Applied Optimization for Wireless, Machine Learning, Big data
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Lecture – 13
Norm Ball and its Practical Applications: Multiple Antenna Beamforming

Hello, welcome to another module in this massive open online course. So, we are looking at the basics of convex optimization. In particular, we have looked at the concept of convex set hyper planes and hyper spaces and we are looking at applications of these concepts in wireless communication, that is practical applications of this concept, alright. So, today, let us look at another application of the same concept that is in Multi Antenna Beamforming, ok.

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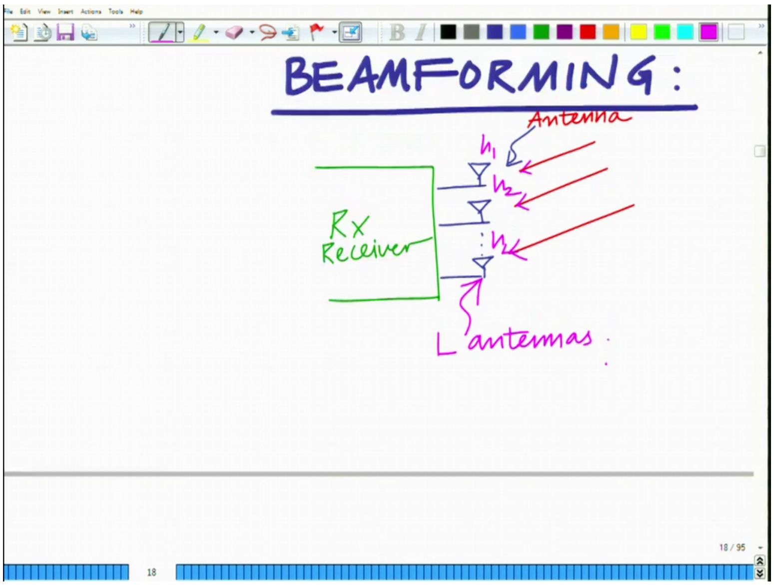

So, what I want to look at is I want to introduce another application for of the concept of hyper planes and half spaces and this is in the context of multi antenna beam forming. And, what happens in multi antenna beam forming is basically you have receiver with multiple antennas, correct and let us say this is a receiver, the wireless receiver in wireless communication system, and you have multiple antennas each of these is an antenna and you have the signals that are coming with various channel coefficients, ok. So, you have the various channel coefficients corresponding to the antennas 1, 2 up to let us say L. So, there are total of L antennas.

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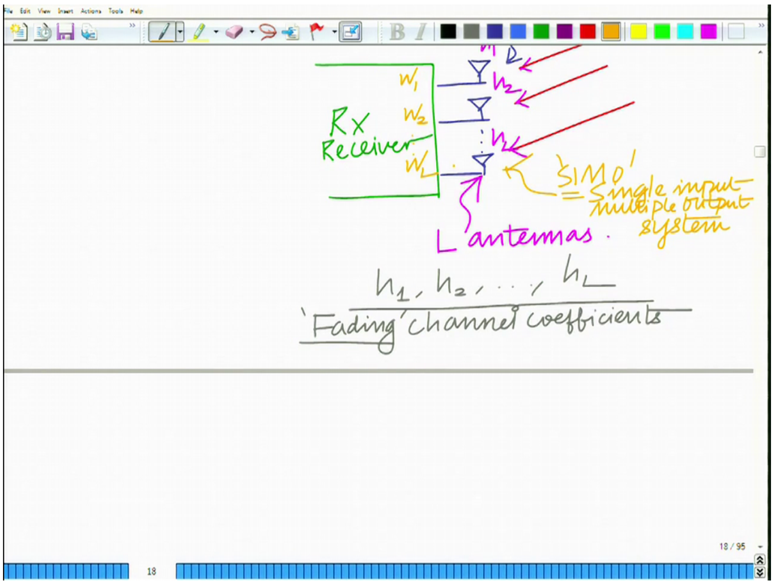

And, this multi antenna system what we have is we have this channel coefficients h 1, h 2, up to h 1, h 2 up to h L, these are the channel coefficients. These are also known as the feeding channel coefficients because the wireless channel is typically a fading channel that is received power of as a channel is varying with time, it is increasing and decreasing. So, the wireless channel coefficients are also known as fading channel coefficients, and h 1, h 2, h L denote the L channel coefficients corresponding to the L antennas in this multi antenna receiver.

