Principles of Signals and Systems
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Lecture – 57
Discrete Time Fourier Transform: Properties of DTFT - Linearity, Time Shifting, Frequency Shifting, Conjugation, Time-Reversal, Dualit

Hello welcome to another module in this massive open online course.

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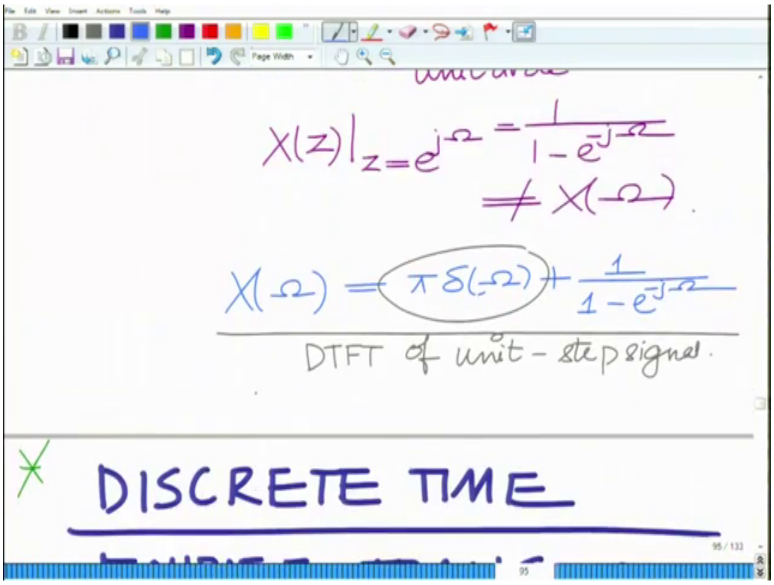

So, we are looking at the Fourier analysis of discrete type signals in particular the Fourier analysis for discrete time aperiodic signals through the discrete time Fourier transform alright. So, we are looking at the discrete time, Fourier transform for discrete time aperiodic signals or what is known as the DFDT the discrete time Fourier transform all right.

So, now, let us look at the properties of the DFDT all right we have looked at the DFDT of a unit step signal.

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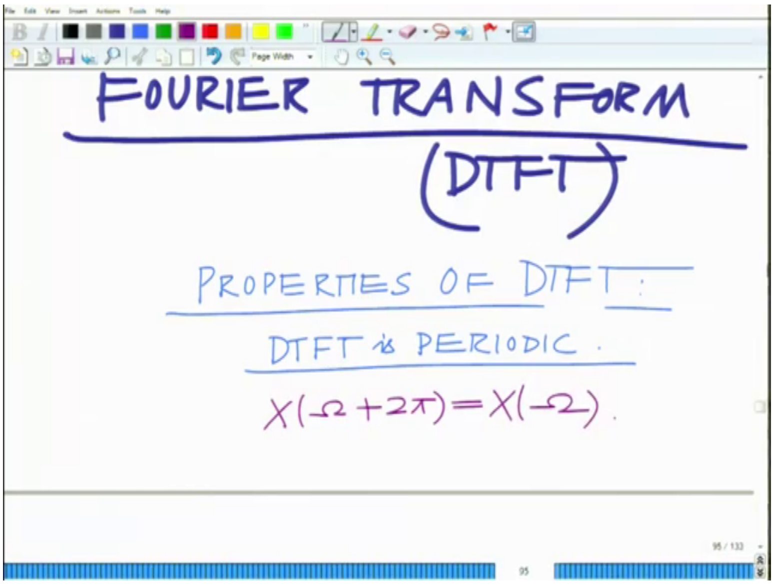

So, the properties of let us continue our discussion with the properties of the DFDT of any aperiodic signal x n. Now, one of the most fundamental properties of the DFDT is that; the DFDT is periodic. So, DFDT is periodic and there in fact, if you can su see X of omega plus 2 pi this is equal to X of omega this can be seen as follows; X of omega equals summation n equals minus infinity to infinity x n e raised to j omega n.

