Principles of Signals and Systems
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Lecture – 58
Discrete Time Fourier Transform: Properties of DTFT - Differentiation in Frequency, Difference in Time, Convolution, Multiplication, Parseval's Relation

Hello welcome to another module in this massive open online course. So, we are looking at the Fourier analysis for discrete time a, periodic signals through the DTFT the discrete time Fourier transform alright. And we are looking at a, at the properties of the DTFT.

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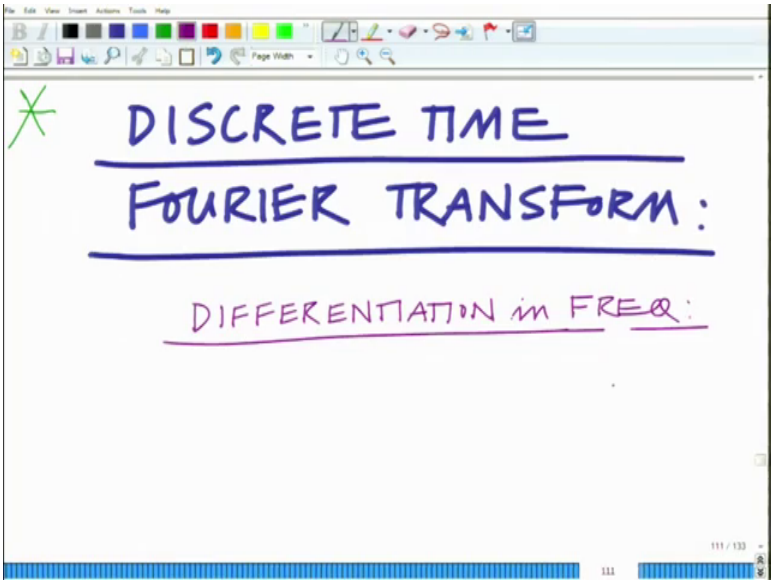

So, let us continue our discussion on that so we are looking at the discrete time Fourier transform. And the next property so we have looked at several properties, we have looked at the duality, the next property that we want to look at is differentiation in frequency of course, in time it is a discrete signal. So, we cannot talk about differentiation in time so this is differentiation in frequency what happens when you differentiate in frequency.

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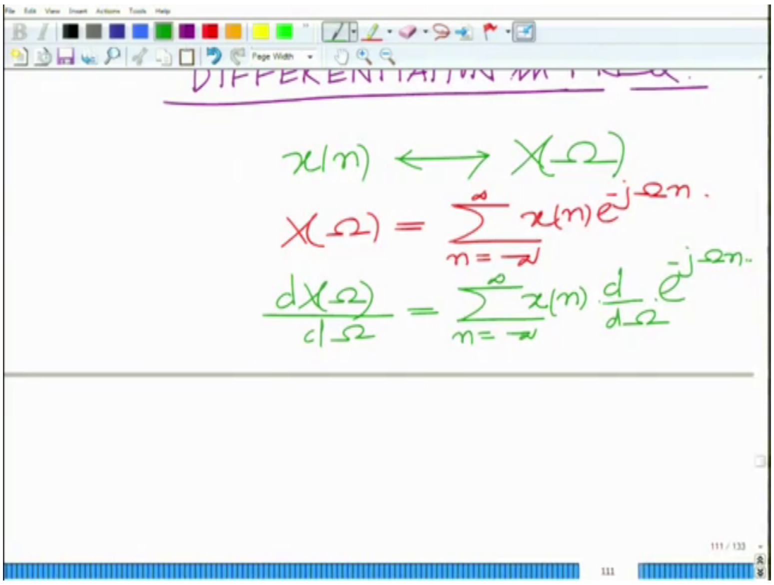

So, we have x of n let us say as the DTFT, capital X of omega alright which means basically your X of omega equals summation n equals minus infinity to infinity x n e raised to minus j omega n. Then if you differentiate this d x omega or d omega that is n equals minus infinity to infinity if you take the differentiation sign inside x n is a constant that is d of d or different derivative d or d omega of e raised to minus j omega n with respect to omega.

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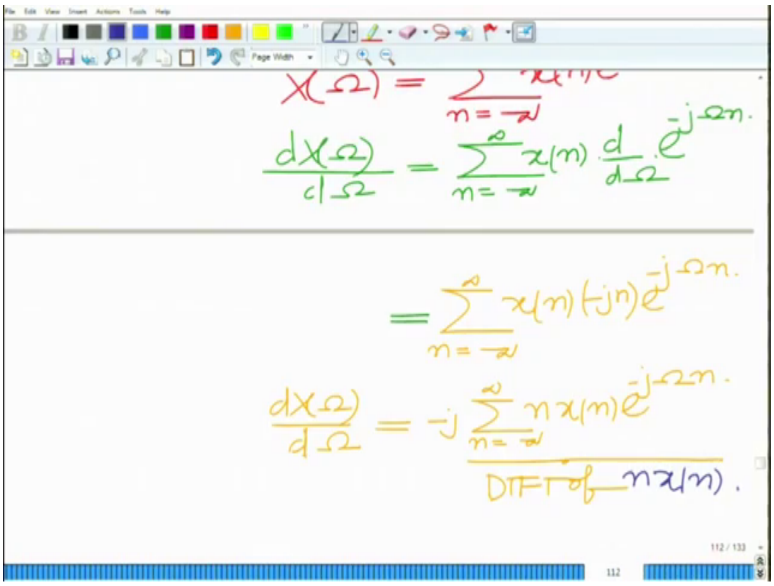

So, or let me just write it that is d over d omega e raised to minus j omega n which is summation n equals minus infinity to infinity x n minus j n e raised to minus j omega n which is minus j summation. So, d x omega over d omega that is n equals minus infinity to infinity minus j summation n equals minus infinity to infinity n x n e raise to minus j omega n. So, this is basically if you look at it this is basically the DTFT of n x n of the signal n x n ok.

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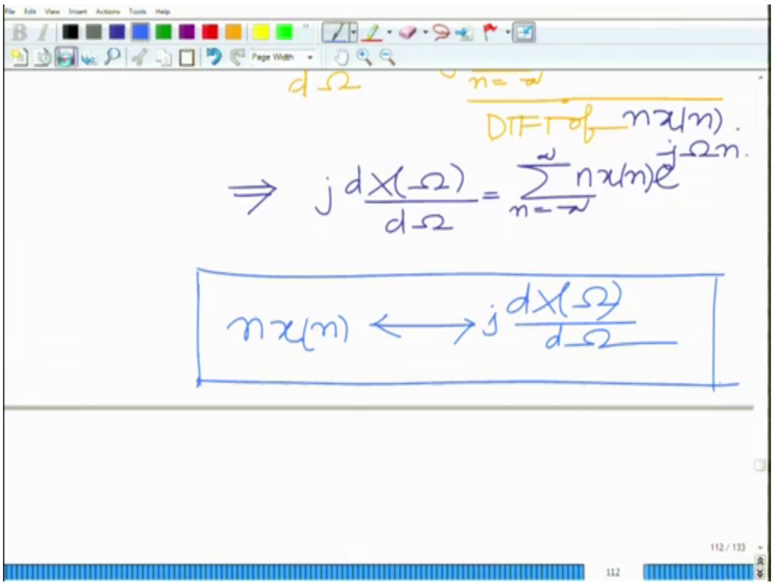

And therefore, this implies if you bring the j to the other side this means j d x omega or d omega equals summation n equals minus infinity to infinity n x n e raised to minus j omega n ok. And therefore, we can conclude that n x n is basically has the DTFT which is j d x omega over d omega alright.

