Principles of Signals and Systems
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Lecture – 69
Group/ Phase Delay - Part I

Hello, welcome to another module in this massive open online course. In this module we will start looking at a different concept that is group and phase delays of an LTI system ok.

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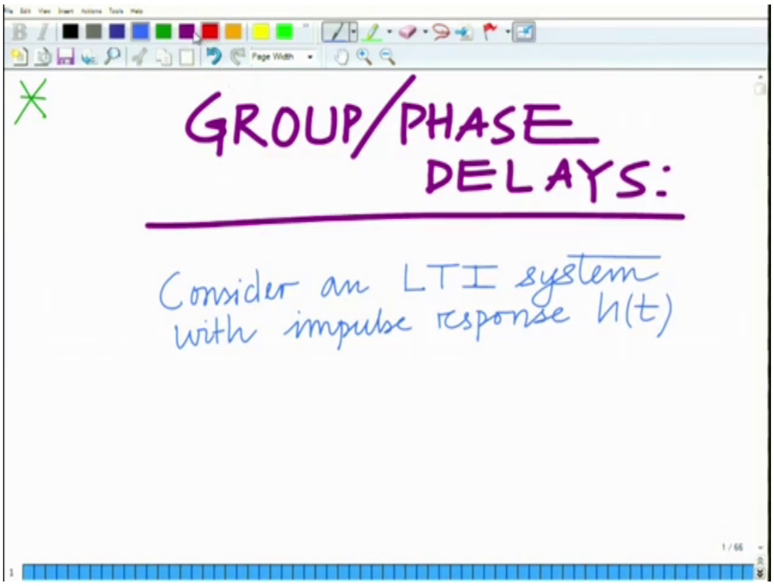

So, want to look at, another important topic a small topic which is known as how to characterize the group delay as well as the phase delays, the group and phase delays.

And so, to do this consider an LTI system considering LTI system with impulse response h t and therefore, what we have is, we have the LTI system with input x of t output y of t the impulse response of the LTI system is h of t. So, this is your input, this is your output ok.

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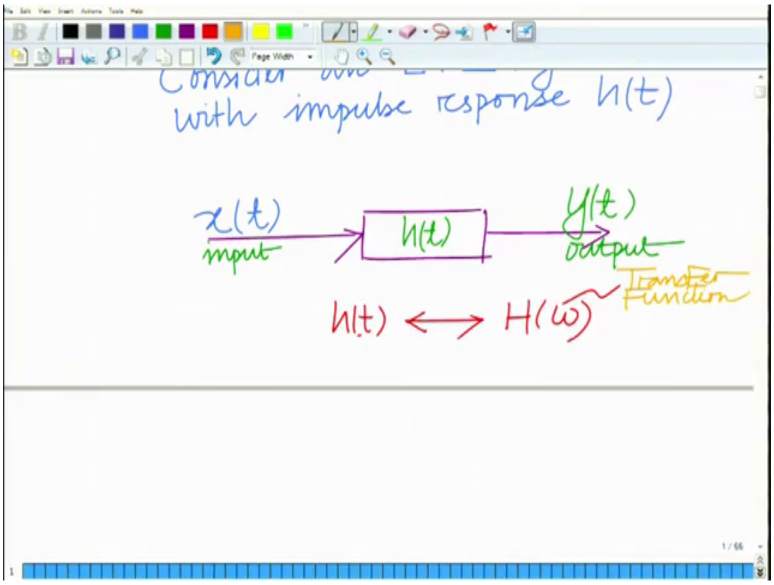

And let us say this transfer function of h of t as the transfer function as the Fourier transform H of omega we know this also known as the transfer function correct of the LTI system.

Now, what happens in this is that we have I can write Y of omega the frequency response of the output is basically X of omega times H of omega, all right.

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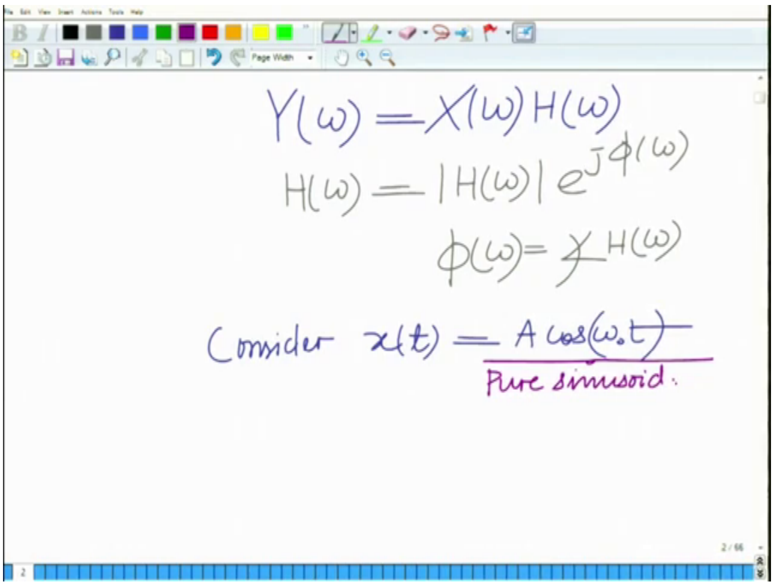