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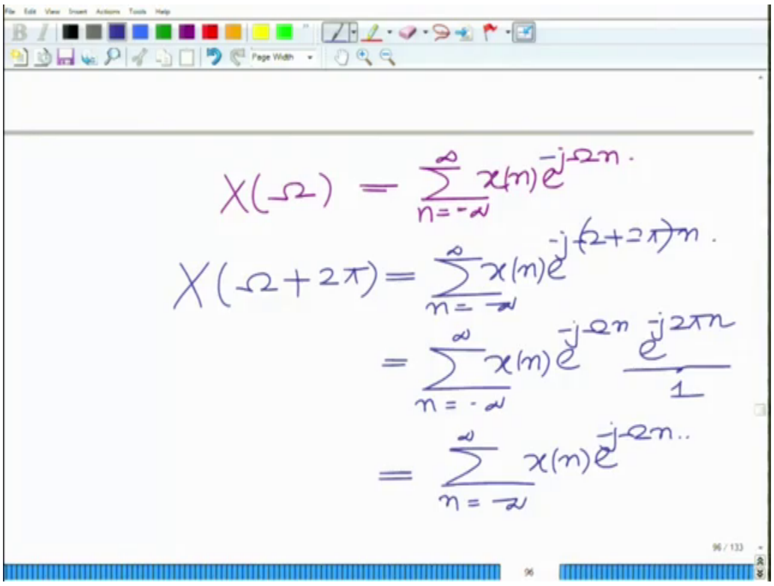

And X of omega plus 2 pi this is equal to sum n equal to minus infinity to infinity X of n e raised to j omega plus 2 pi n which is sum n is equal to minus infinity to infinity X of n e raised to j omega n X of n e raised to j 2 pi n; this is one which is sorry. In fact, this X of n e raised to minus j omega n minus j omega plus 2 pi and this is summation n equals minus infinity to infinity X of n e raised to minus j omega which is nothing, but X of omega ok.

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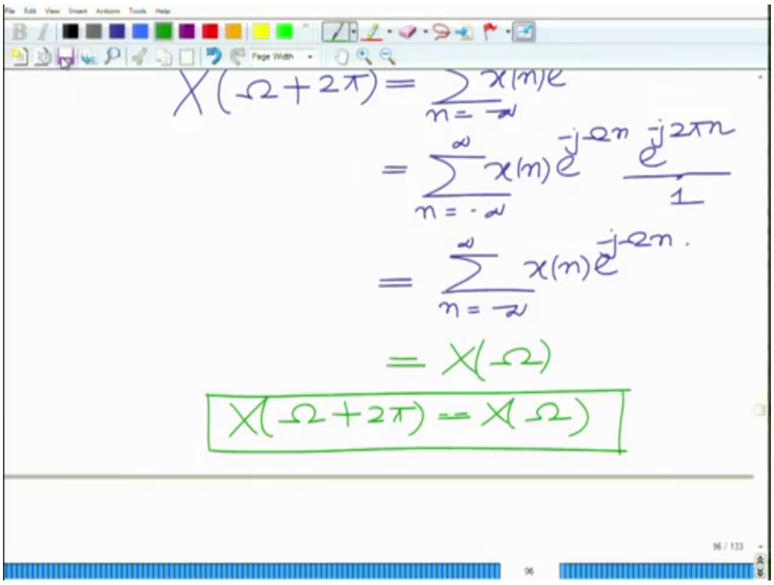

So, this is basically again once again this is X of omega. So, we have X of omega plus 2 pi equals X of omega ok. So, the DFDT is very basically periodic and the period is 2 pi the fundamental period ok, all right.

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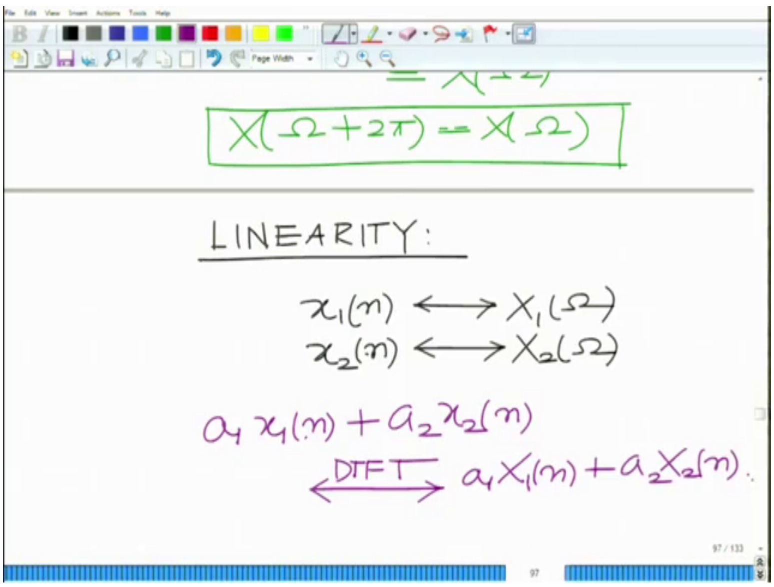