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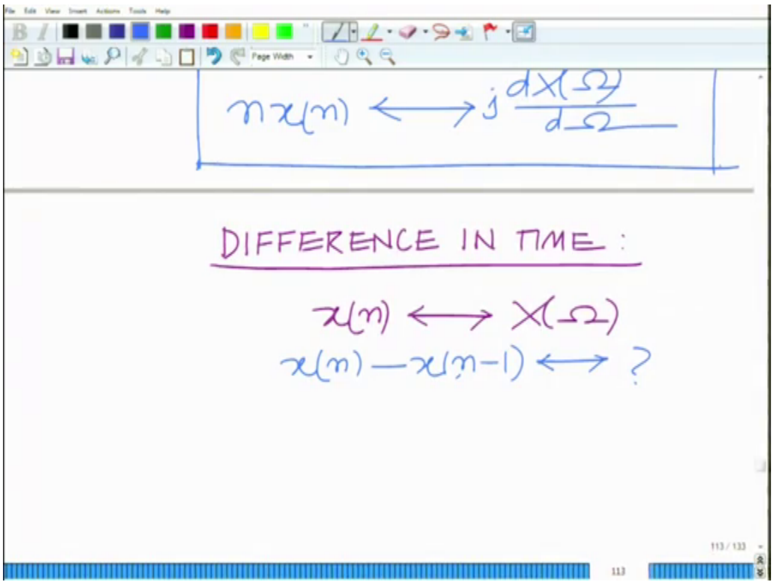

So, next property is the differencing property since we cannot differentiate in time one can difference the differencing in time which means if x of n has the d t f t x of omega what can we say about the difference signal x n minus x n minus one. What can we say about this is the difference signal or the differential signal alright x n the successive differences sometimes also called as a differential signal alright so that is your basically your delta x in time.

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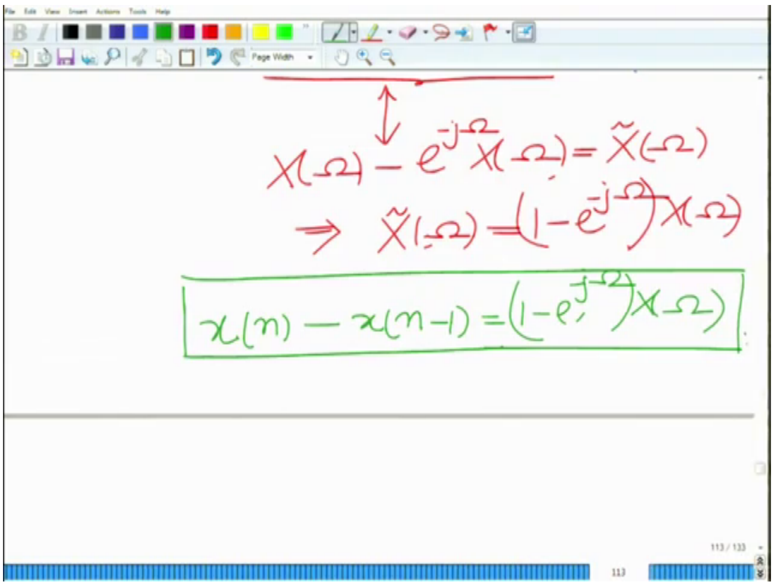

And now if you look at this if you take the Fourier transform of this you can see it is very simple this is x of omega minus the Fourier transform of x n minus 1 that is e raised to that is the delay in time leads to modulation in frequency e raised to minus j omega x omega equals x tilde omega which implies x tilde omega that is the DTFT of x m minus x n minus 1 equals 1 minus e raised to minus j omega into x of omega.

So, basically x n minus so if you look at this you have x n minus x n minus 1 has the discrete time Fourier transform 1 minus e raised to minus j omega into x of omega so this is basically the discrete time Fourier transform.

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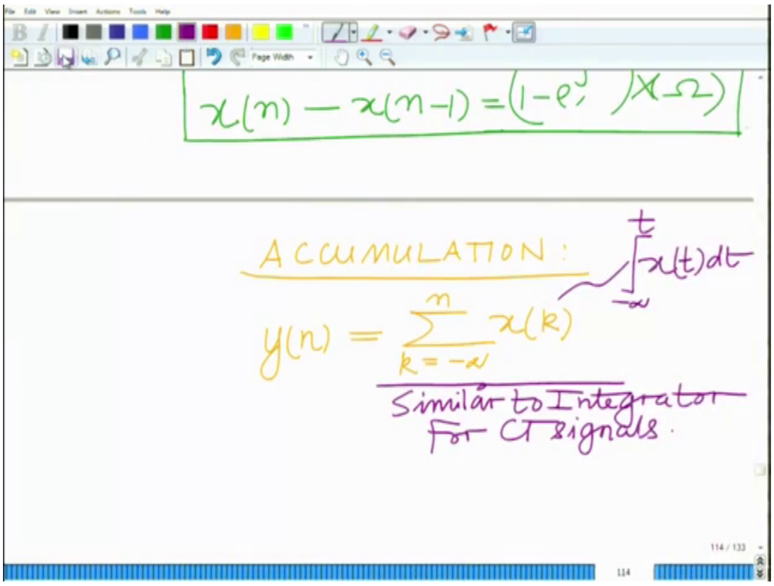

Let us now look at another property. So, all these are small properties, but fairly useful in the manipulation and analysis of DTFT of signals a lot of these properties can be used together to derive the final DTFT or to analyze the system or signal under consideration. So, let us consider the accumulation n equals or k equals minus infinity to n that is you are accumulating all the samples of the signal x k until sample n.

You can see this is very simpler similar to the integrator which basically integrates the input signal x t alright in analog time the analog alright the corresponding system in analog for an analog signal or for a continuous time signal is a integrator ok. So, basically it integrates or accumulates the signal at a certain time in the discrete time we are basically accumulating all the signal samples.

So, basically if you look at the continuous time analog that is minus infinity to t x t d t. So, this is similar to the integrator, integrator similar to the integrator for continuous time signals alright.