And, this is also known as a single input multiple output or a SIMO receiver, ok. So, this is also typically known as such a system is also known as a SIMO or a single input multiple output in the sense we have multiple antennas. Single input multiple output system and let us now assume the combining it h 1, h 2, h L are the channel coefficients.

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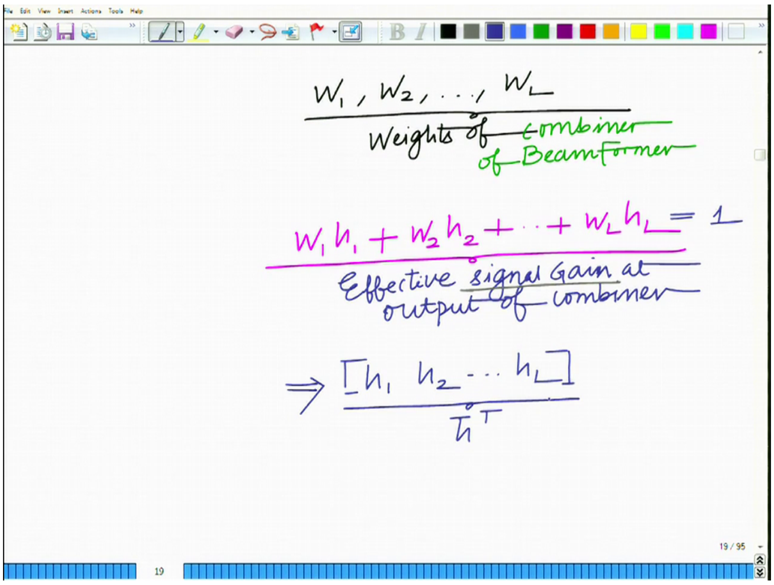

What we are going to do is we are going to combine the signals with weights W 1, W 2, W L, so, W 1, W 2, W L you can think of these as the weights of the combiner; these are the weights of the combiner, these are the weights of combiner or you can also think of this as a beam former, weights of the combiner or also beam former and what we are performing is we would weigh the received signals and add them.

So, we are performing a linear combination of the signals and therefore, if you look at the effective gain of the signal gain across at the output of the combiner that will be the weights times the corresponding channel coefficient and the sum. So, you can think of this as the effective you can think of this as the effective gain what the effective signal gain, effective signal gain at the output of the combiner. And, to normalize this effective signal gain what we do is we set this equal to 1. So, typically what we have what this implies is that we want to design a system such that this effective signal gain at the output of the combiner that is we look at the output of the combiner the effective signal gain is unity, alright.

So, you like to design such a combiner, this is typically a constraint in multiple antenna processing one of the types of constraints that can be employed in multiple antenna processing, ok. So, what we have is this can be written as h 1, h 2, up to h L that is the channel coefficients.

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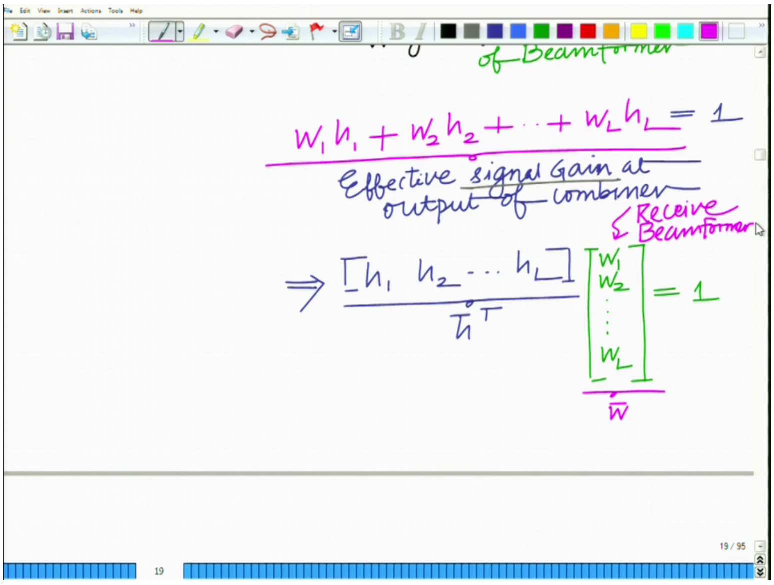

This is the vector h bar transpose times the vector if you look at this W 1, W 2 W L this is equal to 1 you can call this as the beam forming vector or this is basically also your receive beam former. So, h bar transpose W bar where W bar is a vector of combining weights or the combiner or the receive beam former is 1 to ensure unity gain.