Let us look at some other periodics properties again one of the simplest properties which is obeyed by all the transforms that weve seen before and the DFDT is no exception is the linearity that is if x 1 n has Fourier transform or DFDT X of omega X 1 of omega x 2 of n has DFDT X 2 of omega, then x 1 of n. In fact, a a 1 where a 1 is a constant plus a 2 x 2 of n where a 2 is a constant has the DFDT as you can expect a 1 X 1 n and a 2 X 2 n ok. So, this is the linearity that is a linear combination of the signals produces has the DFDT that is the corresponding linear combination of the DFDTs of the corresponding individual signals alright.

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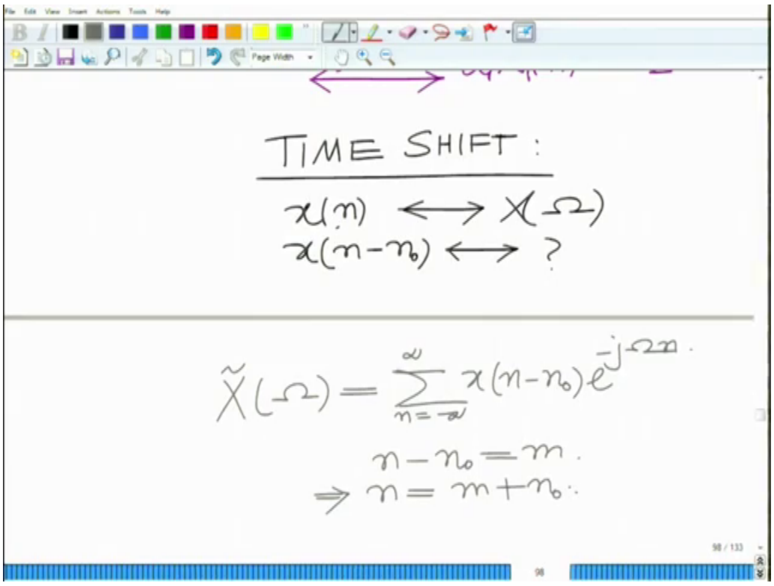

Time shifting again another important property what happens when you time shift a discrete time signal. So, let us say x of n has the DFDT X of omega what happens when your time shift by n naught. So, we are asking what is X tilde omega which is summation n equals minus infinity to infinity x of n minus n naught e raised to minus j omega n set n minus n naught equals m that implies n equals m plus n naught.

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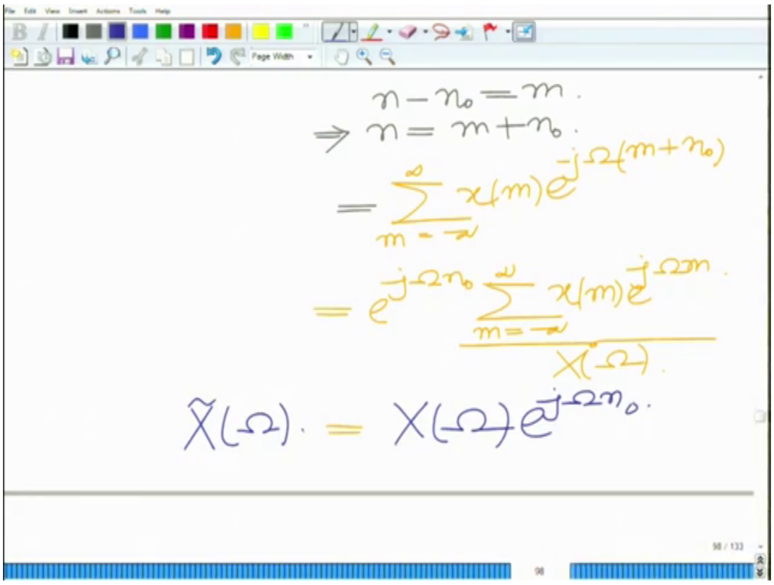

So, this is equal to summation n equals minus infinity m. In fact, m the index will be m e raised to minus j omega m plus n naught which is if you take e raised to minus j omega n naught outside you have summation m equals minus infinity to infinity x of m e raised to minus j omega m; and this is X of omega and therefore, this is basically equal to X of omega times e raised to minus j omega naught that is your X tilde omega.

