Principles of Signals and Systems
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Lecture – 68
Example Problems: DFT – DFT, IDFT in Matrix form

Hello. Welcome to another module in this massive open online course. So, we are looking at example problems in the discrete Fourier transform and a inverse discrete Fourier transform, alright.

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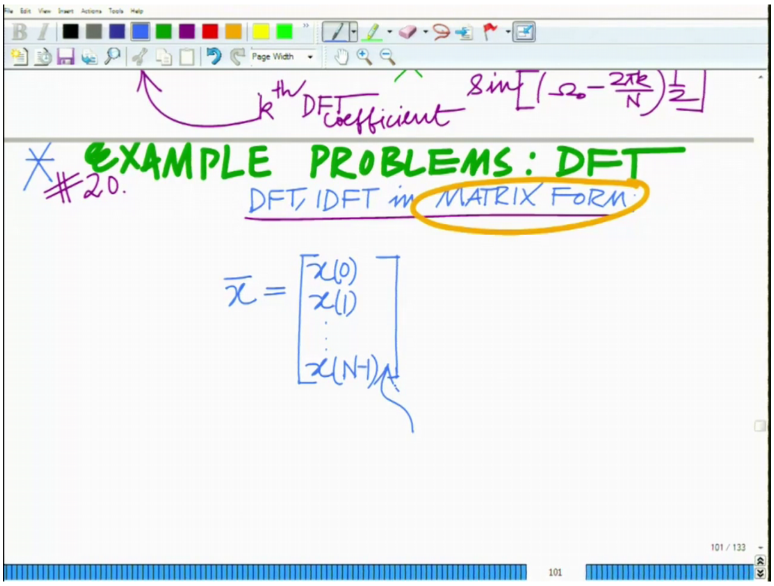

So, we are looking at example problems, for the discrete Fourier transform. In particular, what we would like to explore is? we would like to write, this DFT the discrete Fourier transform and the IDFT in a matrix form, that will be in particular a very convenient for evaluating the DFT and IDFT, as well as representing them as a linear transformations, all right.

So, what we want to do, is given, x bar, the finite length signal, that is x 0, x 1, up to x N minus 1. And, well you can see this is the N samples. Correct?

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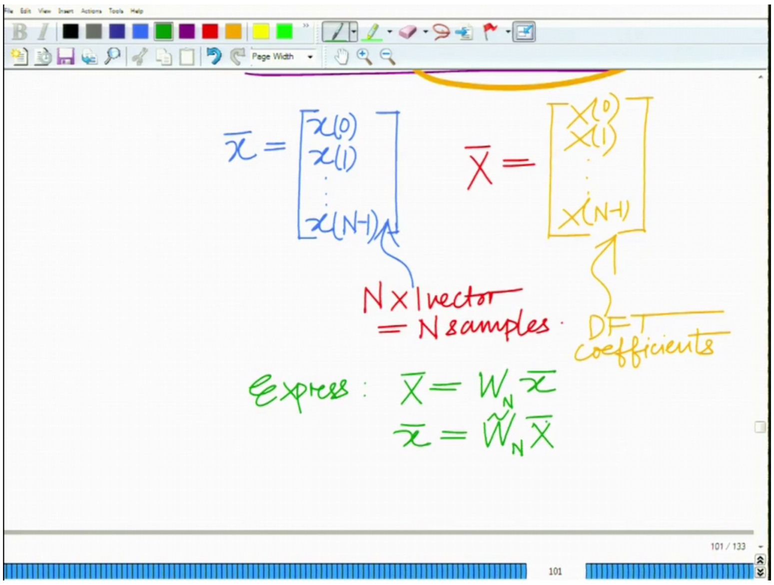

So, these are n cross 1 vector corresponds to N samples, ok N length signal. And, we also have correspondingly x bar, which is the vector of DFT coefficients. That is we have, naturally, N DFT coefficients x 0, x 1, x N minus 1 this is your vector of DFT coefficients. And, what we want to do is, we want to Express the DFT x bar equals vector W, in fact, N corresponding to the N length DFT x bar and also, x bar the signal, as the inverse DFT of length N times X bar.

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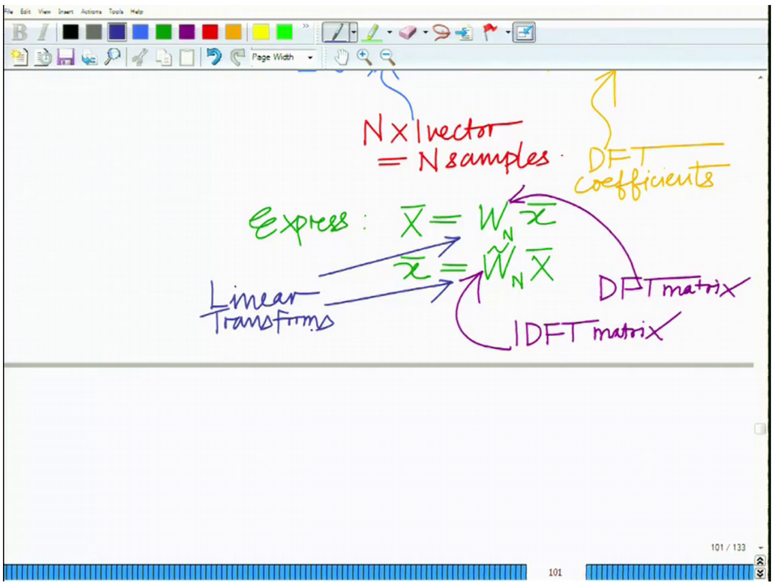

So, this expresses these things, as a transformation. So, these are the matrix. This is called the DFT matrix, and this is the IDFT matrix; the inverse transform. So, we are representing the DFT and IDFT as linear transformations, ok. So, these are basically your, Linear representing the DFT and IDFT as a linear transformation.