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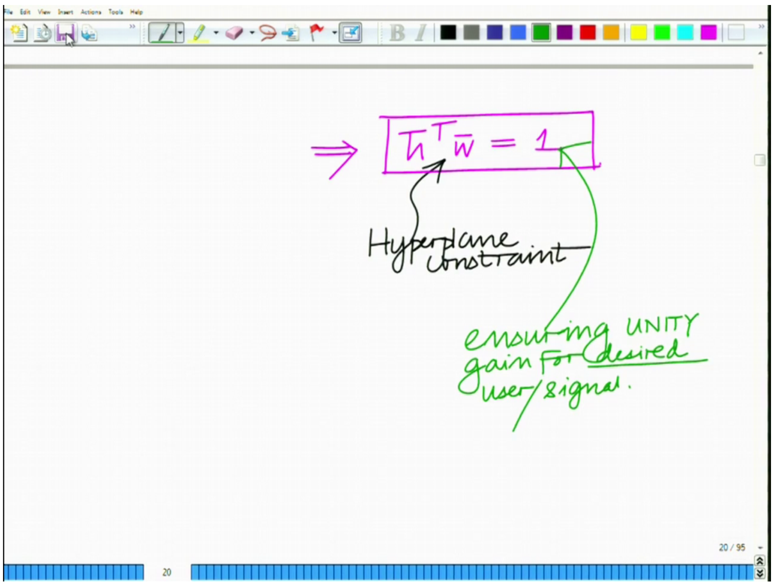

Which basically implies h bar transpose W bar equals 1 and this you can see now is nothing, but a hyper plane constraint, ok. So, this is a practical application of the concept of hyper plane in a wireless communications which says that all this beam forming vectors W bar lie on this hyper plane described by h bar transpose W bar equals 1. This is the hyperplane constraint and what is this is doing is basically this is ensuring unity this is ensuring unity gain for the desired user or desired signal, that is what your doing is your ensuring that the gain signal gain corresponding to a particular desired user or signal is unity at the output of the combiner.

So, this signal is unity and then what you can do is you can either suppress, you can typically either suppress the noise or suppress the interfering signals of the interfering users. So, this is typically constraint that is employed in multi antenna signal processing in a wireless communication system, alright. So, let us so, we have seen the definition the notions of hyperplanes and half spaces. Let us know move on to different key type or different types of convex sets in particular let us look at spherical balls or the norm what are also known as norm balls, ok.

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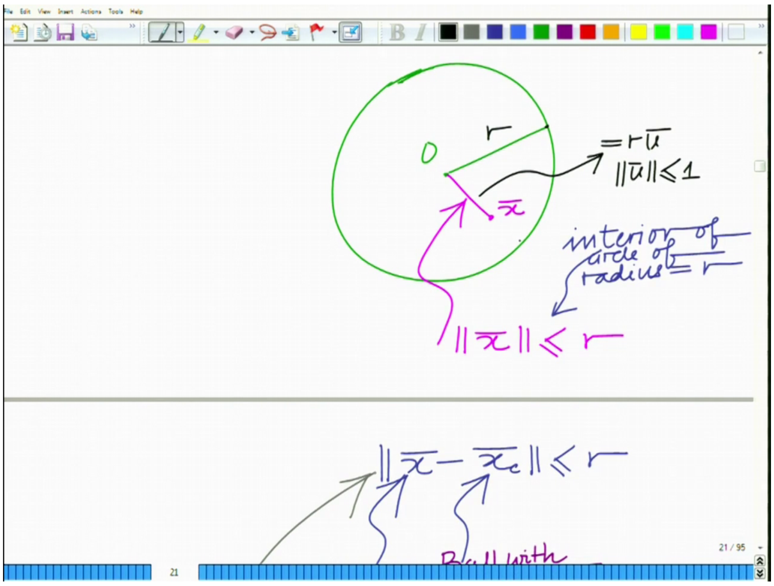

So, the next type of convex set we want to look at is that of a norm ball or basically a Euclidean ball. So, what we want to look at is this constraint of a norm ball in particular a Euclidean ball. And, if you look at this, for instance you look at ball in 2 dimensions is basically a circle with a certain centre let us say at the origin 0 and this is has a radius r, then if you look at any point in the interior of the circle x bar, if you look at norm that is the length of this vector which is norm of x bar that has to naturally be less than or equal to r which is the radius, ok.