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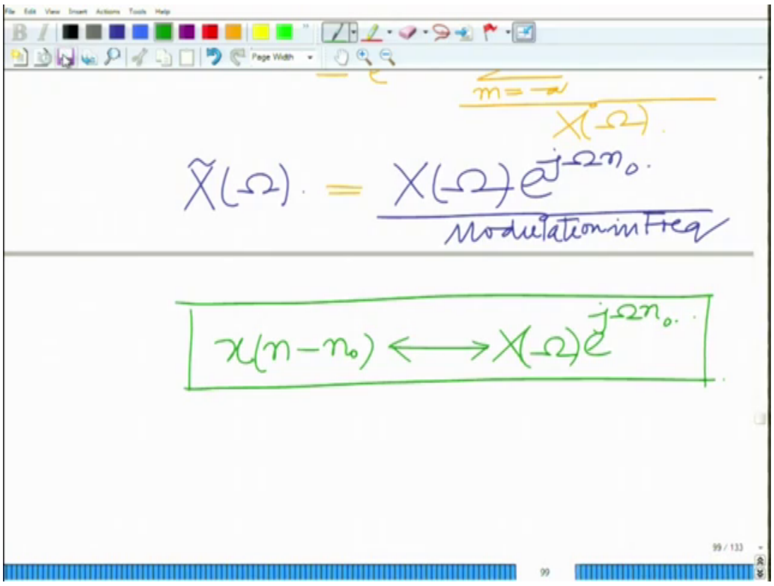

So, when your time shift that is basically a modulation in frequency multiplied by the complex exponential e raised to minus j e raised to minus j omega n naught. So, x of n minus n naught as the DFDT X of omega e raised to minus j omega n naught ok.

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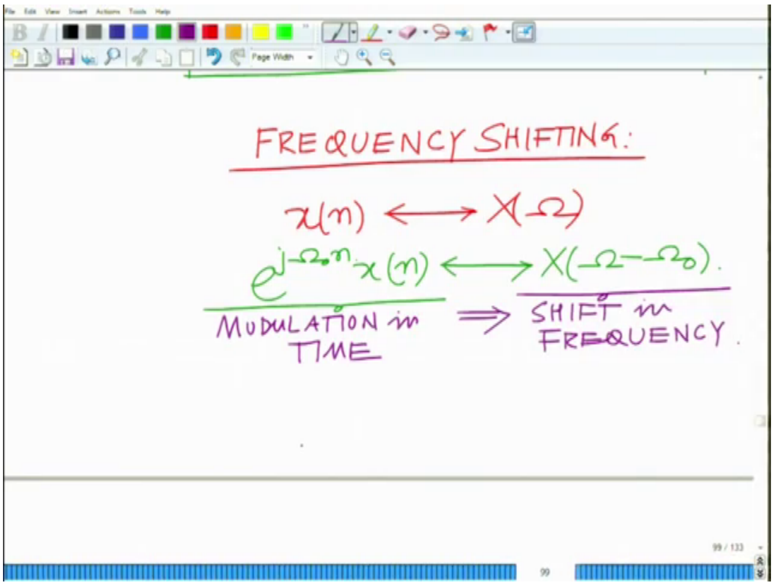

Similarly, we have frequency shifting now in frequency shifting again one can derive this similarly x n; let us say has Fourier transform X of omega then e power e raised to j omega naught n, that is e raised to j omega naught n x n, that is a modulated signal modulation in time leads to a corresponding shift in frequency. So, e raised to j omega naught n x n e raised to j omega naught n x n, that is a modulation in time because you are multiplying it by a complex exponential of frequency omega naught in that in the frequency domain leads to a shift in the frequency by omega naught ok.