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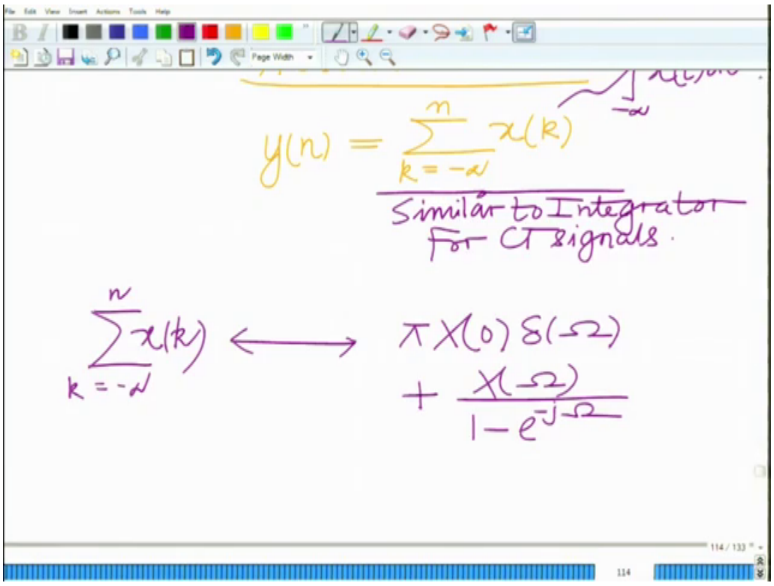

And now the Fourier transform of this is given as pi x of 0 you can show that pi x of 0 into delta omega plus x of omega over 1 minus e power minus j omega, so that is basically your accumulation that is your integral n equals that is your summation k equals minus infinity to n x k that has the DTFT that is given as pi x 0 delta omega plus x of omega, divided by 1 over 1 x of omega divided by 1 minus e raised to minus j omega alright.

So, that is basically the result that is basically the result for accumulation of a discrete time signal ok. Let us now proceed to another important property that is a convolution because we have seen this several times our convolution is always a very important relation very important property in analysis of signals and systems because the convolution describes the output of a linear time invariant system to any input either continuous time or discrete time ok.

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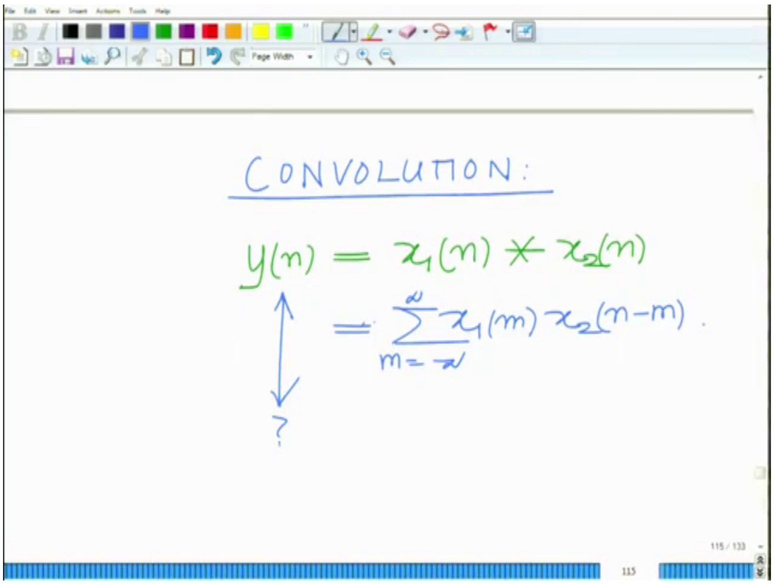

So, the next property that we want to look at is basically the convolution and now the convolution of 2 discrete time aperiodic signals. And convolution is always very important as I said because it is very closely related to the properties of LTI systems ok. So, y n which is the convolution of x 1 n into x 2 n which means this is equal to the summation m equals minus infinity to infinity x 1 m x 2 n minus m.

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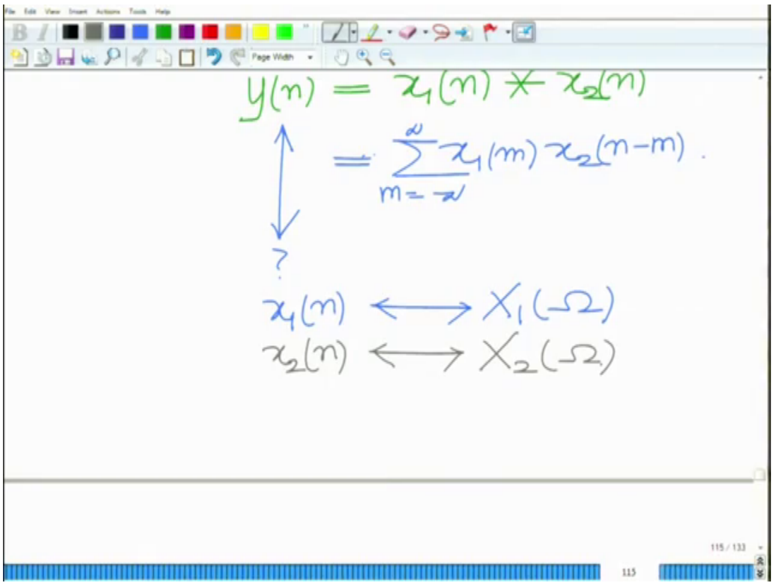

And now we want to find what is the DTFT of y n given of course the DTFT’s of x 1 n has a DTFT that is x 1 omega and x 2 n has the DTFT that is x 2 omega.

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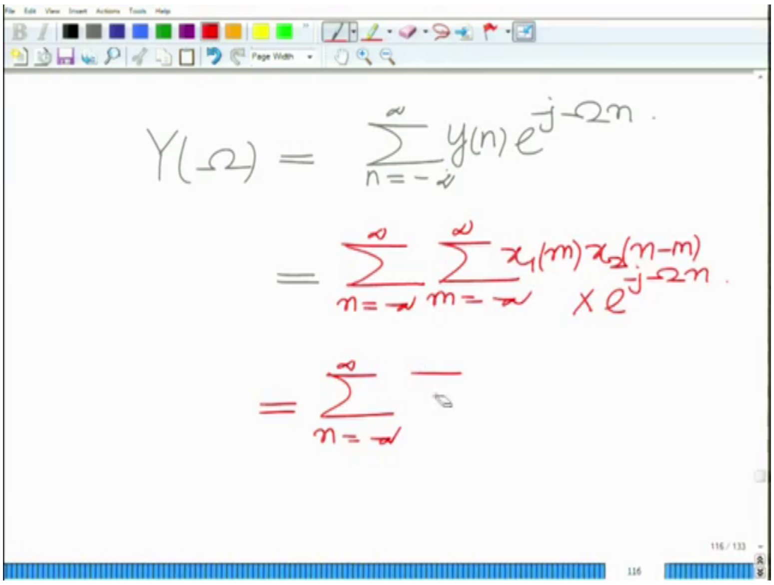

And now you can see this DTFT Y of omega can be obtained as follows. So, Y of omega equals summation that is y of n, n equals minus infinity to infinity y of n e raised to minus j omega n which is basically I can always write this as summation n equals minus infinity to infinity substitute the expression for y of n that is m equals minus infinity to infinity x 1 m x 2 n minus m into e raised to minus j omega n. And now I can always write this as summation n equals minus infinity to infinity summation.