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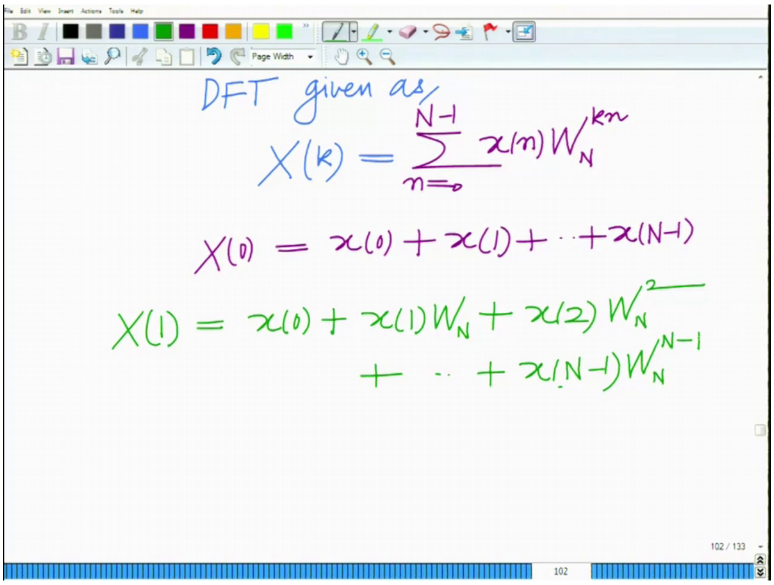

Now, the DFT realized; the DFT is given as, well, the n point DFT is summation n equal to 0 to N minus 1 x n into WN raised to the power k n. So for instance you have your X 0 which is, x 0 plus x 1 plus so on plus x N minus 1 into x N minus 1 that is it WN raised to k is 0, so this will be 1.

And similarly, if you look at X 1, ok. X 1 will be again x 0 plus x 1 times WN k equals 1 N equals 1, so plus WN plus x 2 WN; well, k equals 1 N equals 2 so WN square plus x N minus 1 WN raised to N minus 1.

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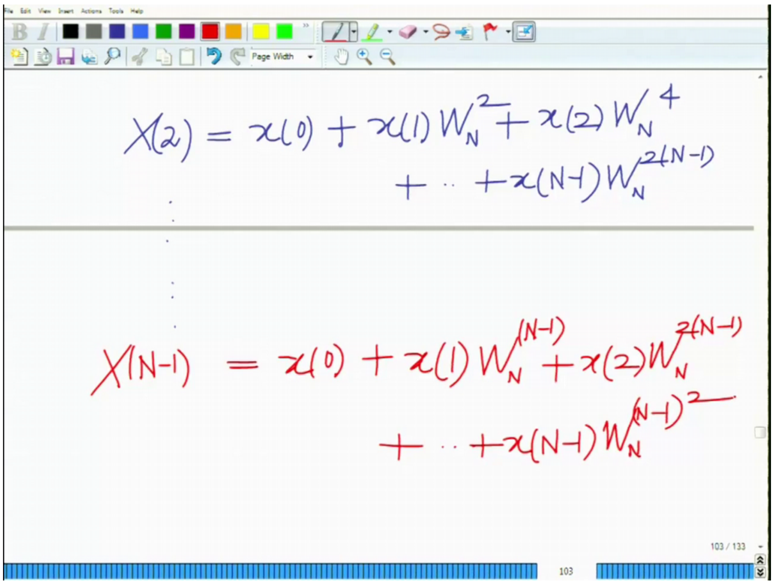

This is X 1. X 2 will be you can see x 0 plus x 1 k equals 2, so WN square plus 2 k equals 2 N equals 2, so WN 4 plus x N minus 1 WN raised to 2 N minus 1 and so on. Correct? And so on, what we are trying to do is we are trying to develop a linear transformation that is a matrix representation for this DFT.

And, if you look at x of N minus 1, that is the Nth DFT point that is going to be x 0 plus x 1 k equals N minus 1, so WN k minus 1 plus x 2 I am sorry x WN N minus 1 x 2 WN twice of N minus 1 plus x N minus 1 k equals N minus 1 N equals N minus 1, so that will be WN raised to N minus 1 square.

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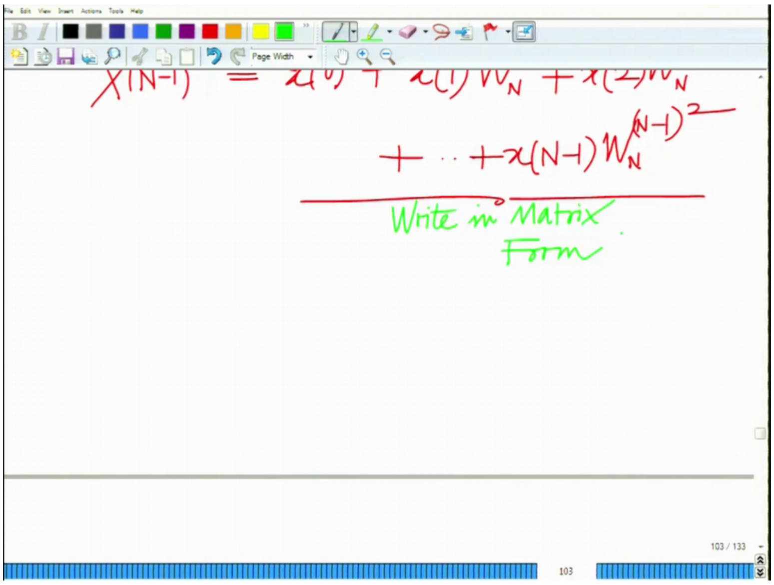

And you put all these things together, as a matrix, so you write a Matrix Form.