The output is basically the convolution of the input signal with the impulse response which means in the Fourier domain it is a multiplication of the input Fourier transform X omega with the transfer function H of omega ok.

Now, let H of omega equals the magnitude of H of omega times phi of omega where phi of omega is the phase response, this is the angle of H of omega. Now, what we do is consider a sinusoidal carrier x of t equals A cosine omega naught t. So, this is basically your pure sinusoid ok.

So, just for the purpose of illustration you consider a pure sinusoid A cosine omega naught t.

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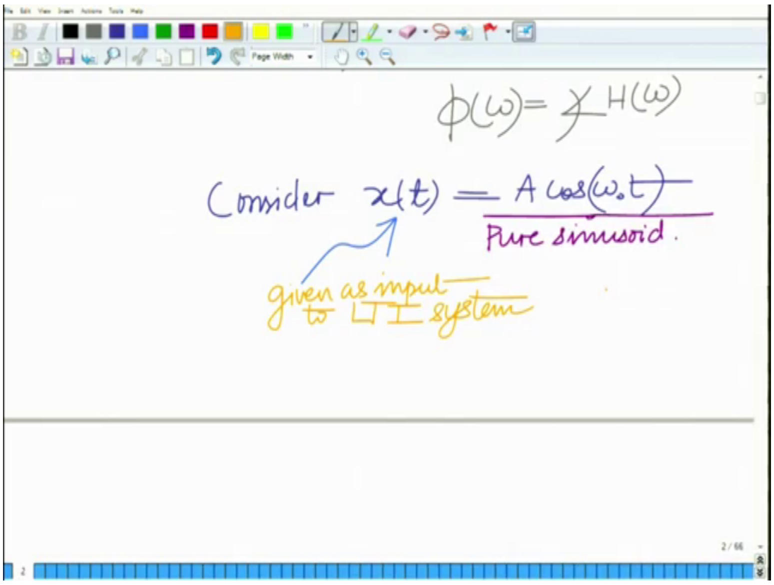

Now, this is given as input to the LTI system ok. It is given at an input to the LTI system.

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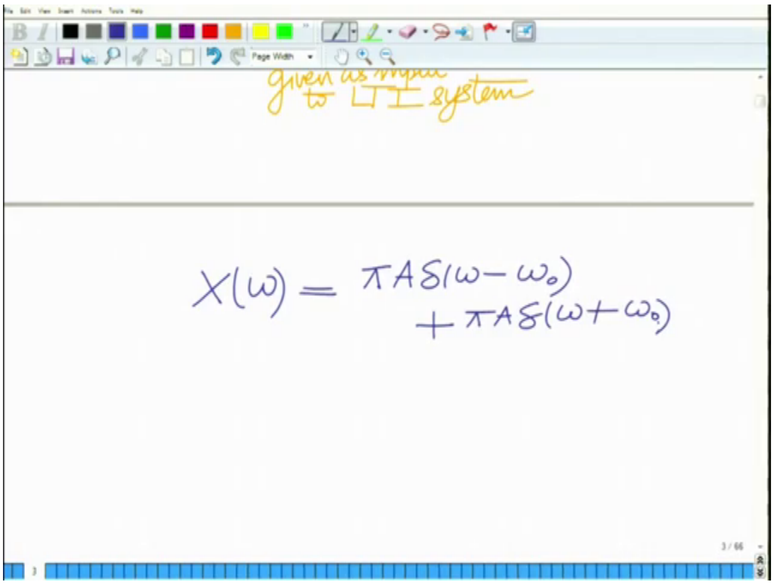

Now, we have already seen Y of omega equals H of omega into X of omega. Now, what is X of omega? X of t is a cosine omega naught t which means X of omega we know from the Fourier transform that is A times the Fourier transform of cosine omega naught t. So, this will be pi A delta omega minus omega naught plus pi A delta omega minus omega naught ok. That is the response or that is a spectrum of a cosine omega naught t 2 impulses at omega and minus omega naught all right omega naught and minus omega naught scaled suitably ok.