So, this describes the interior of a circle which is nothing, but a 2 dimensional ball or a sphere interior of circle of radius equal to r and with center origin.

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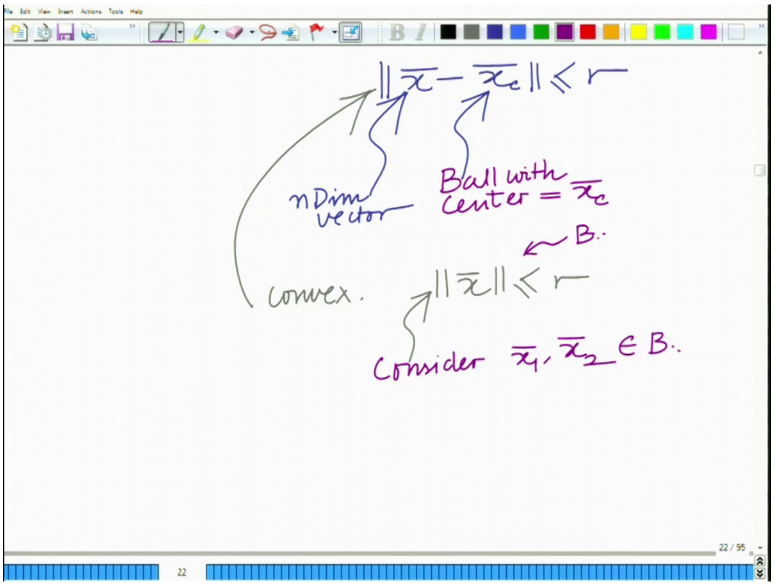

Now, if the center is not the origin then you can simply shifted to origin by considering norm x bar minus x c bar less than or equal to r this is the general this is a circle or this is the circle or a sphere. In fact, you can if x bar is n dimensional vector if you consider as n dimensional vector, this is the sphere or a ball with center at x c bar, ok.

So, this describes interior of n dimensional ball with centre x c bar and radius r, and this can be seen to be convex and that can be briefly justified as follows that is this region is convex for sake of simplicity, let us consider simply the ball with centre at origin. Remember if the ball with centre at origin is convex then naturally if you shift it to any centre x c bar it is also going to be convex because shifting does not affect the convexity, alright. So, if norm x bar less than or equal to r is convex then norm x bar minus x c bar less than equal to r is also convex because a translation does not affect the convexity of the object convexity of the region, or the set, ok.

And, consider two points to verify this simply consider two points x 1 bar. So, we have to demonstrate that given any two points x 1 bar x 2 bar their convex combination lies in the set.

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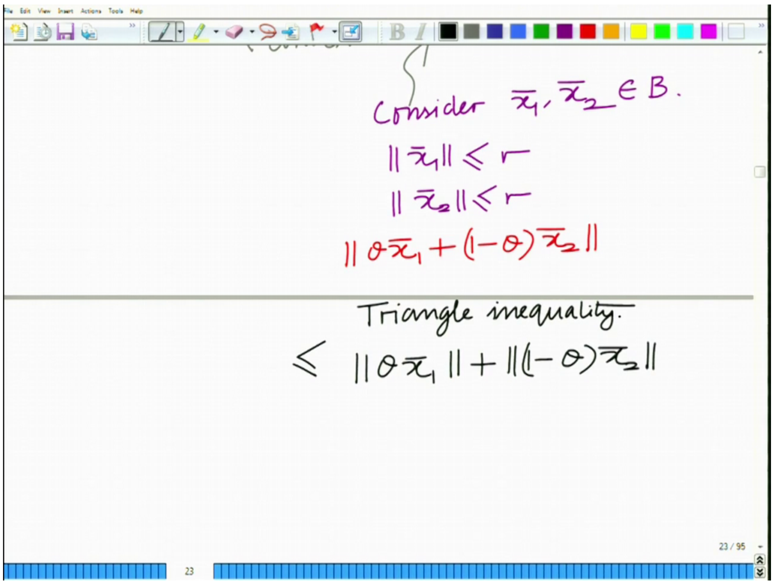

Consider x 1 bar, x 2 bar let us denote this set by B, belong to B. Then what we have is we have norm by definition since they belong to the interior of the ball norm x 1 bar less than equal to r, norm x 2 bar less than or equal to r. Now, let us consider the convex combination we have theta times norm theta times x 1 bar plus 1 minus theta times x 2 bar, we have to show that the norm of this is less than equal to r. So, that this also lies in the interior of the ball.