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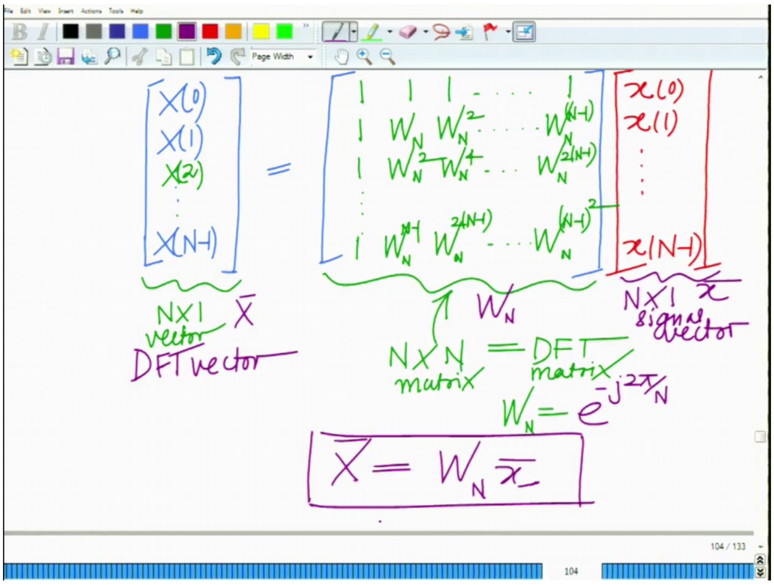

And what you are going to have is. You are going to have on the left you have this n dimensional vector of DFT coefficients X 0 X 1 up to capital X 0; these are the capital X's, ok. These are the n dimensional DFT coefficients, this is this matrix. Correct?

1 1 the first will be all 1's 1 times X 0. So, you look at this into what is multiplying each element of the signal. You look at what is multiplying each component of the signal. The first entries will all be 1; 1 times x 0 plus 1 times x 1 so on up to 1 times x 2 plus so on;

so on 1 times x N minus 1 x 1 will be 1 times x 0 plus W N times x 1 plus W N square times x 2 plus W N raised to N minus 1 times x N minus 1. X 2 will be 1 times x 0 plus W N square times x 2 plus W a W N square times x 1 plus W N 4 times x 2 plus W N raised to 2 N minus 1 times X N minus 1. And the last entry, last row will be X N minus 1 will be W N raised to N minus 1 times x 1 W N raised to 2 N minus 1 times x 2 and W N raised to N minus 1 square times N minus 1 square times X N minus 1.

And therefore, if you look at this; this is an N cross 1 vector and if you look at this; this is an N cross N matrix and this is your DFT matrix of size N. Where recall, W N equals e raised to minus j 2 pi over N And this is your N cross 1. This is your N cross 1 signal vector and this capital X bar this is the n cross one DFT vector. So, this is your X bar, this is your W N and this is your X bar. So, we are able to write; we are able to write X bar equals W N times X bar. Ok? Where, W N is basically your DFT matrix. Ok? So, that is basically, the structure of the DFT matrix.

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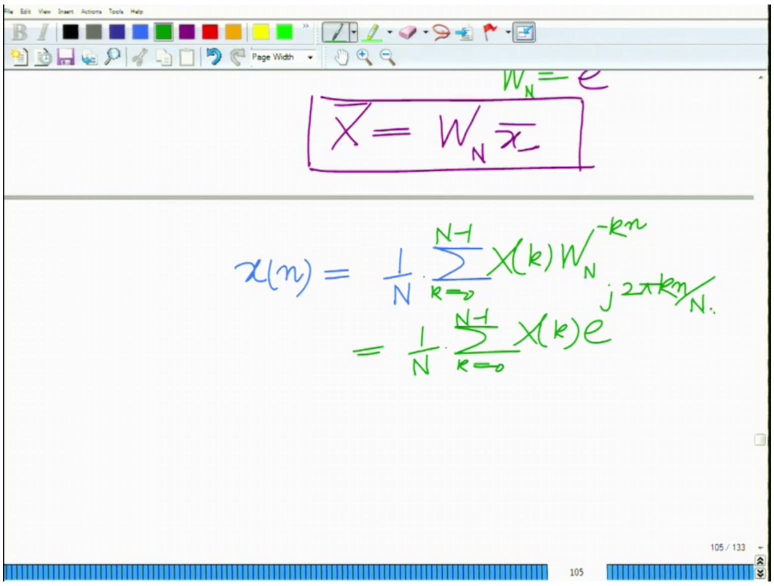

Now, you can do similarly for the IDFT matrix. Now realize, that the inverse discrete Fourier transform is also, is given as; if you look at each coefficient x of k, if you look at each coefficient, we have the expression for the IDFT, x n or I am sorry x small n equals 1 over capital N summation k equal to 0 to N minus 1 X k W N raised to minus k n, ok, e raised to j 2 pi k small n or capital N. Ok, W N raised to minus k which is also as I said, this is 1 over N summation k equals 0 to N minus 1 X k e raised to j 2 pi k small n over capital N. Ok.