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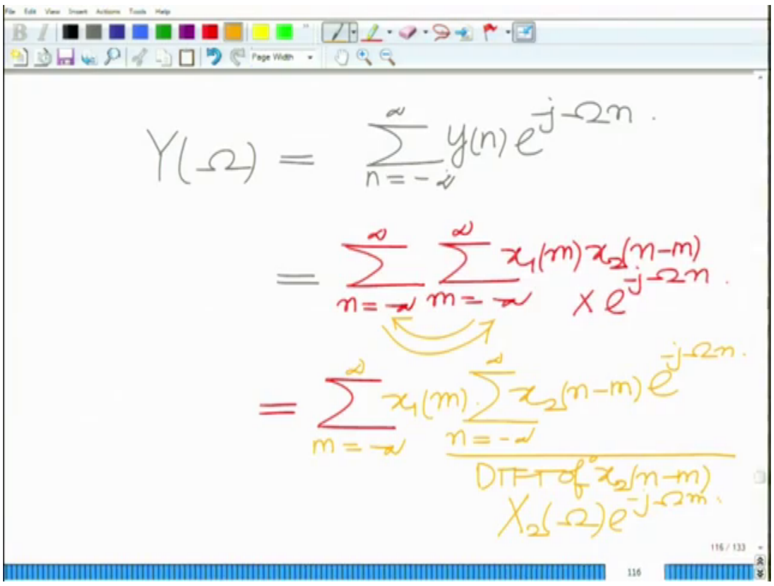

Now I can bring I can interchange the summation with respect to n and m so interchanging the summation I am going to describe this by this arrow. So, first you have the summation m equals minus infinity to infinity then x 1 m will come out because it depends only on m n equals minus infinity to infinity x 2 n minus m e raised to minus j omega n this is DTFT of x 2 n minus m that is x 2 n delayed by m samples; x 2 n delayed by m samples. Hence the corresponding DTFT is naturally x 2 omega e raised to minus j omega m that is modulation in frequency ok.

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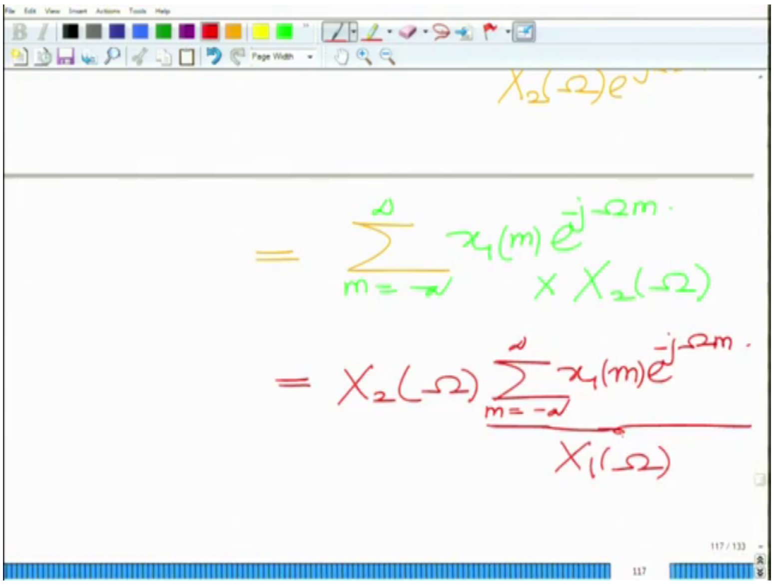

So, that basically gives me summation m equals minus infinity to infinity of x 1 m e raised to minus j omega m into x 2 omega and now if you look at x 1 m into e raised to so I can write this as x 2 omega m equals minus infinity to infinity x 1 m e raised to minus j omega which is nothing but basically x 1 omega.

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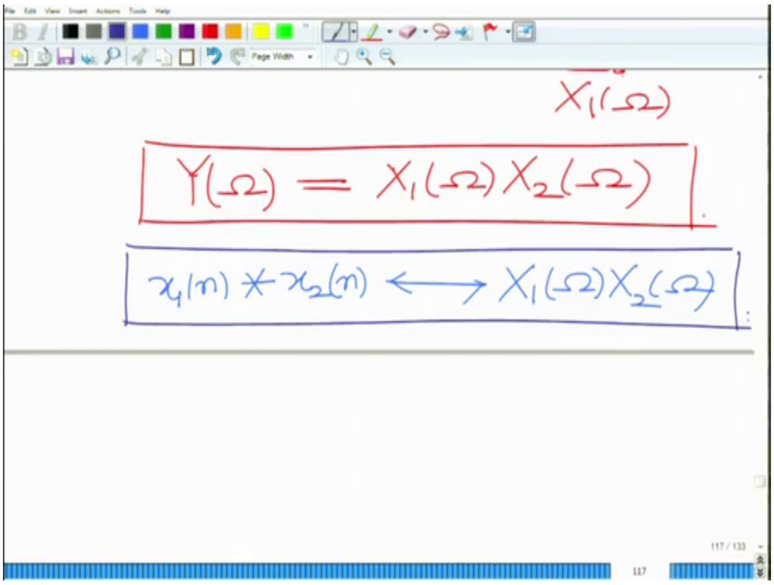

And therefore, what we get is y of omega which is a convolution y n which is the convolution of x n x 1 and x 2 n has the Fourier DTFT that is x 1 omega into x of omega. So, the net result is that x 1 n convolved with x 2 n has the DTFT x 1 omega into x 2 omega. So, convolution in time implies multiplication and frequency similar to what we have seen several times if equation of 2 discrete time aperiodic signals leads to a multiplication of their discrete time Fourier transform.