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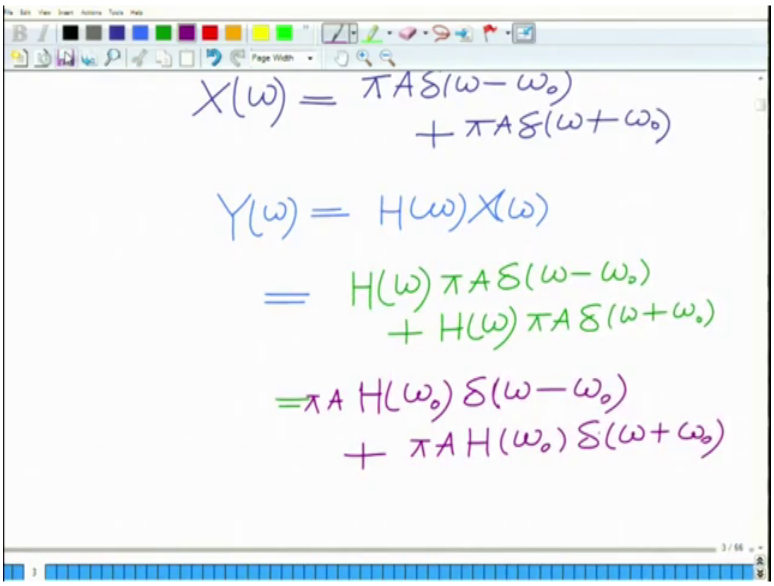

Now, Y of omega equals H of omega into X of omega which is equal to when H of omega into pi A delta omega minus omega naught plus H of omega pi A into delta omega plus omega naught which is equal to well when H of omega into delta omega minus omega naught that is simply you know H of omega naught. You can say pi H of omega naught delta omega minus omega naught plus pi A H of omega naught into delta omega plus omega.

Now, let us consider a real filter all right, filter with real impulse response h of t which implies H of omega naught is h conjugate of minus omega naught oh I am sorry this has to be H of minus omega naught ok. So, considering the real impulse response h t we have H of omega naught is H conjugate of minus omega naught, ok.

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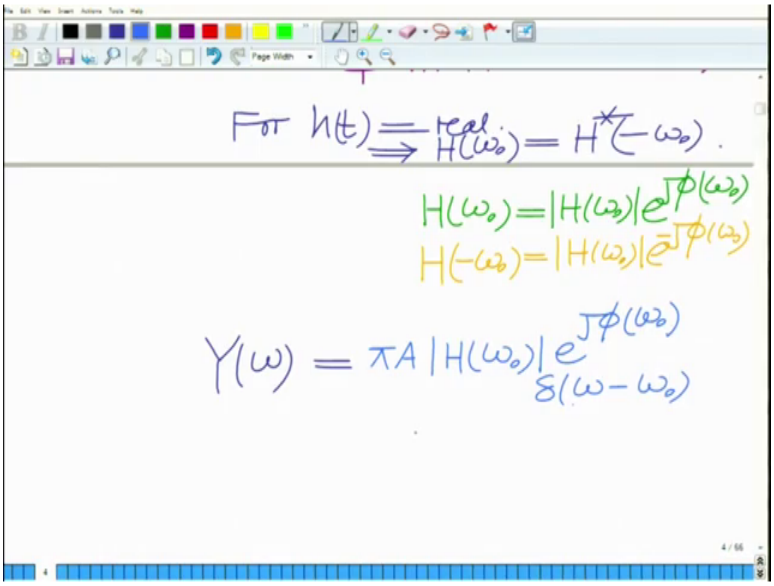

So, we have 4 real h t. When ht equals real this implies H of omega naught is H conjugate of minus omega naught which implies if since H of omega naught is magnitude H of omega naught e raise to J phi omega naught. H conjugate or H of minus omega naught is H conjugate of omega naught which means it has the same magnitude and negative phase e raise to minus J phi omega naught.

So, now, substituting this above this implies your Y of omega equals pi A magnitude H of omega naught e raise to J phi of omega naught that is H of omega naught into delta omega minus omega naught plus pi A into magnitude H of omega naught times e raised to minus J phi omega naught into delta omega plus omega naught.