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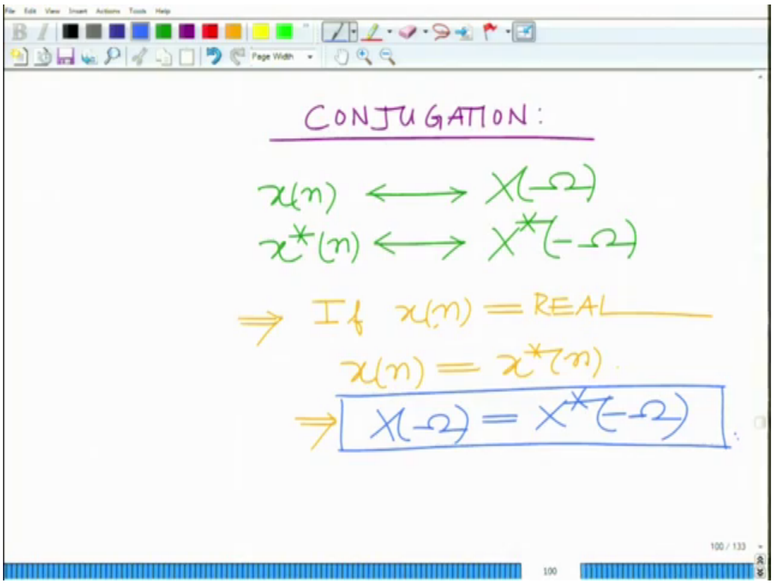

So, this is modulation in time implies a shift in the frequency domain ok. Conjugation when you have x n conjugate, that is x n has DFDT X of omega, then x conjugate n has DFDT X conjugate of minus omega and therefore, this naturally implies if x n is real then we have x of n equals X conjugate of n and that implies the corresponding DFDTs are equal that is X of omega equals x conjugate of minus omega this is something that we have already seen before; that is for an even that is for a real signal X of omega X of omega is basically x conjugate of minus omega. So, the magnitude spectrum is an even function of omega and the phase spectrum is an odd function right.

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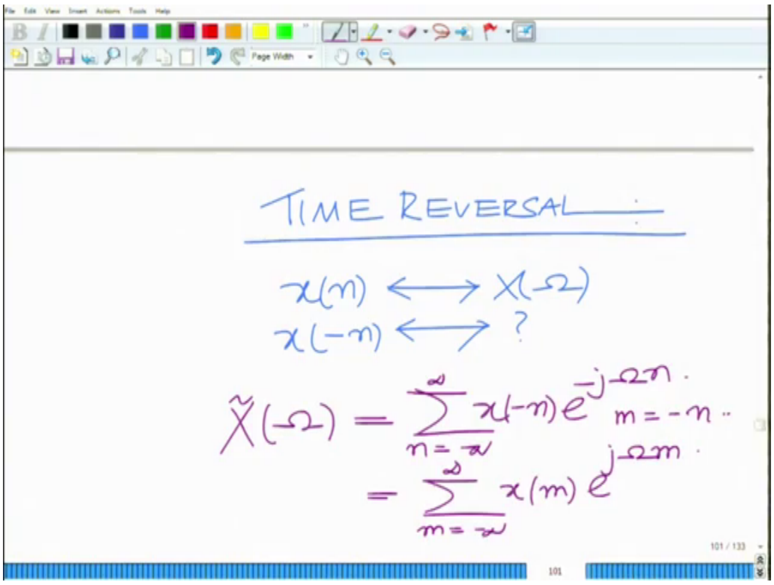

Time reversal again similarly one can look at time reversal, if x n has DFDT X of omega X minus n has DFDT well what is the DFDT that would be X tilde omega equals sum n equals minus infinity to infinity x minus n e raised to minus j omega n equals summation setting m equals minus n that would be minus infinity to infinity x of m e raised to j omega m where m equals minus n and therefore, X tilde minus omega equals summation m equals minus infinity to infinity x m e raised to minus j omega m that is X of omega.