Now, you can readily see what needs to be an first we can use the triangle inequality that is norm A bar plus B norm of A bar plus B bar is less than equal to norm A bar plus norm B bar. So, that gives me this is less than equal to theta times x 1 bar norm plus norm of 1 minus theta times x 2 bar.

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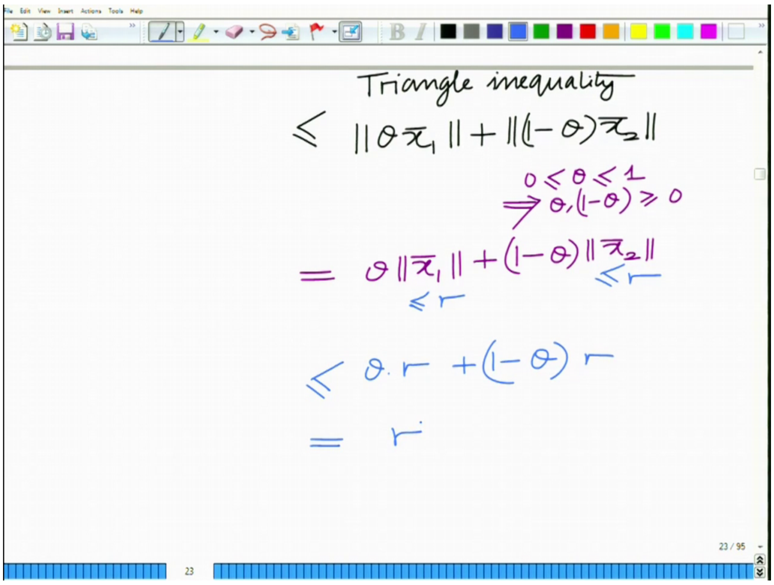

Now, remember theta is positive because this is the convex combination. So, we have 0 less than equal to theta less than or equal to 1 which basically implies that theta coma 1 minus theta are both greater than equal to 0. So, a norm of theta times x 1 bar is theta times norm of x 1 bar because theta is greater than or equal to 0 plus 1 minus theta times norm of x 2 bar since, 1 minus theta is also greater than equal to 0. Now, observe that norm x 1 bar norm x 2 bar are less than or equal to r. Remember, both these quantities lie in the interior of the ball therefore, they are less than or equal to r. So, this is less than or equal to theta times r plus 1 minus theta times r which is nothing but r, ok.

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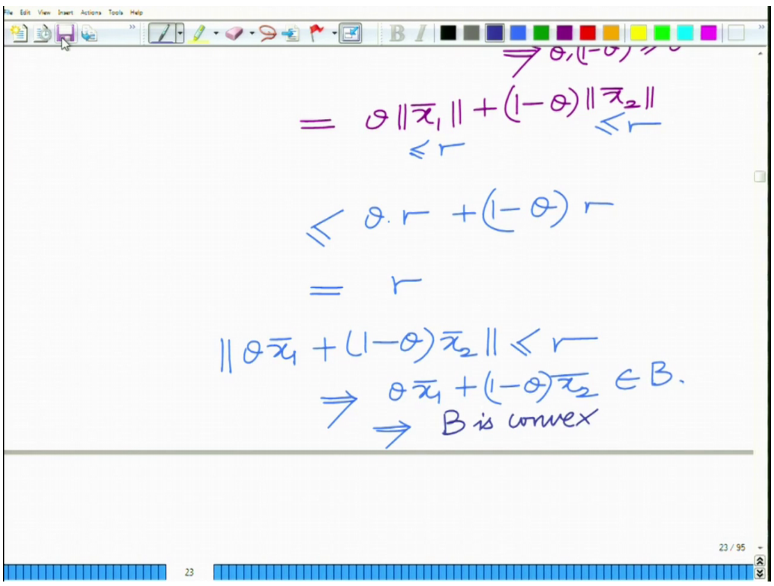

So, that implies you have norm of theta times x 1 bar plus 1 minus theta times x 2 bar less than or equal to r implies theta times x 1 bar plus 1 minus theta times x 2 bar less than equal to r which implies this is essentially belongs to the set B, which implies B is convex, ok. That completes the proof, and that is obvious what we have been able to show that if x 1 bar for any two points x 1 bar, x 2 bar belong to the ball their convex combinations, all their convex combination also belong to the ball. Therefore, the norm ball is convex, ok.