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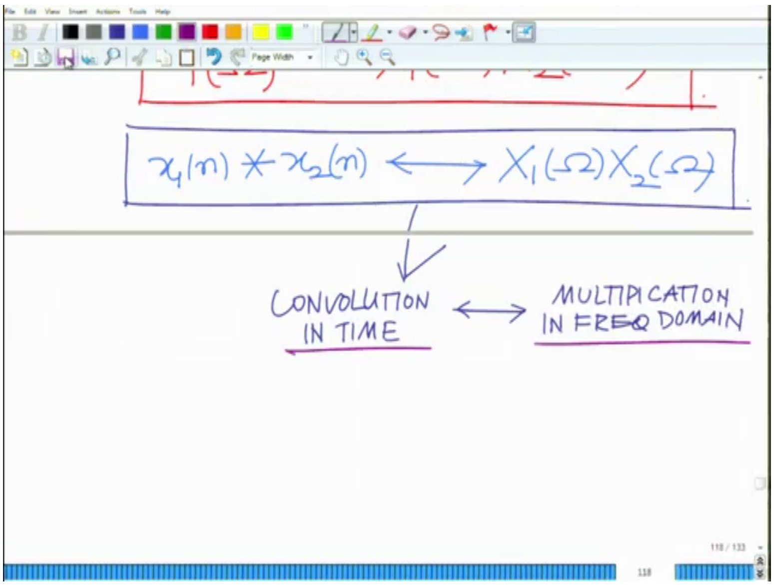

So, basically I can summarize this as convolution in time multiplication in the multiplication in the frequency domain ok. So, when you convolved to signals in time the corresponding DTFT's correct in these case are basically convolved in the frequency alright.

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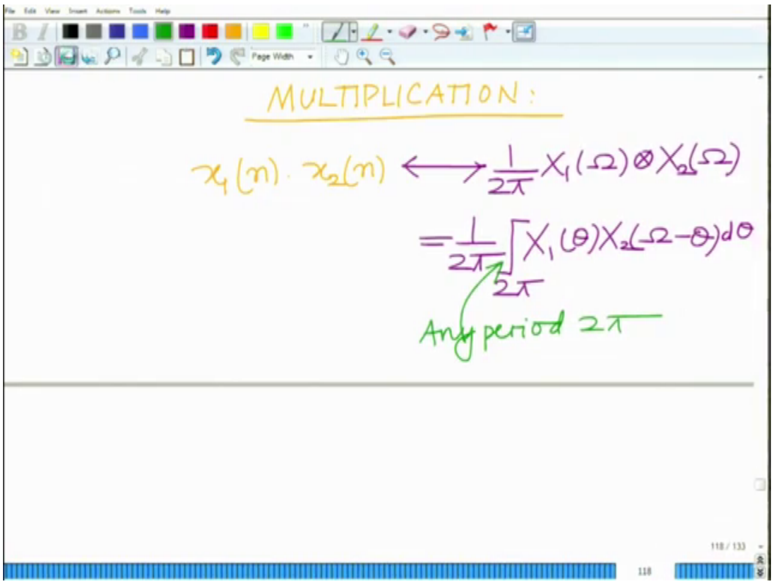

Let us look at similarly for multiplication in time again the dual of that is basically it have it has convolution in the; similarly let us consider multiplication in time that is if you look at 2 signals x 1 n into x 2 n.

Now remember the x 1 omegas are periodic hence this is the periodic convolution 1 over 2 pi x 1 omega periodically convolved with because remember the net DTFT, also has to be periodic correct when you convolve to a periodic thing remember the DTFT, is always a periodic signal so you cannot perform a general convolution, but you have to convolve in such a way that the resulting output is periodic and that is given by the periodic convolution ok.

So, this is basically the periodic convolution of x 1 the periodic convolution of x 1 omega with x 2 omega which is basically 1 over 2 pi over 2 pi x 1 theta x 2 omega minus theta d theta I am sorry this is d theta and this is for any contiguous, this is for any period 2 pi, this is a periodic convolution. This is over this is over any period 2 pi ok.

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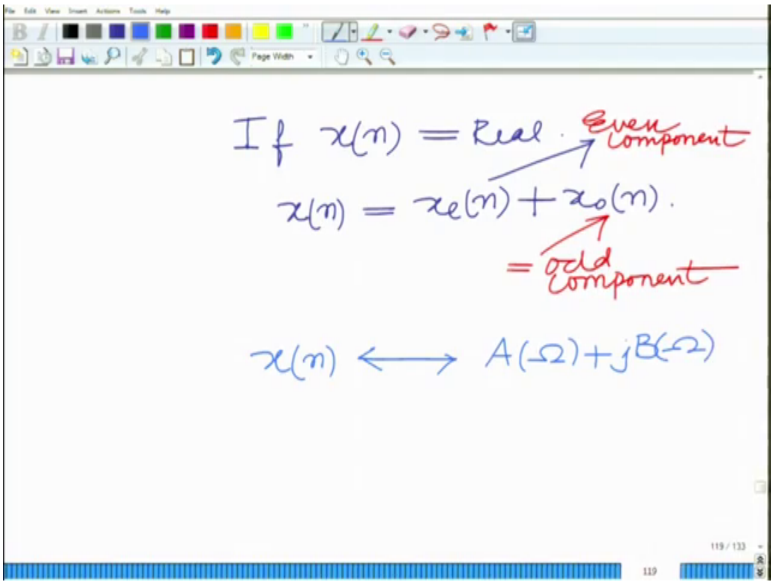

Now some other properties if x n is real let us consider a real signal. Then I can always express a real signal as the sum of even and odd components so this is the even component x e n the even component x o n, this is the odd component we have already always we have seen that you can always do this that is x e and the even component of a signal is x n plus x of minus n by 2, the odd component is x n minus x of minus n by 2 we can always do this for any real signal ok.

And now further if I can express the DTFT of x n as a omega plus. So, in general the DTFT of x n is complex you can express this as a omega plus j times b omega.

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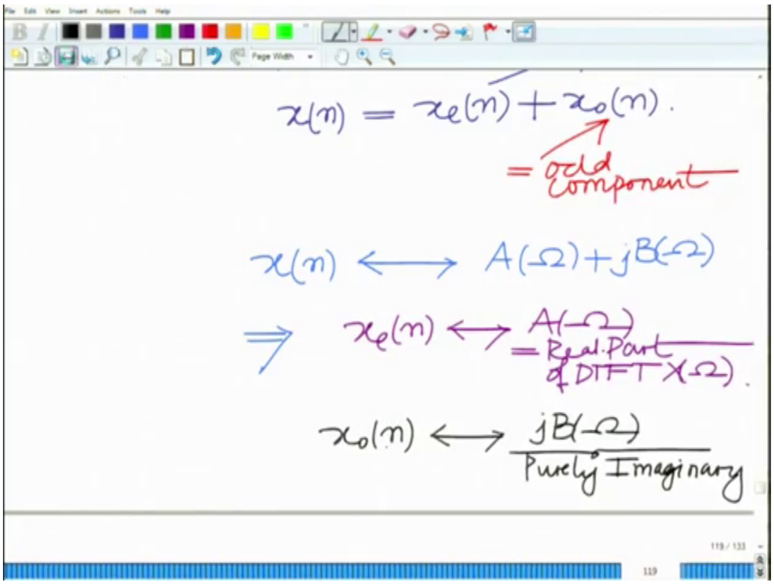

Then it can be shown that and this is something that can be shown that is x n has the DTFT that corresponds to the real part of the DTFT of x n that is the even component has the DTFT a omega that is real part of the DTFT x of omega. And similarly as you can expect the odd component has the DTFT that is given by j times b omega ok.