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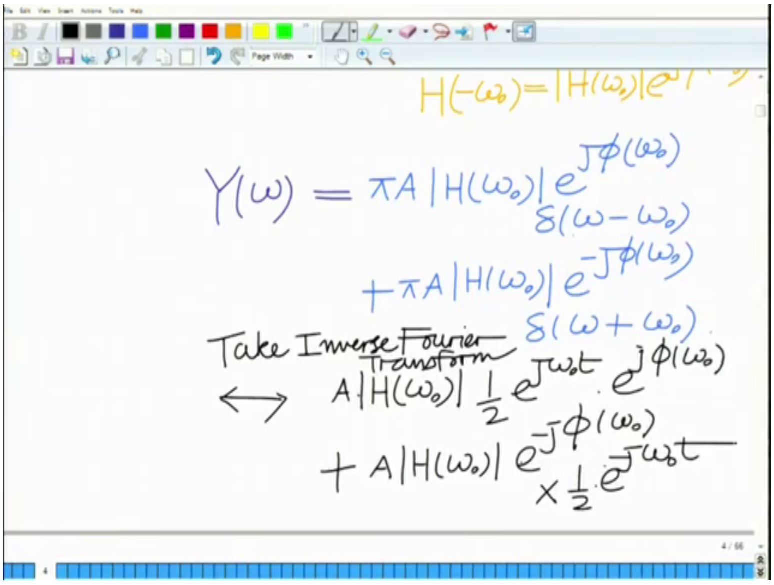

And now, if you look at this response this is interesting it can be simplified as follows this will be pi A take the magnitude of H of omega naught common. If you take the inverse take the inverse Fourier transform and you will observe something interesting.

This pi A magnitude H of omega naught times e raised to J. Now, look at this is delta omega times omega delta omega minus omega naught. So, pi delta omega minus omega naught that will have the inverse Fourier transform half e raise to J omega naught t times e raised to J phi omega naught plus A times magnitude H of omega naught e raise to minus J phi omega naught times pi delta omega plus omega naught that has a inverse Fourier transform half e raise to J or minus J omega naught t.

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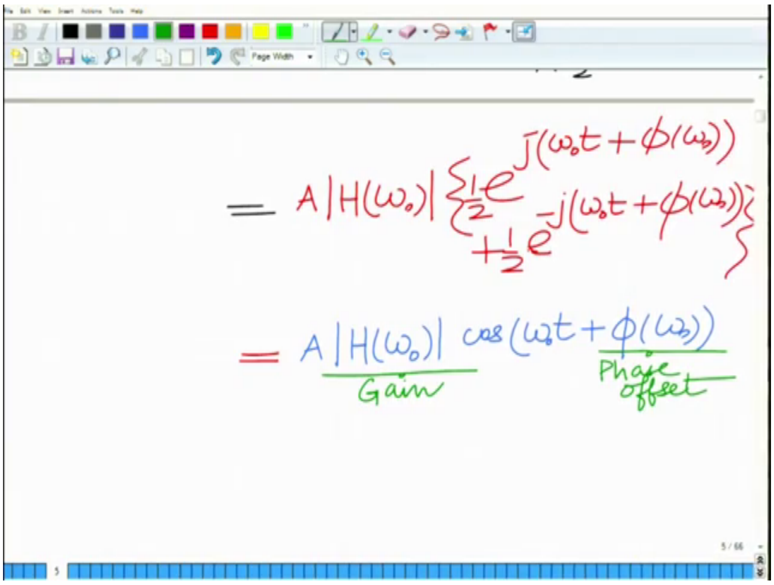

So, as a result this will simply be your this will be your a times magnitude h of omega naught times e raise to J half, e raise to J omega naught t plus pi omega naught plus half e raise to minus J omega naught t plus phi omega naught ok. And therefore, this will be equal to A magnitude H of omega naught times. Now, half e raise to J omega naught t plus phi omega naught plus half e raised to minus J omega naught t plus 5 omega naught is simply cosine omega naught t plus phi omega. So, that I am going to write out here. So, that is cosine omega naught t plus phi omega naught.

And now, therefore, you can see this is simply has a phase difference. So, original carrier is. So, this is your scaling ok. This is your gain you can say this is the gain due to the LTI system and this is the phase offset. So, when you input a pure sinusoid as you know to an LTI system you get output is also pure sinusoid this is the game magnitude H of omega naught and this is the phase offset phi omega naught.

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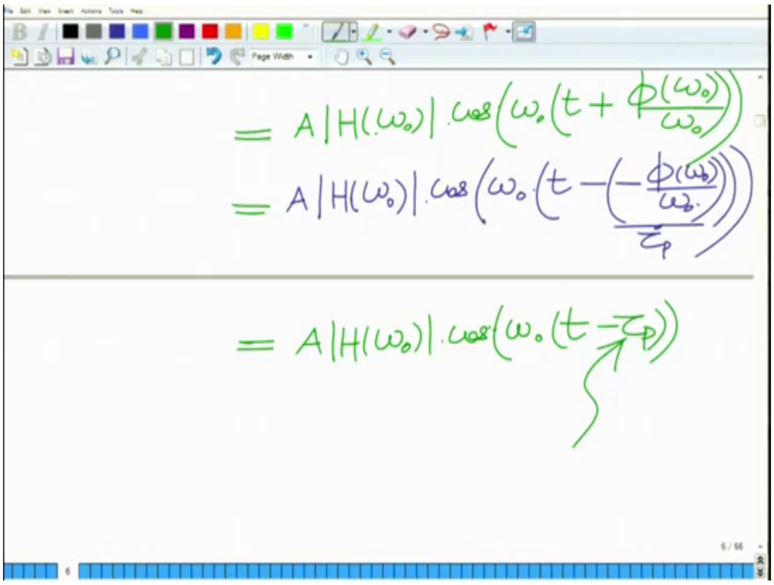