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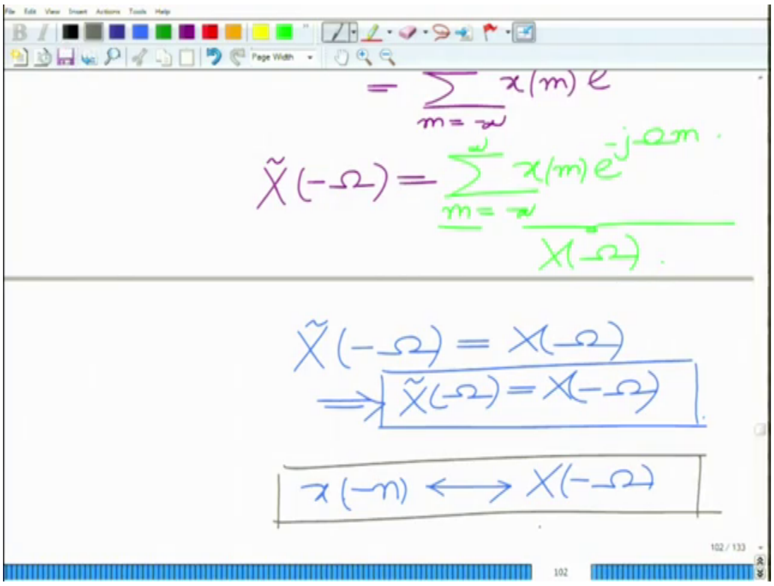

So, we have X tilde minus omega equals X of omega which implies X tilde omega equals X of minus omega ok. So, basically x of minus n has the Fourier transform as the DFDT X of minus omega ok. So, x of minus n has the DFDT X of minus omega alright.

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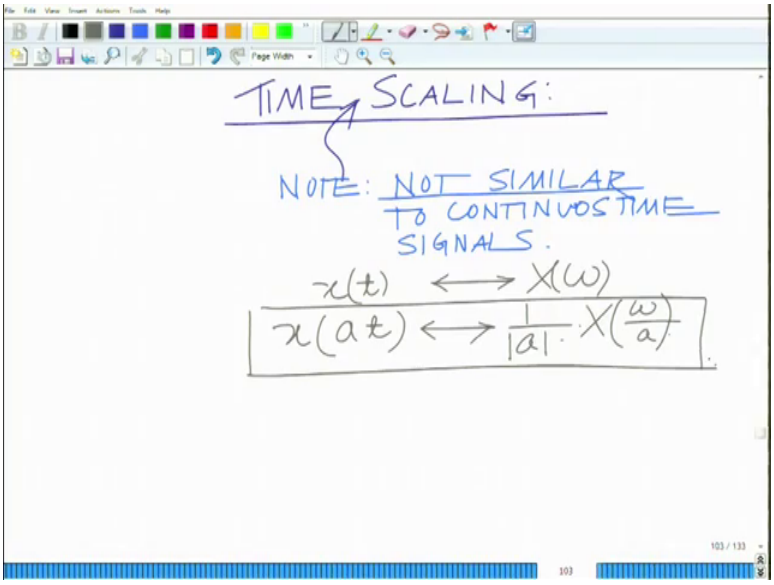

And another interesting property is that of time scaling; now time scaling is interesting, because it does not directly follow from the, and this is a very interesting property because this is one aspect which does not have a direct which does not directly follow from the continuous time.

So, one would be mistaken if one would substitute directly the result from the Fourier transform of a continuous time signal and the reason is as follows. So, time scaling caution is important note not while most are similar to the continuous time properties is not similar this is not similar to that for continuous time signals the reason is as follows; that is if you look at a continuous time signal then you have x of a t as the Fourier transform that is if x of t has the Fourier transform X of omega, then x of a t has the Fourier transform one over magnitude of a X of omega over a this is the result for time scaling ok.

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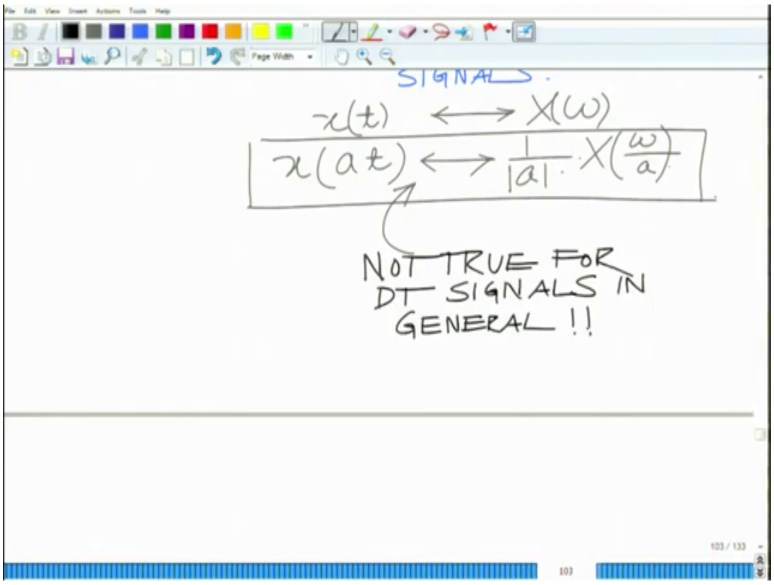