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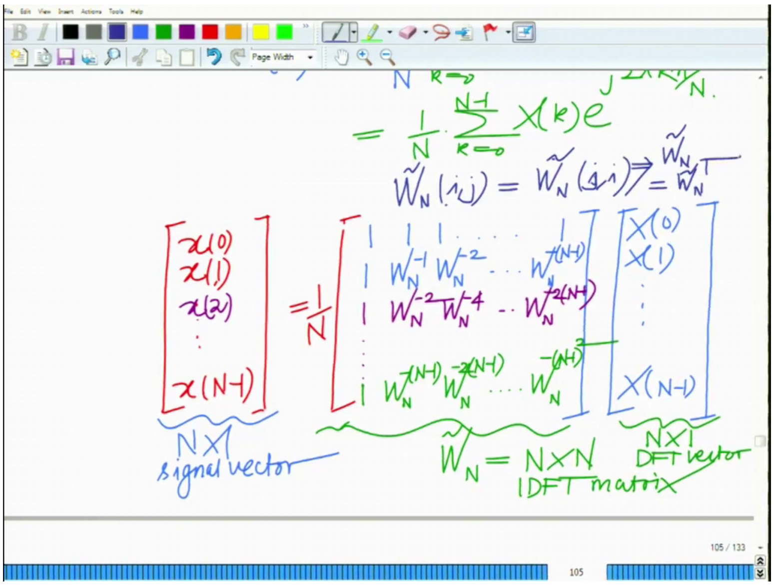

And now you can again write this as the following matrix. Now, the outputs will be the signal samples, the inverse DFT, and the corresponding IDFT matrix. First you have this scaling factor 1 over N, that is there and, the input will be the DFT coefficients x 0 x 1 up to x N minus 1. And the IDFT matrix will be, 1 1 1 1 that is small x 0 equals capital X 0 plus capital X 1 plus capital X minus 1 divided by capital N. Small x 1 is 1 plus W N to the power minus 1 into capital X 1 W N to the power minus 2 into capital X 2 plus WN to the power minus N minus 1 into capital X N minus 1.

Similarly, x 2 is and overall divided by 1 of divided by capital N, ok. And capital X 2 or small x 2 will be 1 over N times 1 capital X 0 W N minus 2 capital X 1 WN minus 4 capital X 2 W N minus 2 N minus 1 capital X N minus 1 and finally, X N minus 1 will be 1 over N times 1 capital X 0 W N minus N minus 1 capital X 1 W N minus 2 N minus 1 capital X 2 plus and so on. W N minus N minus 1 square times capital X N minus 1.

And, if you look at this, now this is in fact, this including the scaling factor 1 over N, this is W N tilda. This is your N cross N IDFT matrix. Ok? And, this is your N cross 1 DFT coefficient vector X bar. Ok, the input is the DFT vector and output is the N cross 1 signal vector.

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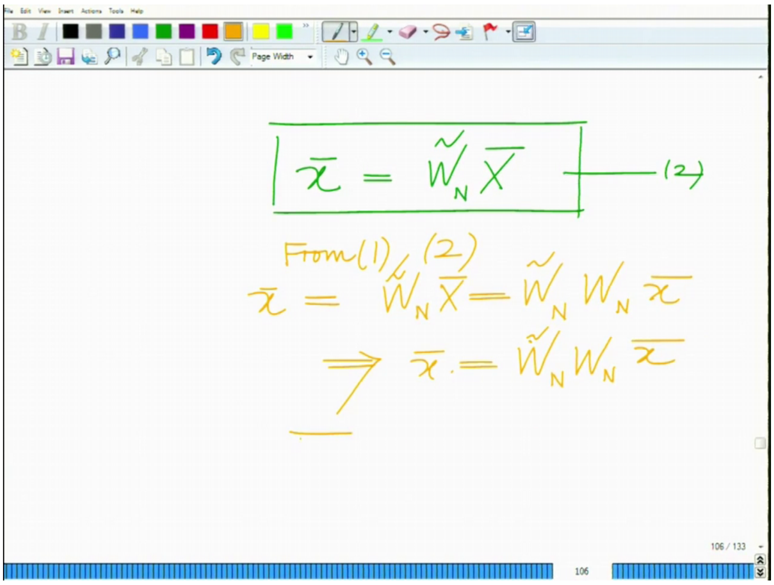

And you can see therefore, that I can write this x bar equals W N tilda times x bar and if you call this equation 2 and you call this equation 1, it is but natural to see that, you have x bar equals now, from 1 and 2 equals W N tilde into X bar equals now substitute for X bar capital X bar from 1, that is WN tilde into WN into small x bar.

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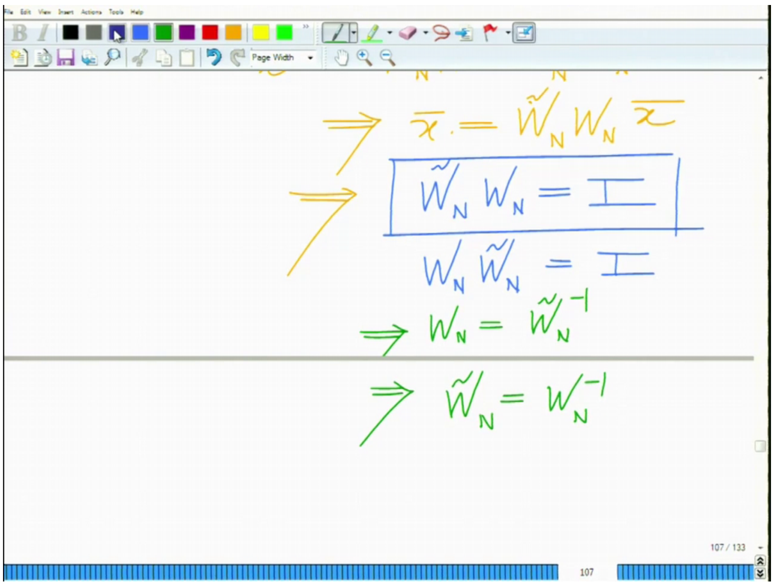

So, we have x bar equals W N tilde into W N into x bar. This implies that W N tilde into W N equals identity must be equal to identity, right? Correct? You have x bar equals, so you have the matrix, x bar the vector remaining unchanged x bar equals a x bar and that happens for every vector x bar, ok. Remember it is not specific to a certain vector x bar, happens for every vector x bar which means; that the matrix a must be identity. All right, which means, W N tilde is the inverse of W N. Similarly, W N is the inverse of W N tilde and that is what? And that is natural. Correct? Because, IDFT is the inverse of the DFT, the DFT is the inverse of the IDFT. Ok.