And this I can write as A times magnitude H of omega naught times cosine omega naught t minus or a plus phi of omega naught divided by omega naught which is equal to a magnitude H of omega naught cosine omega naught t minus minus phi of omega naught divided by omega naught and close the brackets. And therefore, now, if you denote this quantity by tau p by the delay tau p this will be a times magnitude H of omega naught cosine omega naught into t minus omega naught into t minus tau p ok.

At this tau p is therefore, a delay. So, what you are observing is you input a pure sinusoid what you have get at the output is another pure sinusoid with a suitable gain that is magnitude H of omega naught and a delay tau p. This tau p is known as the phase delay of the system. So, it means input a pure carrier the delay in the resulting carrier is known as tau p that is the phase delay of the system.

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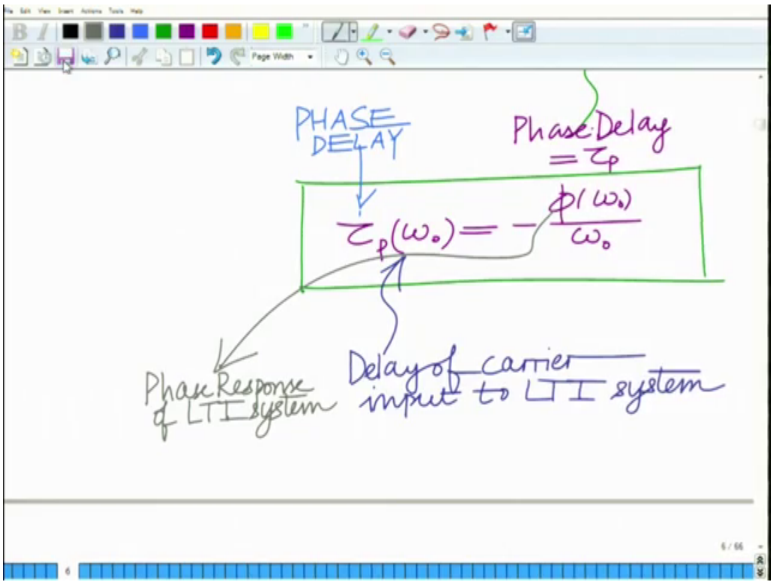

So, this is basically your and the phase delay tau p equals minus pi of and in fact, this phase delay you can see is a function of omega omega naught its function of the frequency. So, this is minus phi of omega naught divided by omega naught ok. So, this is basically the delay of a carrier this is delay of the carrier input to LTI system and this is system as your phase delay ok. This is termed as this is termed as the phase delay of the system ok.

So, the phase delay of the system of an LTI system is basically minus pi omega naught divide. Phase delay at carrier frequency omega naught is minus phi omega naught divided by omega naught where phi omega naught remember this is the frequency this is the phase response of the LTI system ok. So, phi omega naught the other thing to remember is this is the phase response of the LTI system.

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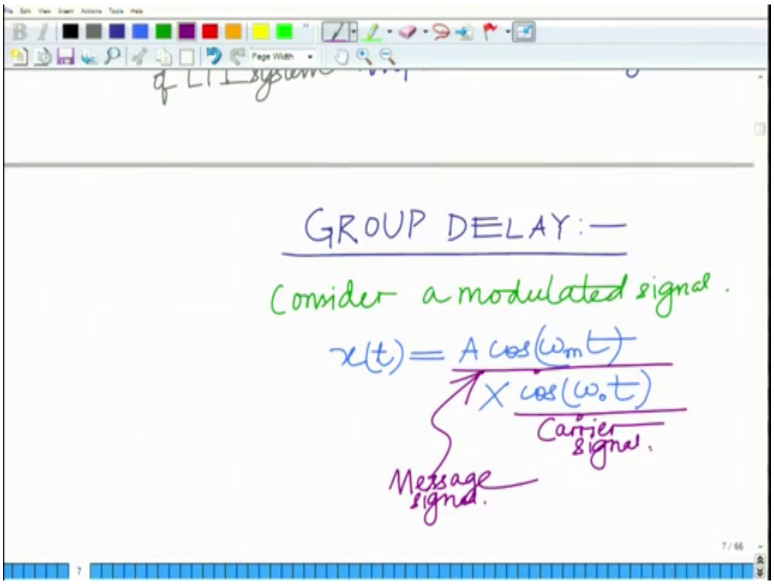

And group delay, now let us now, consider the other aspect that is the group delay. Now, the group delay to illustrate the group delay we considered a modulated signal that is a pure carrier modulated by a message signal. So, consider a modulated signal.