However, this is not true for discrete time signals in general; however, not true for discrete time signals the DT signals; that is if you have x of n has the DFDT X of omega.

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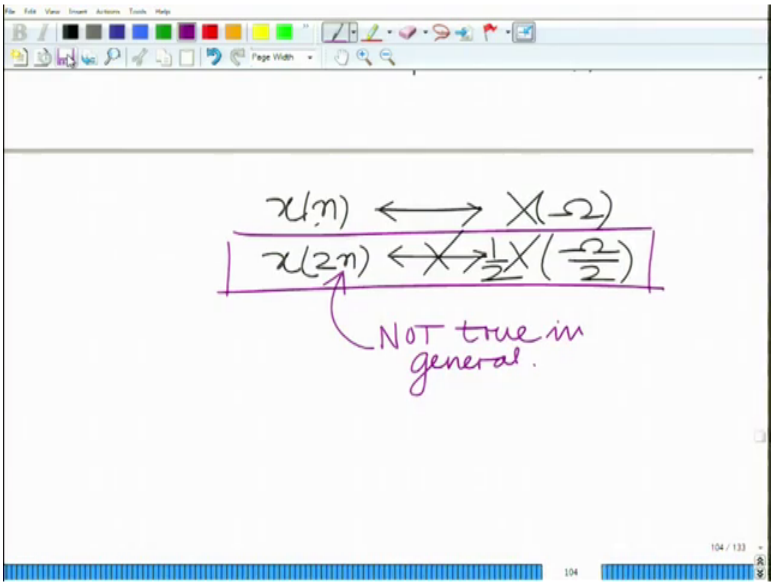

Then x of 2 n will not have the DFDT half X this is not true; in general this statement is not true this is not true this is not true in general that is x of 2 n preferring to the sequence that does not have that is I cannot use the formula 1 over a 1 over magnitude of a X of omega over a that is I cannot get the DFDT by 1 over by half X of omega divided by that result is not to true in general.

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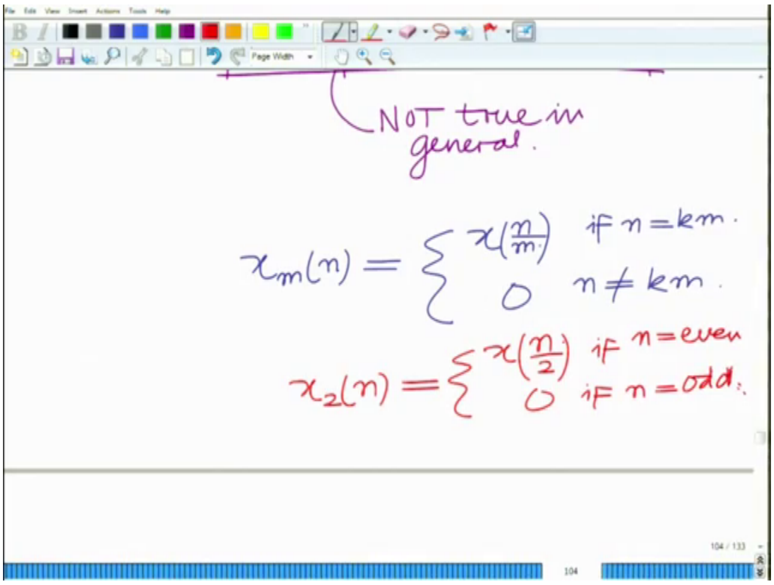

However; the following result is with this for the following result is true that is if I define a signal x m of n equals or any integer m x n over m if n is a multiple of m, that is if n equals k times m if n is that; is n is divisible by m equals 0 n naught equal to k m for instance for example, x of 2 of n x of m equals x of that is basically equals x of n by 2; if n equals even and 0 if n is odd let us consider this for instance ok.