So therefore, W N tilde W N equals identity similar. Similarly, W N into W N tilde equals identity, the inverse of a square matrix when exists is unique and is both; that is if a b is identity for square matrices, then b a is also identity, ok. So, W N tilde W N W N tilde equals identity. The DFT properties DFT and IDFT properties matrices satisfy this property. Ok.

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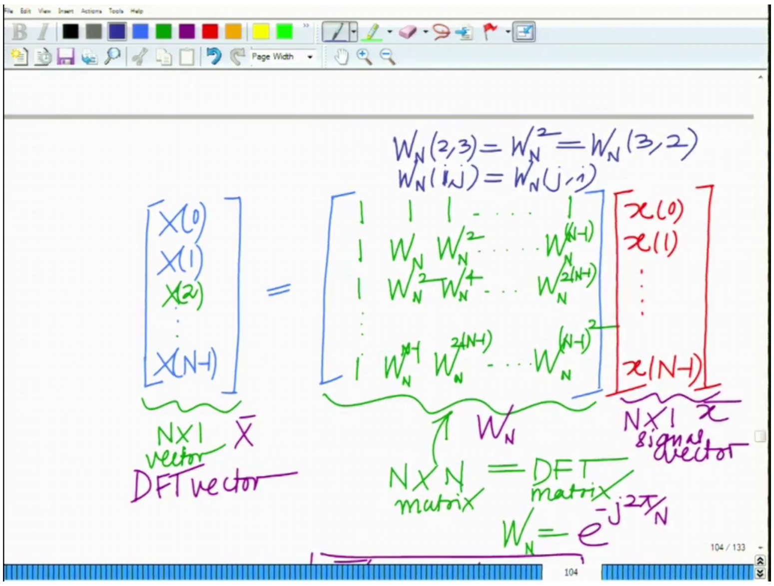

So, this implies; W N equals WN tilde inverse and this also implies; W N tilde equals WN inverse. Ok. So, DFT and IDFT matrixes satisfy these properties. And, further observe something very interesting if you observe this, you can see that, if you look at the elements, look at this; this is, if you look at the DFT matrix, this is the element, if you look at W N, the element in the second row, third column is W N square, which is also the element in the third row, second column. Look at this that is W N square. Correct?

Similarly, you can compare that WN of I comma j equals W N of j comma I for all j you can see that. This means; that if you take the transpose of this matrix W N it is equal to itself. Correct? Because, W N i comma j is equal to that is; W N I comma j equals W N j comma I. If you make the rows into columns and the columns into rows then that matrix remains unchanged, which means; that the DFT matrix is equal to the transpose of itself.

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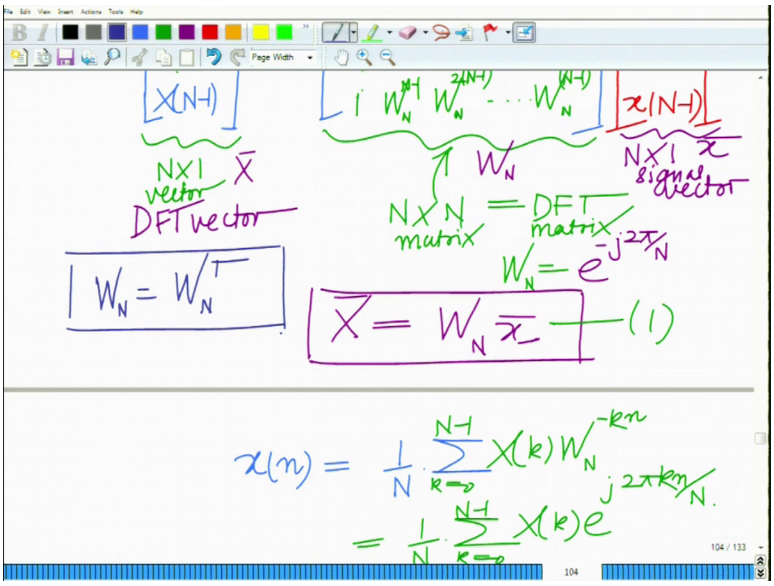

So, there is an interesting property W N equals W N transpose. Is an important property so observe that; so property, this is property 1, let us call this property 1. The inverse property and property 2 equals is basically W N equals W N transpose and similarly you can also observe, if you look at the IDFT matrix once again, you can see that W N tilde, for instance we let W N tilde of 2 comma 3 second row third column that is W N raised to minus 2 divided by N and W N tilde of 3 comma 2 is W N raised to minus 2 divided by. So, we again have W N tilde of i comma j equals W N tilde of j comma i. This implies, W N tilde equals W N tilde transpose.

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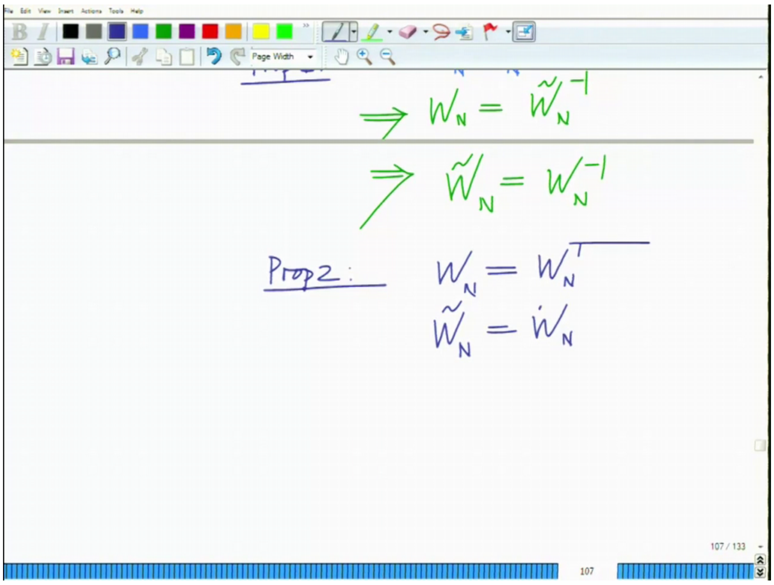