So, to illustrate the group delay basically consider a modulated signal consider a modulated signal we have x of t we have x of t equals A cosine cosine omega m t times cosine. So, previously we simply had a pure sinusoid that is A times cosine omega naught t.

Now, we are also considering a message signal correct. So, this is your message signal and this is in fact, very similar to a communication system in which you modulate a message signal with a carrier frequency. So, you have A cosine omega empty that is a message signal and cosine omega naught t you can think of it as the carrier ok. So, this is basically your carrier, carrier signal or simply the carrier at a very high frequency compared to the.

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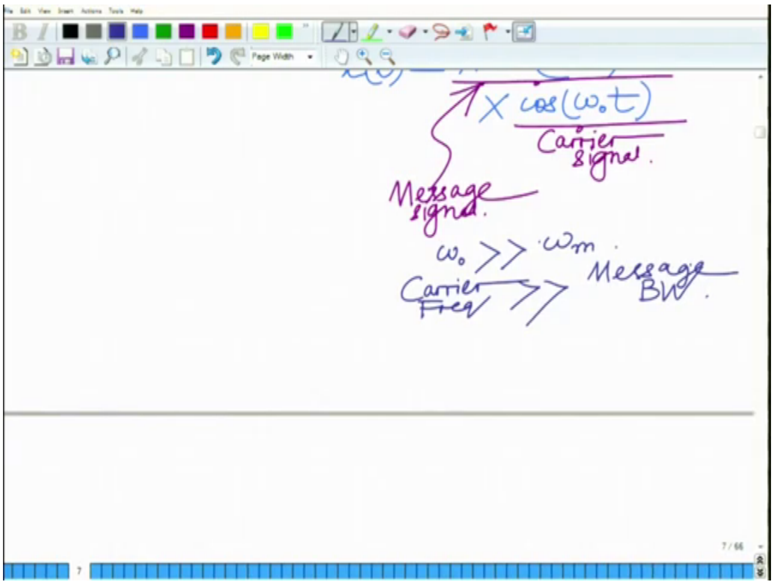

Now, the thing about a communication signal where a modulation is that omega naught is typically much larger than the message frequency the carrier has a much carrier is much higher than the bandwidth of the message signal ok.

So, carrier frequency is significantly larger than the message bandwidth. For instance you can look at a typical am signal which is carrier frequencies of carrier frequencies of 1000s of kilohertz the message bandwidth is typically only of 10 kilohertz multiples of ten kilohertz sorry 10s of kilohertz ok. So, the carrier frequencies typically 100 of times larger than the message the bandwidth of the message signal all right.

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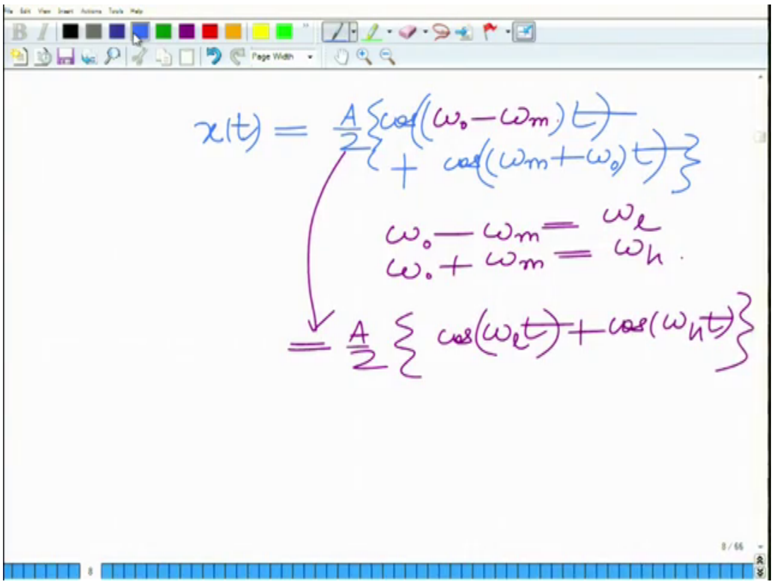

And therefore, now, I can write this signal x t using the properties of trigonometric functions I can write this x t as A by 2 cosine. Well this will be omega m minus omega naught t plus cosine omega m plus omega naught omega m plus omega naught times t ok.