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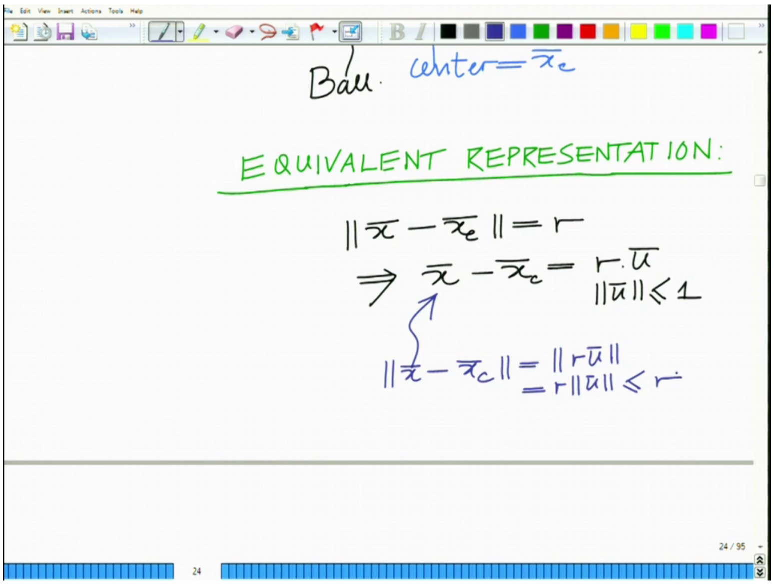

And, this norm ball now remember another equivalent so, we want to a de another equivalent way an interesting way to represent this is now we can represent this norm ball as B of x c bar comma r. So, this denotes a ball for norm ball its center equal to x bar c, I am sorry, this is not r bar, but this is r which is the radius equal to r. So, this is the norm ball another equivalent way to represent the norm ball that is equivalent representation is as follows.

The equivalent representation of the norm ball is as follows. that would be remember norm of x bar minus x c bar equal to r and this implies, you can write this as x bar minus x c bar is some is r times some vector u, correct? Where, norm of u bar is less than or equal to 1. For instance I can always write this if you look at this x bar, I can always write this x bar equals r times some vector u bar, correct where u bar is unit vector in this direction or u bar norm of u bar is less than or equal to 1, ok. So, the norm of u bar is less than or equal to 1, ok.

So, what that means is, that is what this and therefore, now you can see and you can write readily verify this. This implies that norm of x bar minus x c bar equals norm of r of u bar, since r is positive this is r times norm of u bar and norm u bar less than or equal to 1 which means this is less than or equal to 1, which is the same thing which is setting the same thing in a different way that is your finding a vector u bar and which is norm less than or equal to 1 and your saying that x bar minus x c bar equals r times u bar. And, this is true for any such vector you bar since this norm of x bar minus x c bar less than equal to r. So, such point x c x bar will lie in the interior of the ball.

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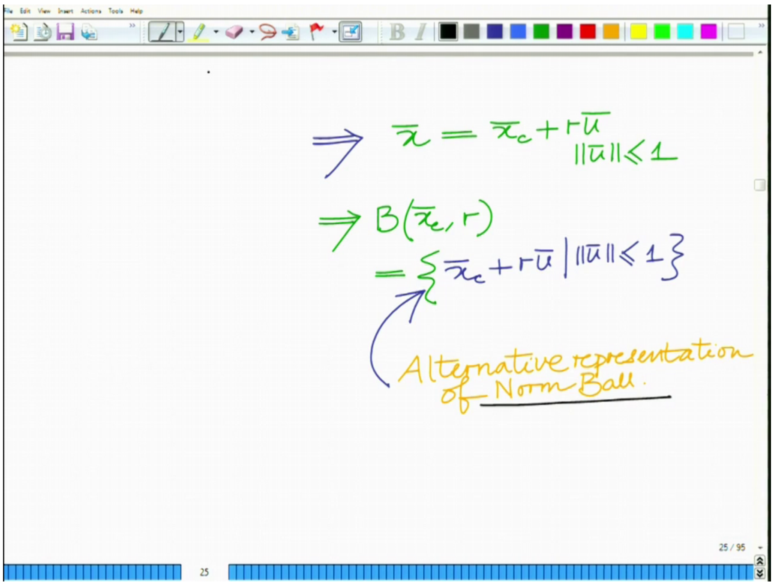

Which implies now, that if you look at x bar equals x bar c plus r times u bar such that norm u bar less than equal to 1, this lies in the, well this lies in the interior of this interior of the ball this implies that I can represent the ball with center x c bar comma radius r also in the following form that is equal to the set of all vectors x bar c plus r u bar such that norm u bar is less than or equal to.