So, we have W N tilde equals W N tilde transpose. And now, we come to another very interesting property. If you look at this, each element, now you look at W N of 2 comma 3, ok, that is W N square. All right, that is e raised to j that is; W N square that is; e raised to j 2 pi into 2 or that is; e raised to j 4 pi over N. Now, if you look at W N tilde of 2 comma 3 that is; W N raised to minus 2 divided by N that is; e raised to e raised to j 4 pi over N whole divided by N.

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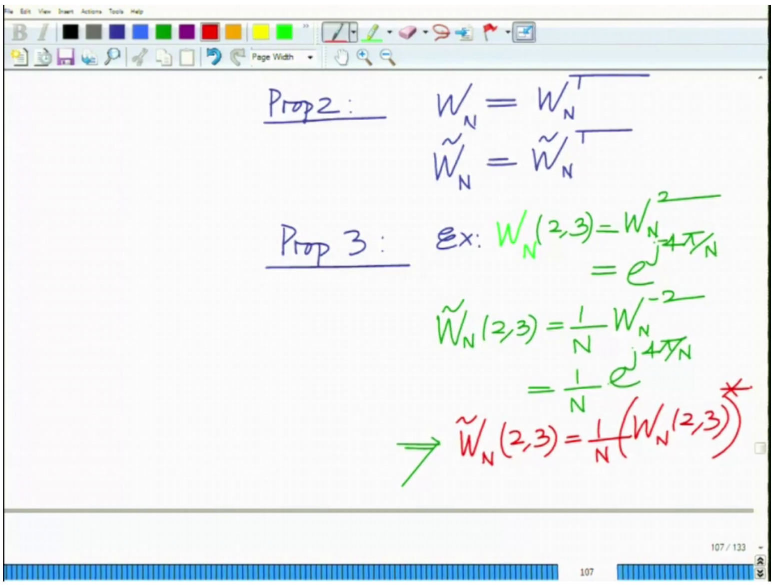

So, you can see that these elements are complex conjugates of each other. So, if you look at this example and these are interesting properties, so what we have is? W N of 2 comma 3 equals W N this is the small w n, this is the matrix W N, this is W N raised to 2 equals e raised to j 2 pi over N. 2 pi over N into 2 that is; e raised to j 4 pi over N, I am sorry e raised to minus j 4 pi over N.

Now, W N tilde, the matrix WN tilde of 2 comma 3 equals 1 over N W N raised to minus 2 equals 1 over N e raised to j 4 pi over [noise] which implies; W N tilde 2 comma 3 equals 1 over N. W N 2 comma 3 conjugate and that is the interesting property. So, each element of the IDFT matrix is a corresponding conjugate of the element of the DFT matrix divided by capital N.

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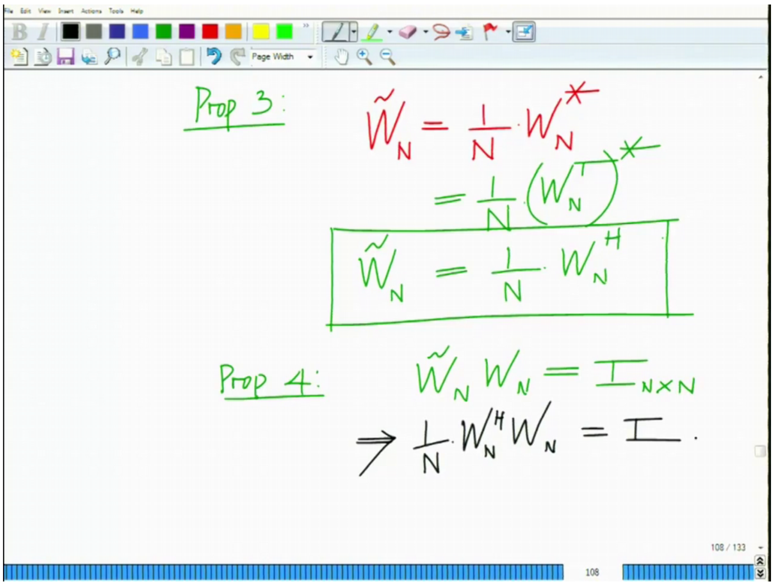

So, you can say, basically the entire matrix W N tilde equals 1 over N, W N conjugate and this is the interesting property and since, W N conjugal W N is W N Hermitian, so the you can also say, this is also 1 over N W N transpose conjugate, which is the same as 1 over N W N Hermitian.

Remember, if you take the transpose and the conjugate that is; we take the conjugate that is the conjugate of each element transpose rows become columns, columns become rows. We can transpose and conjugate of each element that becomes the Hermitian. So, what you have is? W N tilde is 1 over N times W N Hermitian. Ok?

And therefore, since W N tilde, so this is your property you can collectively call this as property 3 and therefore, now you come to property 4, which is basically, W N tilde into WN equals identity N cross N identity. This implies that; now you see W N tilde is 1 over N W N Hermitian. Which means; 1 over N W N Hermitian, WN equals identity, which implies; W N Hermitian into W N equals N times identity matrix. Ok.