So, the even component has a DTFT that is a times omega that is a real part. And the odd component as there is so the DTFT of the real part even part of the purely real signal you can see is basically real and the DTFT of the odd part odd component of this real signal is purely imaginary you can see that j times B of omega this is purely imaginary ok, this is purely imaginary.

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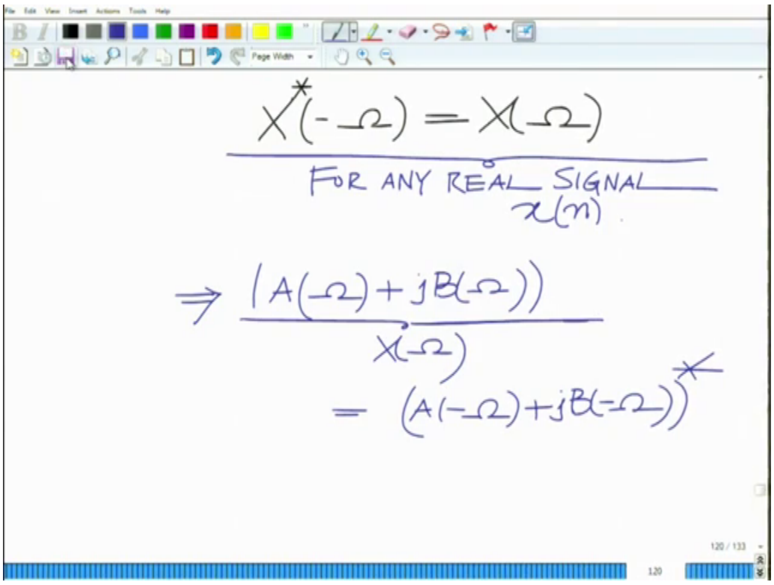

And now further since we have a real signal recall that for a real signal x of conjugate of minus omega equals x of omega for a real signal x n where x omega is the DTFT of this is true for any remember this is true for any real signal x n.

So, this implies that if I take the DTFT a of omega plus j B of omega this is x of omega this must be equal to x of minus omega conjugate so a minus omega j b minus omega conjugate.

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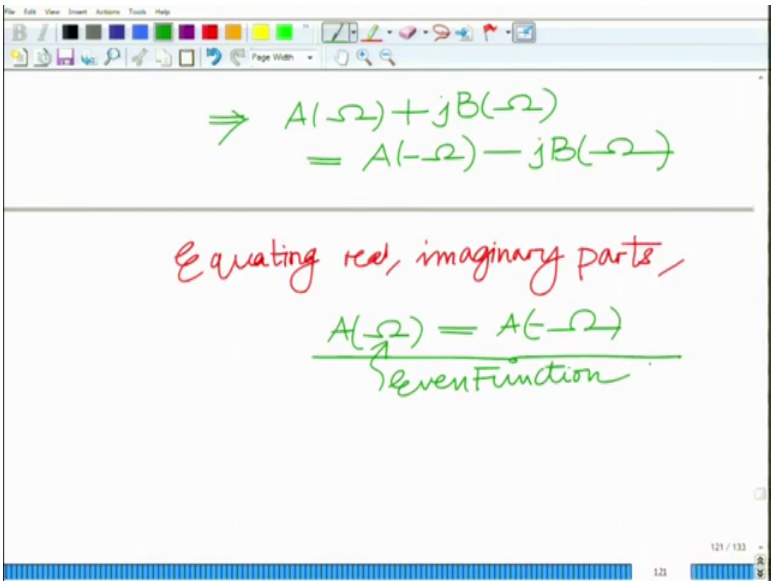

So, this must be equal to a of minus omega plus j b minus omega conjugate and this implies that A of omega plus j b minus omega equals well a of minus omega minus j b of minus omega. Now equating the real and imaginary parts equating real comma imaginary parts we have A of omega equals A of minus omega which implies A of omega is a even function.

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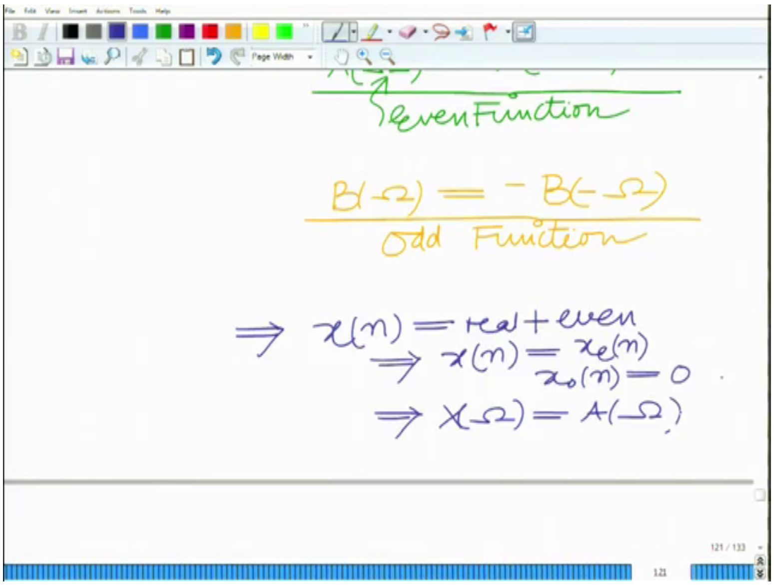

And further we also have B of omega equals you can see waiting the imaginary parts where B of omega equals minus I am sorry this has to be B of minus omega. So, this is basically B of minus omega ok. So, B of omega equals minus B of minus omega which means this is a this is an odd function.

So, A of omega for a real signal the real part is an even function over a real part of the DTFT and the imaginary part B of omega is an odd function of omega ok. Now, therefore, if x n is real and even now what does this imply this implies two things; one is x n equals real plus even this implies x n equals x e of n comma x o of n the odd component of n is 0. So, this implies x of omega equals A of omega which is basically real and even.