And this you can think of this as this omega m minus omega naught you can think of this as basically the low frequency all right. So, you have two frequency components omega m minus or in fact, you can write this as omega naught minus omega n, does not make a difference because cosine cosine of minus x is cosine of x. So, omega naught minus omega m equals the low frequency omega l omega naught plus omega m is the higher frequency component equals omega h. So, this is basically you are A by 2. So, what we have over here is A by 2 cosine omega l t plus cosine omega h t ok.

So, basically now, what you have because you are modulating a sinusoidal omega m t with much smaller bandwidth with a carrier that has a much higher bandwidth that gives you two resultant sensors, one is at the lower frequency omega l which is omega naught minus omega m and the other which is a higher frequency omega naught plus omega m ok.

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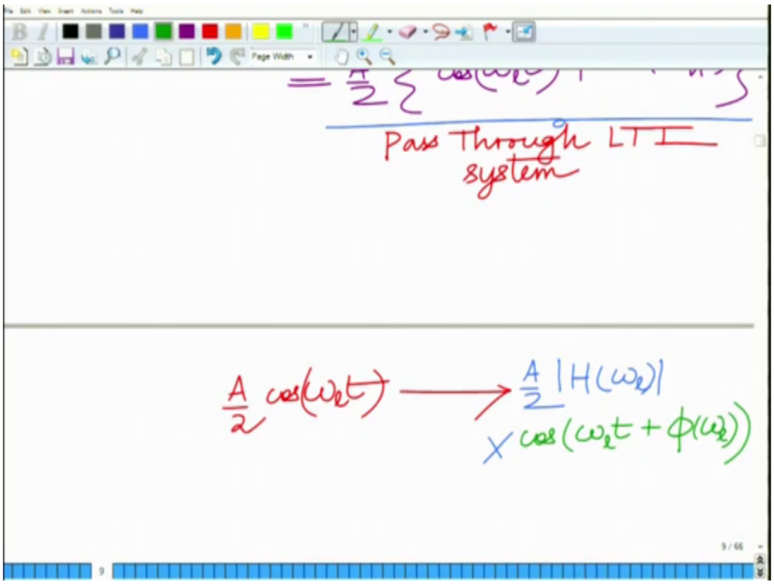

And now, we pass this through the LTI system, pass through the LTI system. Now, you pass this through the LTI system. Now, the output corresponding to a remember we have already seen this, so all we have here is previously we have omega naught previously we had A cosine omega naught t and we are passing it to the LTI system. Now, we simply have a A by 2 cosine omega l t all right, a pure sinusoid at a different frequency omega l.

And therefore, the output corresponding to this pure sinusoid of the LTI system can be easily written as the output will be A by 2 as we have seen before magnitude H of omega l times cosine into cosine of omega l t plus phi of omega l, where phi omega l as you know is the phase response of the phi of omega is the phase response of the LTI system all right. This we have this follows from what we have already seen before ok.

Now, similarly the output corresponding to well the output corresponding to the higher frequency component; similarly if you look at A by 2 cosine omega h t.

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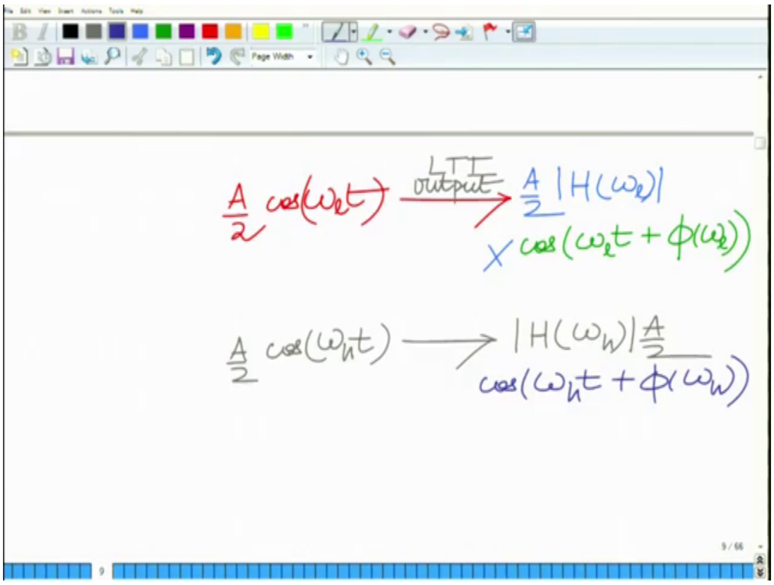

So, the LTI output the LTI output correspond to A, A by 2 cosine omega ht is magnitude h of omega h into A by 2 times cosine omega h t plus phi of omega h.