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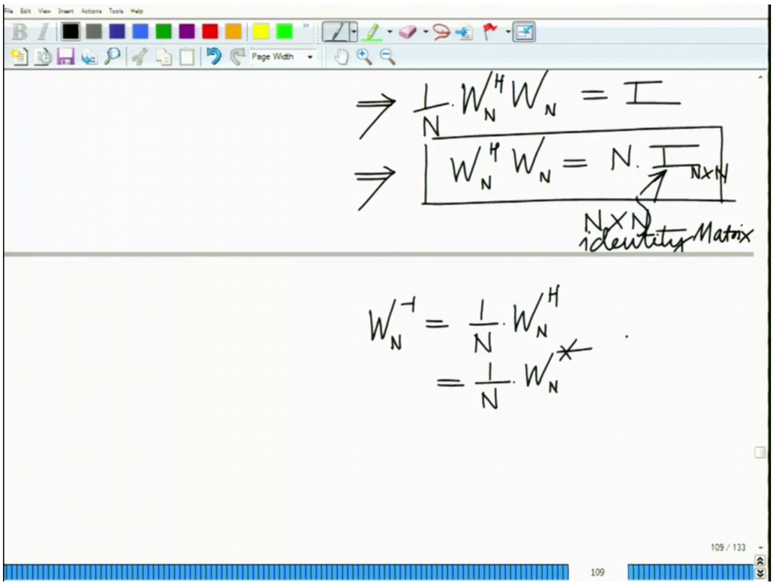

And this is your property number 4 that is; W N Hermitian into W N is N times identity matrix. Ok. This is an N cross N identity matrix. Ok. This is an N cross N identity matrix. Ok. So, basically you can also say, W N inverse equals 1 over N W N Hermitian, which is also basically 1 over N W N conjugate because, W N is W N the transpose of itself. So, this is 1 over N WN conjugate. There is also another way to say the same relation.

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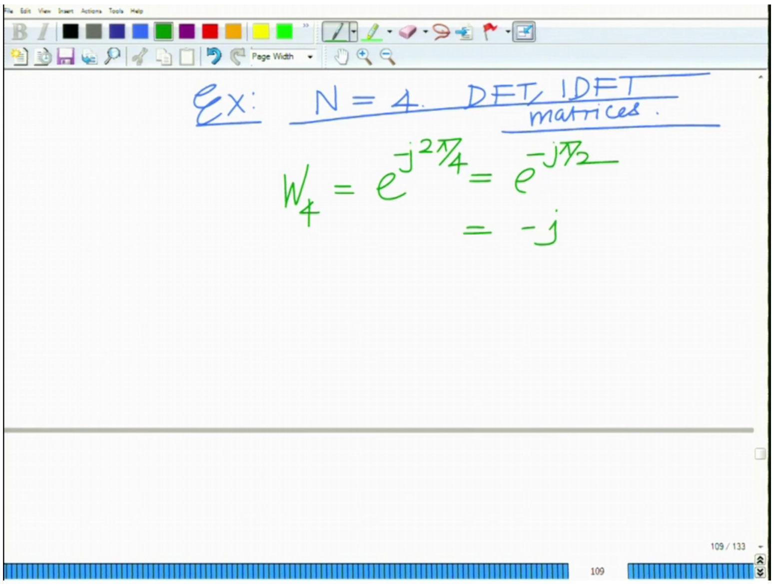

Now, what we want to do is? We want to write, what the DFT and IDFT matrices look. N equal to 4 DFT comma IDFT matrices that is; to get a better idea. What we want to do is? We want to write do a simple example, where we write the DFT and IDFT matrices for N is equal to 4. That is the length of sequence n equal to 4.

Now, realize that when N equal to 4, we have the W 4 which is equal to e raised to j 2 pi over e raised to minus j 2 pi over N e raised to minus j 2 power 4 that is e raised to minus j pi over 2 that is minus j, ok.

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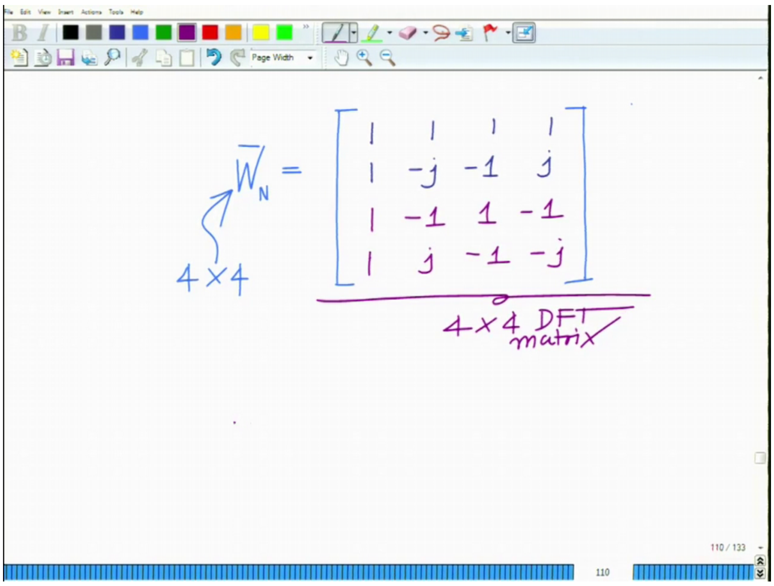

And now, if you look at this; if you look at W N, so the matrix just to distinguish this matrix I am going to write it as the W bar N, to distinguish that this is a matrix because, W N we are using it for the coefficient. This will be first, this will be N cross N, this will be 4 cross 4 matrix. So this will be a 4 cross 4 matrix and the elements are going to be, first elements are all ones, as we have seen. 1 this is W N, raised to 1 that is minus j W N raised to the second 2 cross 2 element will be W N square that is; minus j square that is; minus 1 third element will be W N cube that is; minus j cube minus j cube. You can see that will be minus 1 into minus j, so that will be j.

