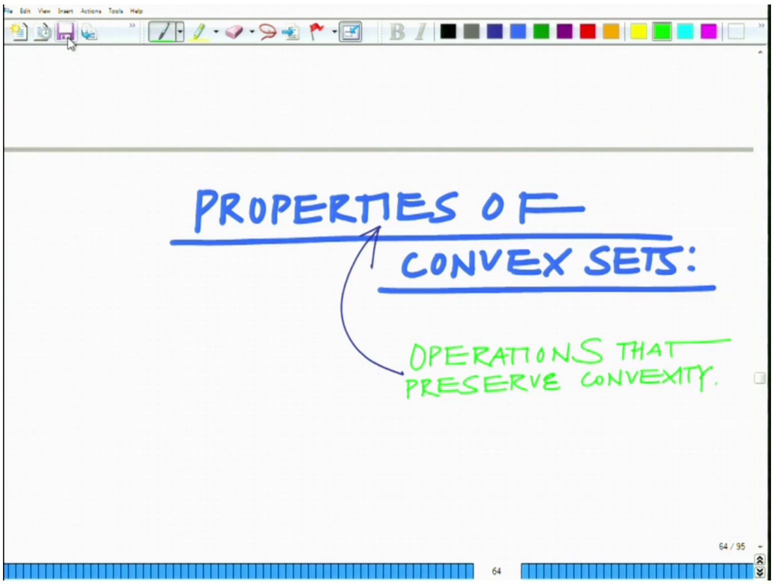

And, let us now move on to another important concept which is explore the properties of convex sets. So, you want to also explore this notion of properties of convex sets or basically operations that preserve convexity or basically you can also think of this as operations that can be performed on convex sets that preserve the convexity.

Various operations that can be performed on convex sets that preserve the convexity various operations that can be performed on convex sets that preserve convexity. Now, the first property in this is very simple.

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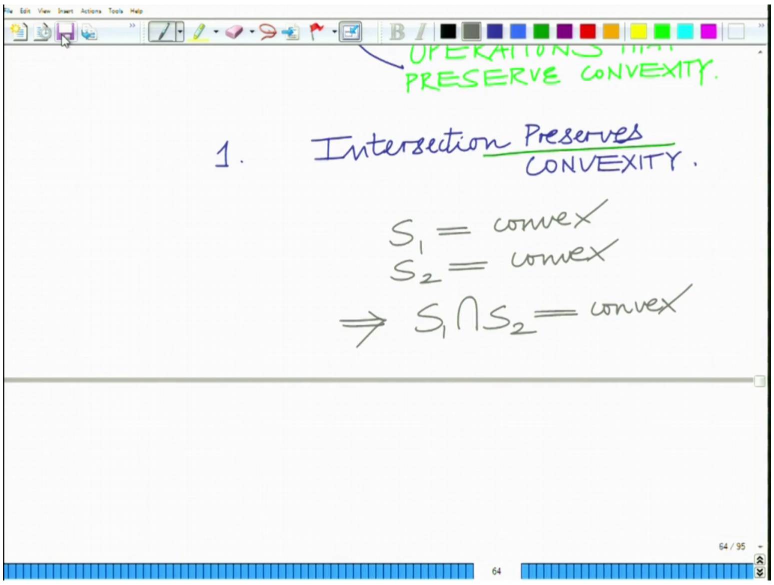

First property is that intersection preserves convexity that is we look at the intersection of two convex sets the result is convex. The intersection preserves convexity. What this implies is that if S 1 is a convex set and S 2 is a convex sets both S 1 and S 2 are convex set. In fact, what you can see is S 1, intersection is S 2 is also convex that is in intersection. In fact, this can be extended to any arbitrary number of sets, that is, we take convex set if each set is convex, then the intersection of all these sets is also convex, that is other interesting property and also very simple to verify you can also prove it formally.

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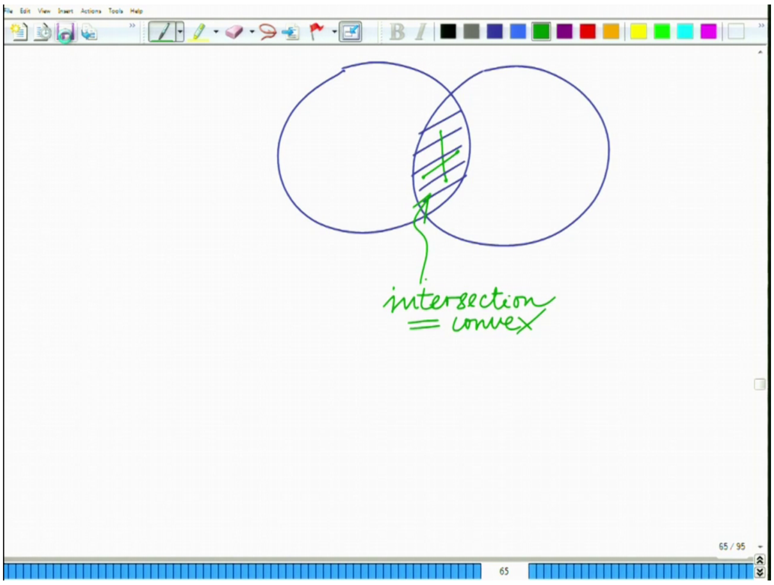

For instance you take two circles look at the intersection region this looks nothing like a circle, but yet you can see that this is a convex set that is if you take any two points beyond the line. So, this intersection so, the two circular regions are convex so the intersection is also the intersection of these two circles is also convex, ok.

And, we already seen an example in this regard that is a intersection of hyperplanes and half spaces is also convex.

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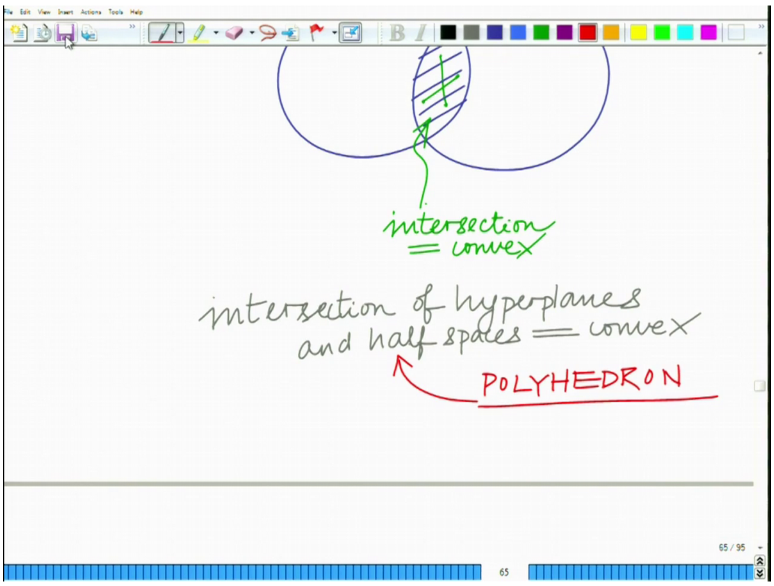

For instance, if you look at the intersection of hyperplanes and half spaces are also convex. In fact, each hyperplane is convex, half space is convex, the intersection of this is convex. In fact, that has a special name this is termed as the polyhedron, we are already seen this, ok. So, this intersection of hyperplanes and half spaces which is convex has a special name it is termed as a polyhedron, alright.

We have looked at several interesting aspects first we have looked at positive semi definite matrices, verified that the set of positive symmetric positive semi definite matrices is a convex set and we also started looking at the properties of convex sets. We will continue this discussion further in the subsequent modules.

Thank you very much.