So, this is an alternate representation of the norm ball or this is basically an alternative representation of the norm ball. This is alternative representation of the norm ball, alright. That is x bar x c bar that is center plus r radius times u bar, where u bar is any vector such that norm u bar is less than or equal to 1, ok. So, this is an alternative representation which is very convenient to represent often times represent the norm ball ok, alright.

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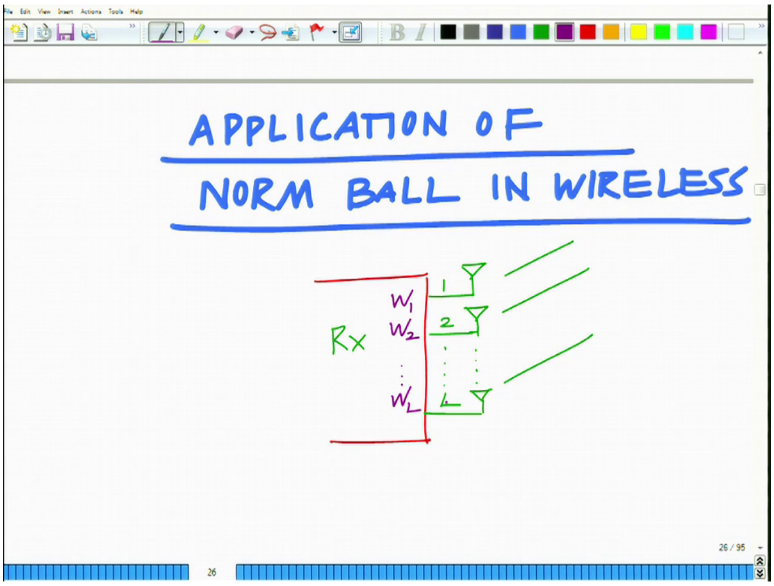

So, let us look at the another application similar to if your seen for the hyper planes and half spaces. Let us again look at the application of the concept of norm ball for wireless application of the concept of norm ball in a wireless system and this can be seen as follows.

For instance, again let us go back to our multi antenna beamforming problem and we again have the different signals. So, this is again your multi antenna receiver and these are antennas 1, 2 up to antennas L and which we are combining using our beamforming using the weights W 1, W 2 up to W L.

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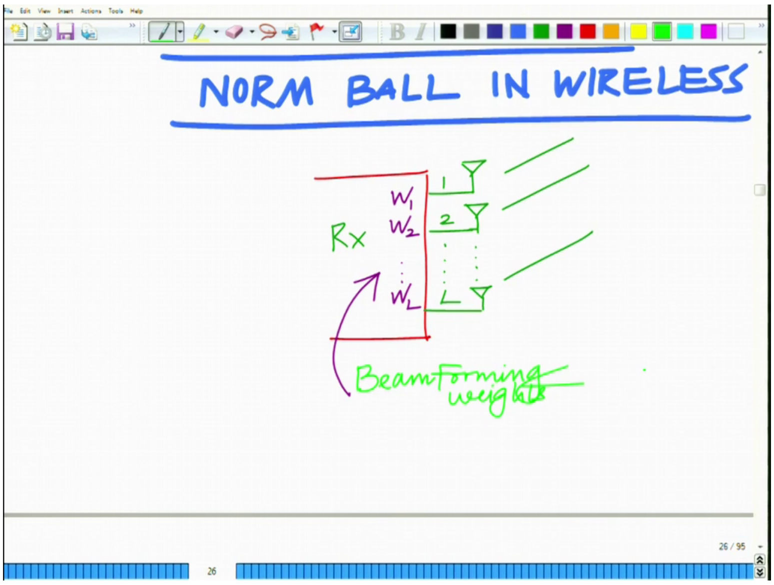

Remember or recall that these are the beamforming weights; these are the beamforming weights.