Similarly, this element a third row will be, 1 again, remember this is transpose symmetric, so I can just write it without looking at it. So, this is going to be minus 1, this element will be W N raised to 4. So, this will be minus j raised to 4, though that will be 1 and again, this element will be W N raised to, I think, if I am not mistaken, W N this is k equal to 2 N equal to this k equal to 2 N equal to 3. So, W N raised to 6 that will be minus 1, ok. And the last row, again you can write it by transpose symmetry, this element will be j because, remember this is the 4 comma 2 element, that will be same as 2 comma 4 element. This element is 4 comma 3, that will be the same as 3 comma 4 and this element, last element is W N raised to N minus 1 square.

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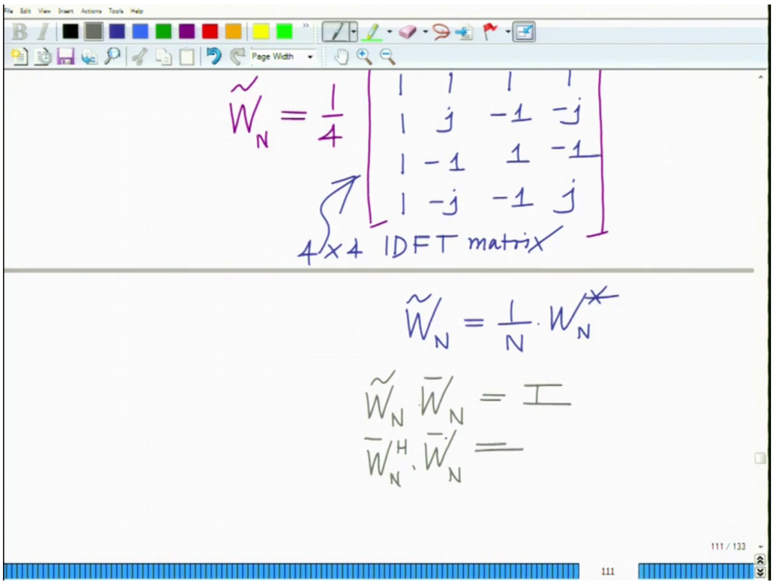

So, that will be W N raised to the power of 9. So, you can see that will be minus j. You can check this, ok. So, this is basically, your 4 cross 4 DFT matrix. Now the IDFT matrix, in fact; here N is equal to 4. Let me call this W N tilde W tilde you can write this as 1 over 4 and remember to do the IDFT matrix, I don't need to go through the entire computation once again. I can just consider the conjugate of the DFT matrix that gives me the IDFT matrix, ok. So, that is very simple. So, the IDFT matrix is indeed. Once you have the DFT matrix, the IDFT matrix itself is very simple. So, that will be 1 1 1 1 second row take the conjugate of each element, minus j conjugate is j minus 1 conjugate is minus 1 j conjugate is minus j, third row will be 1 minus 1 conjugate is minus 1 1 conjugate is 1 minus 1 conjugate is minus 1.

Last row will be 1 j conjugate will be minus j minus 1 conjugate is minus 1 and minus j conjugate is j. And this will be your 4 cross 4 IDFT matrix. And where we have use the property, W N tilde equals 1 over N W N conjugate. In this case, specifically N is equal to N is equal to 4. And now, you can also verify that and this is something that you can do by yourself, you can enter it into MATLAB and you can verify that W N tilde into W N is, in fact; identity and you can also check that W N Hermitian into W N or W bar N rather. These are all W bar and these are the matrices, W bar N Hermitian into W bar N.

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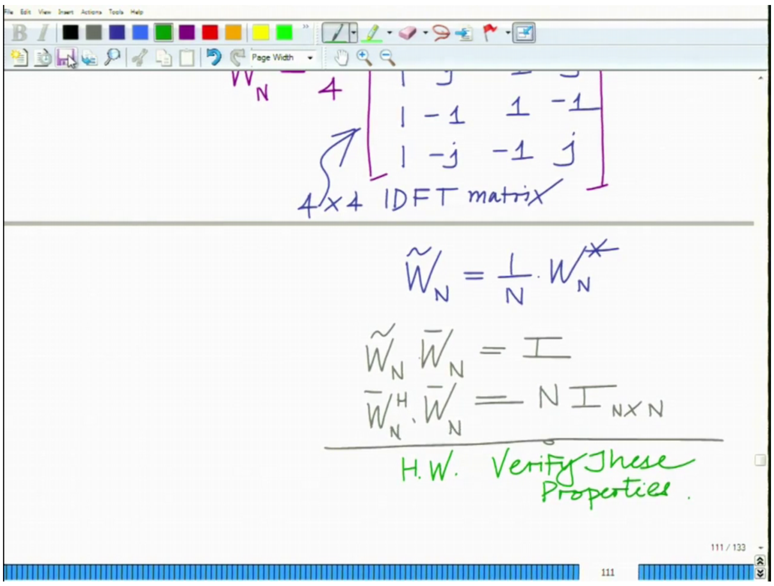

Remember, we said W bar W bar and Hermitian W bar N this is equal to N times identity. So, you can check these things, you can verify this N times N cross N identity matrix. So, you can verify these properties. So, you can do this as homework, you can enter this into MATLAB. MATLAB uses a very convenient interface to enter matrices, so you can quickly enter these matrices and verify this property.

So, this is; this in this module we have dealt with an important aspect that is, representing this IDFT and DFT operations as matrices. And, this is a very important I would like to say tool, because it allows us to represent these operations as a linear transforms. And therefore, look at these as matrices and use these tools to carry out these operations in analysis, right?

So, representing the signals and outputs as vectors and representing these DFT and ID IDFT operations as matrices and that helps us significantly simplify a analysis and represent it in a much more compact and sort of intuitive fashion, alright?

So, we will stop here and continue in the subsequent modules.

Thank you very much.