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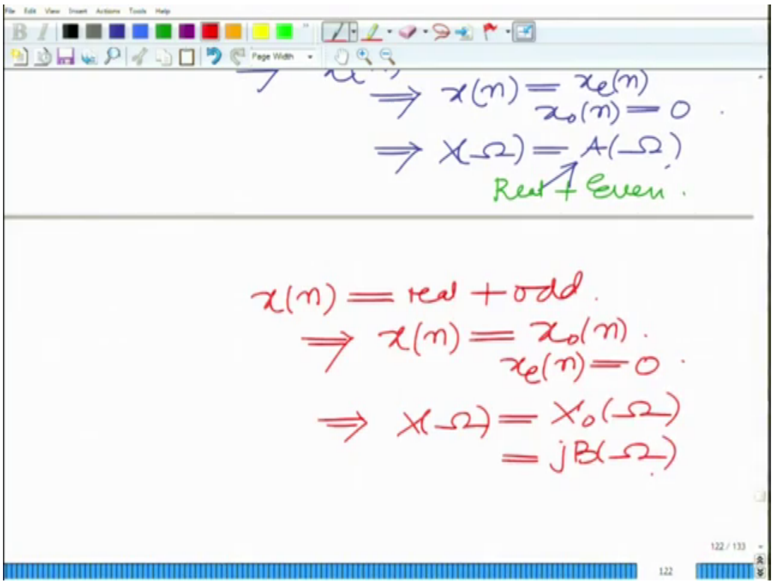

So, for a real and even signal the corresponding DTFT is so if the signal is real plus even the corresponding DTFT is real plus also even on the other hand if the signal is real and odd ok. Now look at the other thing if x n now this implies that x n equals odd component ok, the even component is 0, if x n is an odd signal and x e of n equals 0.

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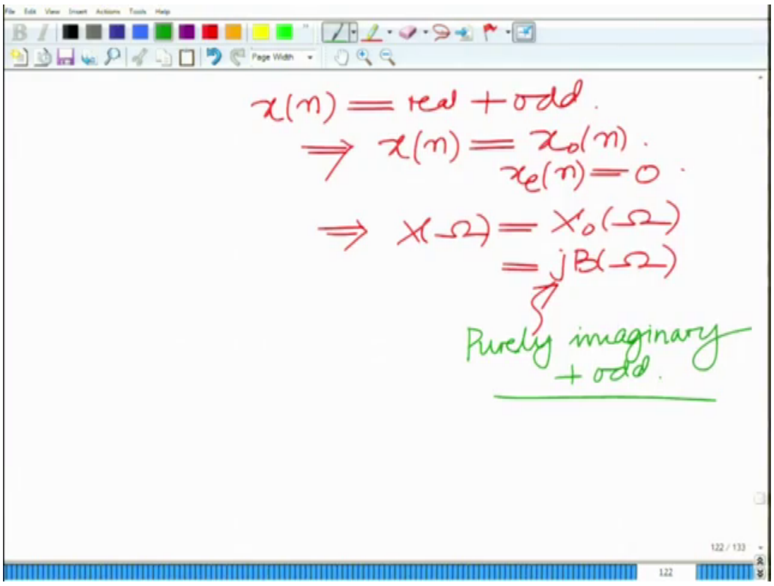

And this implies x of omega equals x odd component of omega which is equal to j b of omega which is purely imaginary and odd remember B of omega is odd and this is a purely imaginary functions. So, this is this is purely imaginary plus odd, so if the signal is real even signal its DTFT is pure is purely real and even.

If x n is real and odd signal then its DTFT is purely imaginary and odd ok. So, this is the important thing to keep in mind alright so that basically sums up the properties of the DTFT of real signals real and even real and odd signals alright.

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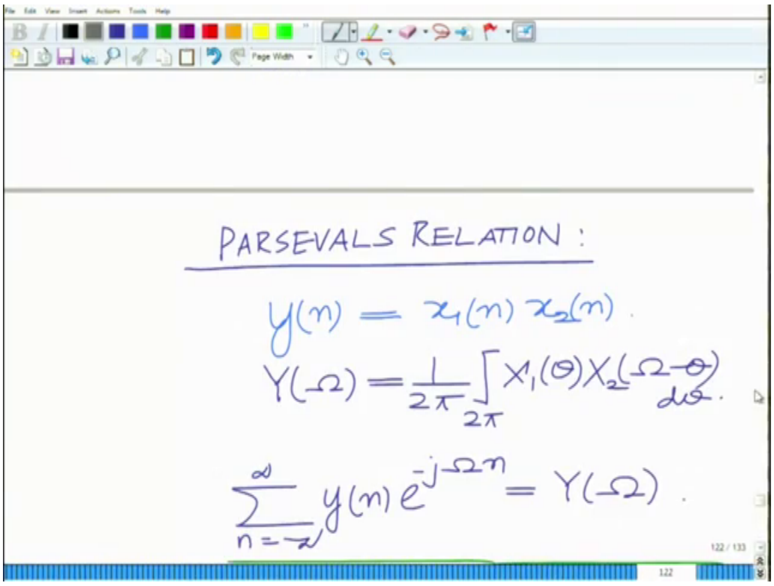

And then what we can also do so as to sort of to complete this we can look at the Parseval's relation that is the last topic one of the last properties that we can look at for the DTFT that is the Parseval's relation.

And this is also very simple remember previously we have said let us say we have y n equals x 1 and that is the multiplication of x 1 n and x 2 n then we know that Y of omega is the periodic convolution of x 1 omega and x 2 omega. So, that is given as integral 1 1 over 2 pi, integral over any 2 pi region x 1 of theta; x 2 of omega minus theta d theta which implies basically that now if I look at summation.

So, now, we have basically what do we have we have basically summation n equals minus infinity to infinity y n e power minus j omega n this is equal to y of omega.

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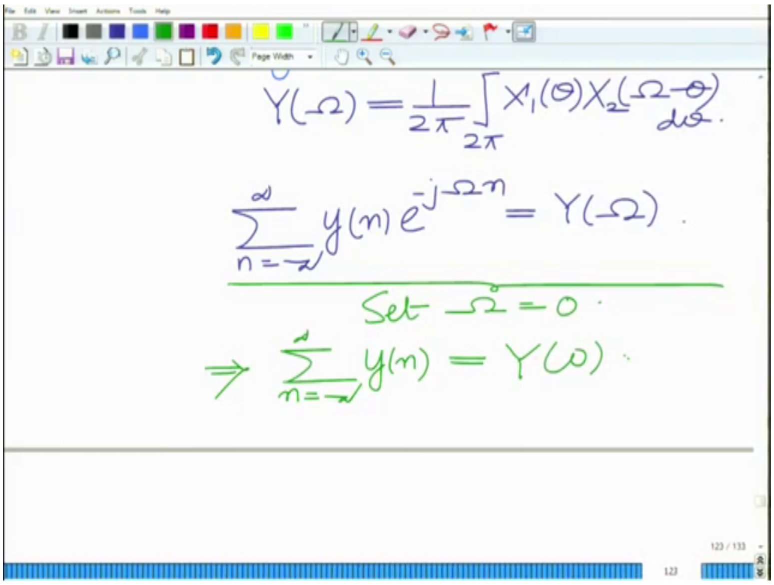

Now if you set omega equal to 0. Now, in this set omega equal to what this implies is summation n equals minus infinity to infinity y n equals y of 0 implies summation n equal to minus infinity to infinity the product x 1 into minus infinity to infinity x 1 n into x 2 of n equals y of 0 and y of 0 is nothing, but set omega equal to 0 on the right.