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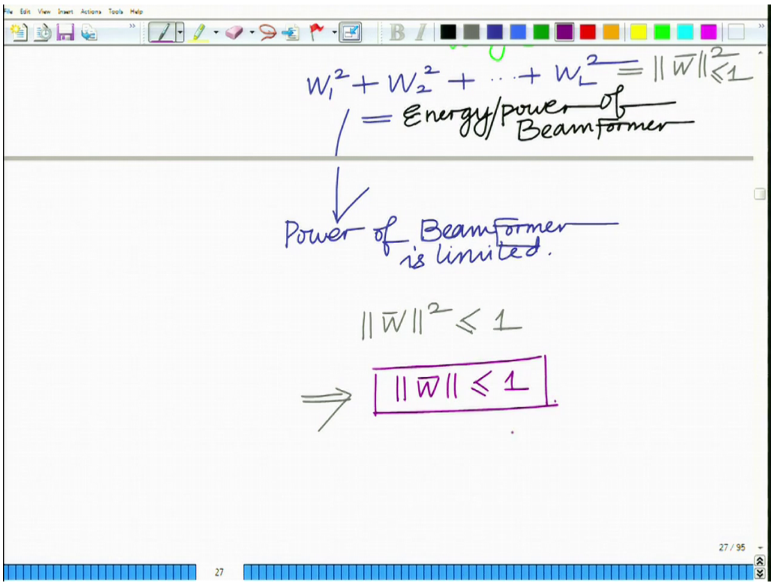

And, what we will ensure that if you look at the energy of the beam former that is W 1 square plus W 2 square because this is also this also influences the power output power of the combiner. So, we restrict the energy of the beam former which is equal to the energy of the beam former and often also called the power of the beam former because this is what is applied at every time instant energy slash power of beam former.

Now, in any wireless communication system this has to be should be restricted because this influences the power of the signal at the output of the beam former. If the energy of the beam former unbounded then the power of the output signal can also be unbounded therefore, to ensure stable beam former restrict this energy of the beam former typically to unity, alright. So, in any wireless communication system energy of the beam former has to be limited. Let us call this as the power because that is typically the nomenclature that is used. The power of the beam former has to be re represented limited.

Now, if you can look at this, this is nothing, but norm of W bar square. This is norm of W bar square. Typically, this is less set less than or equal to unity. So, we have the constraint norm of W bar square less than or equal to unity which implies that if you look at this, this implies that norm of W bar has to be less than equal to unity and this is nothing, but a norm ball constraint.

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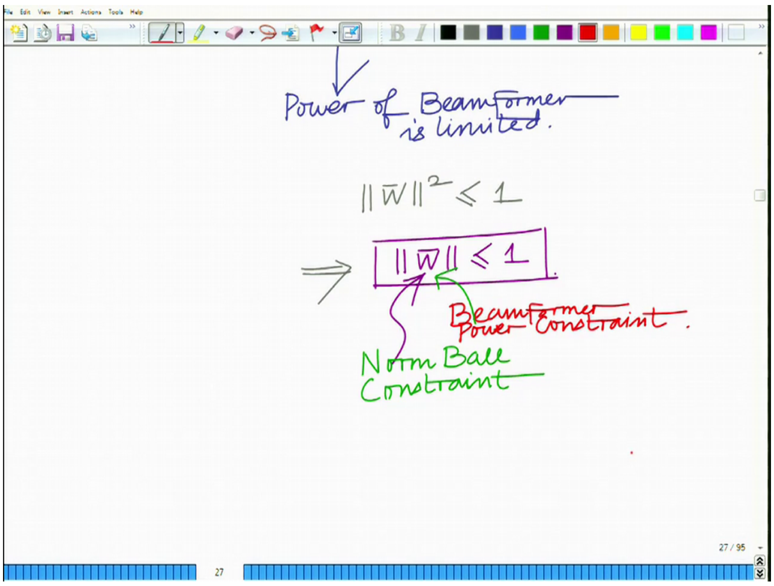

So, this is the beam former power constraint which is a norm ball constraint norm of W bar that is constraints the set of all beam formers to a norm ball with radius 1, interior of a norm ball with radius 1. So, this is normal ball constraint. So, this is the constraint on the beam former power constraint. This can also be thought of as the this can also be thought of as the beam former power constraint which is basically nothing, but a norm ball or a Euclidean ball constraint, alright. So, that is an interesting application of the concept of a norm ball to a wireless communication scenario to design the constraint for a receive beam former, alright.

So, we have seen various other concepts in this module namely that of the norm ball or the Euclidian ball and its application in the context of a wireless communication system. So, we will stop here and continue in the subsequent modules.

Thank you very much.