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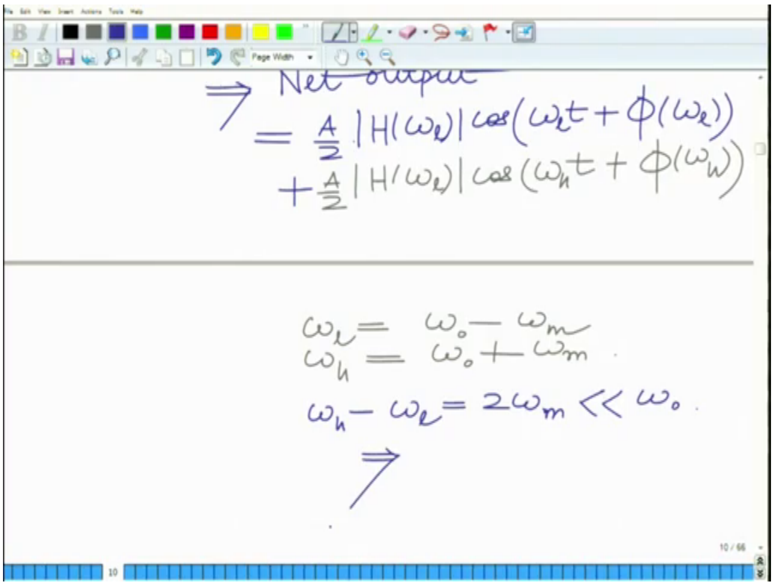

Now, the net output of the LTI system will be the sum of these two outputs implies. Net output equals A by 2 well that will be magnitude of H omega l cosine, well omega l t plus phi omega l plus magnitude A by 2 for cosine omega h t plus phi omega h ok. So, this is what we obtain.

Now, notice that omega l equals omega naught minus omega m and omega h equals omega naught plus omega m. And further what we have is omega m is much smaller than omega naught. So, the difference between omega l and omega h you can see 2 omega naught. So, omega l and omega h are separated by 2 omega naught, omega l and omega h are separated by 2 omega m, but omega m is a very small quantity compared to the frequency. Therefore, the a frequency response that responds them amplitude the magnitude response of the LTI system magnitude H of omega can be assumed to be can be assumed to be constant over this interval that is omega l omega h.

Meaning to say that is you have omega l minus omega h, omega h minus omega l is 2 omega m which is much smaller than omega naught which implies that, you can assume that magnitude H of omega l is approximately equal to magnitude H of omega magnitude H of omega h approximately equal to magnitude H of omega naught.

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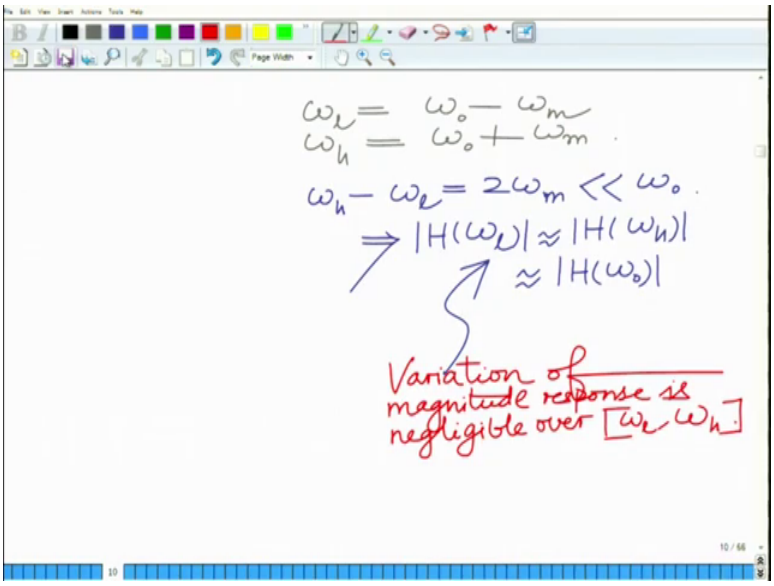

So, one can assume that the magnitude response correct. So, one can assume that the magnitude response is the variation of the magnitude response of the LTI system is basically negligible over this small frequency interval that is from omega l 2 omega h ok, all right. So, this is a, that is basically variation what we are assuming is the variation of the magnitude response, this is negligible over minus is negligible over the interval omega l to omega h ok. So, this is negligible over the interval omega l to omega h, all right.

So, basically what we have seen in this module is we have looked at the phase delay and also started our discussion on the group delay. We will complete this derivation all right then developing this framework in the subsequent module.

Thank you very much.