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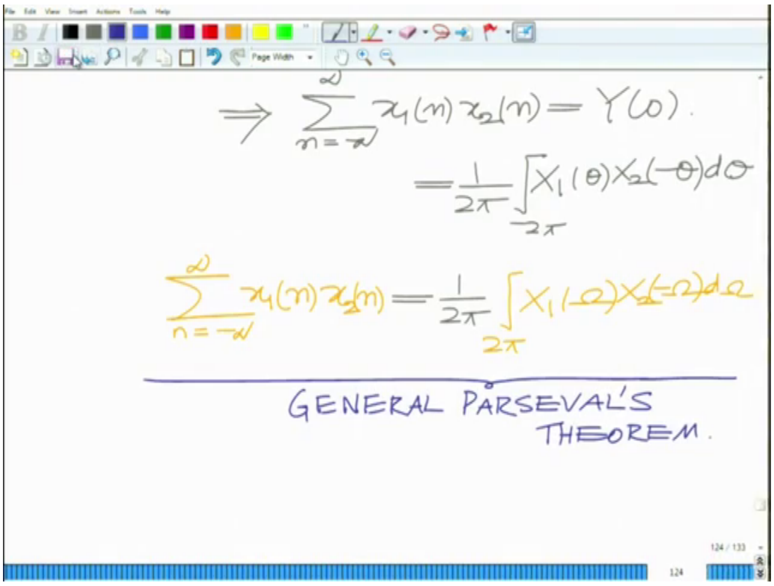

So, that is integral x 1 of theta x 2 of minus theta d theta and you can also in fact, write that as theta is just an index. So in fact, you can also write that as 1 over 2 pi integral over 2 pi; x 1 of omega x 2 of minus omega d omega ok. This is summation x 1 of n x 2 of n ok.

So you can always write this thing summation n equal to minus 1 this is nothing, but the correlation between these two discrete time sequences x 1 n x 2 n summation n equal to minus infinity to infinity x 1 n x 2 n that is integral 1 over 2 pi integral over any 2 pi x 1 omega into x 2 minus omega d omega I have simply replaced the integration variable theta by omega ok.

Now if we set x 2 n now in this so this you can think of this as the generalized Parseval's relation. In fact, this is a step ahead of the Parseval's relation it is much more general it talks about two different signals x 1 n and x 2 n. So, you can think of this as a generalized or a general form. So, even think about this as a general form of Parseval's relation.

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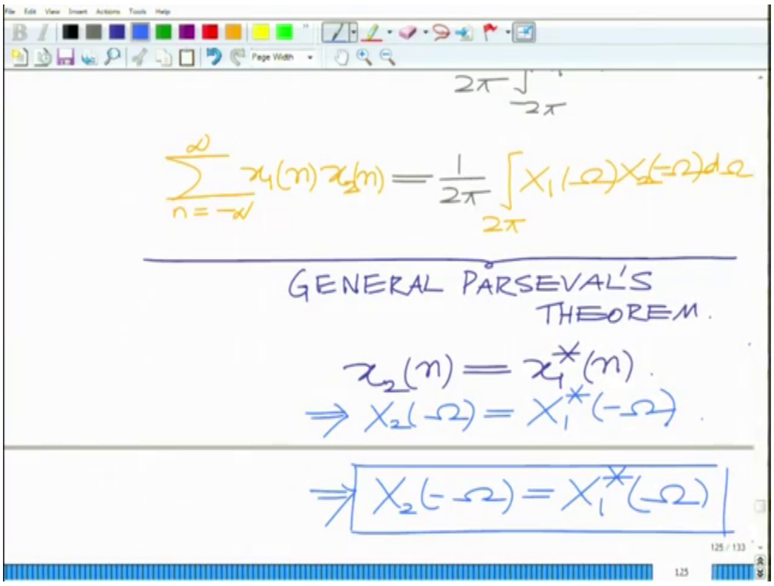

Now if you said x 2 n equals x 1 conjugate n. Then what you have is x 2 of omega equals x 1 conjugate of minus omega because conjugate sequences is x 1 conjugate of minus omega which means x 2 of minus omega equals x 1 conjugate of omega. So, if you substitute if you substitute x 2 n equals x 1 conjugate on n.

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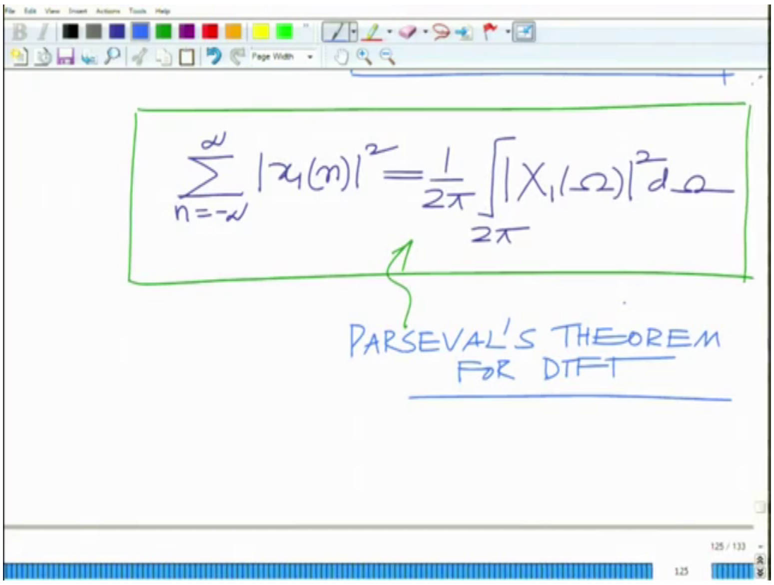

On the left hand side what we have a summation x 1 n into x 1 conjugate n that is magnitude x 1 n whole square that is equal to summation 1 over 2 pi integral over 2 pi x 1 omega into x 2 of minus omega which is x 1 conjugate of omega that is x 1 omega and x 1 conjugate of omega which is magnitude x 1 square into d omega that is the sum of the energy over 1 period in that this is not the energy.

But you can think of this as the power because you are dividing by 2 pi so this is the power of the you can think of this as the power of the DTFT ok. So, this is the Parseval's relation for the, this is the Parseval's theorem for DTFT. The summation n equal to minus infinity to infinity magnitude x 1 n square equals 1 over 2 pi integral over any 2 pi region magnitude x 1 of omega whose square t omega alright.

So, basically that completes our discussion of the properties of the DTFT. In the subsequent when the next module will start looking at the DTFT and its properties with relation to the with relation to LTI systems alright. So, we will stop here and continue in the subsequent modules.

Thank you very much.