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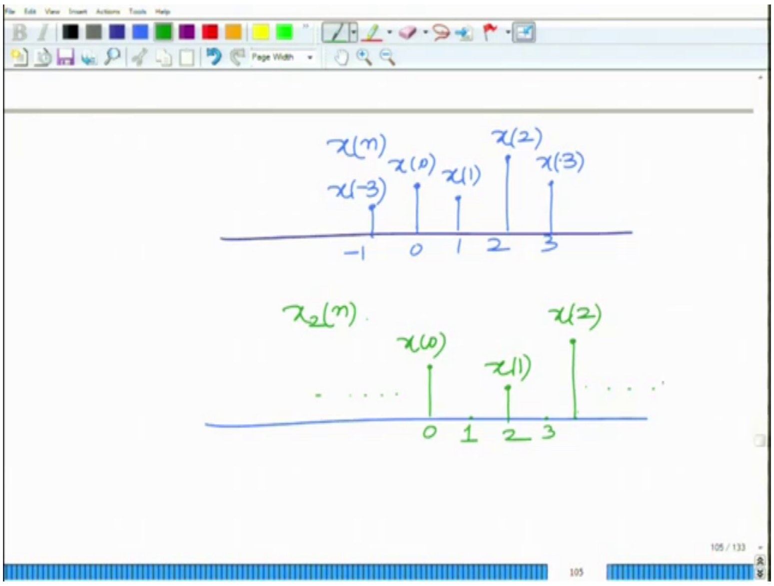

And for instance this signal can be visualized as follows let us take a simple example; let us say I have this is x of 0, this is x of 1, this is x of 2, x of 3, this is x of minus 3, and then if I consider the signal x 2 of n which is basically; so x 2 of 0 is simply x 0, x 2 of 1 because 1 is not a multiple of 2 this is 0, x 2 of 2 is x of 1 ok. So, x 2 of 2 is x of 1, similarly x 3 of 3 is 0 and x 2 of 4 is x of 2 ok. Similarly on the negative axis this is x of n.

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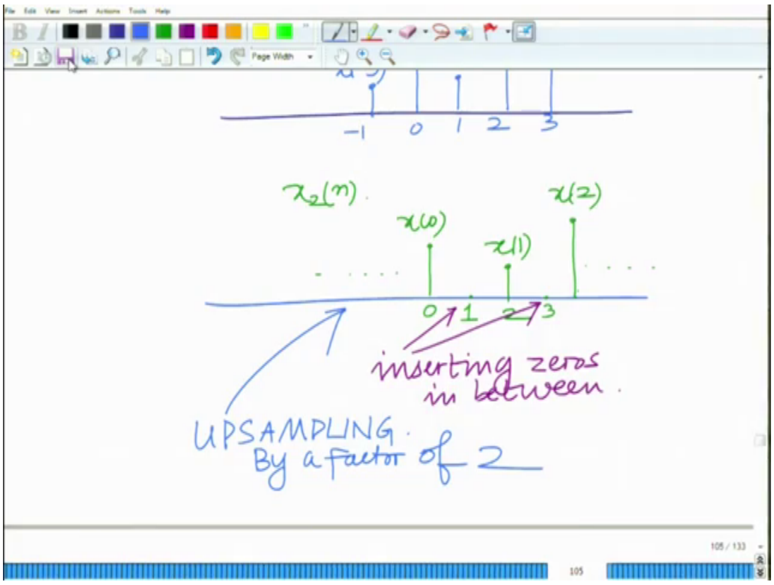

So, you are interspersing so, you are inserting zeros in between so, first thing realize is basically you are inserting you are inserting zeroes in between and you are spreading the signal out all right. So, x 2 of n, x 2 of 0 is 0, x 2 of 0 is x of 0, x 2 of 1 is 0, x 2 of 2 is x of 1, x 2 of 3 is 0, x 2 of 4 is x of 2 and so, on.

So, we are spreading the signal out and basically you are inserting zeros in between this is known as up sampling. In fact, this is known as up sampling by a factor of 2 ok. So, this operation is basically termed as up sampling; in fact, this is equal to up sampling by a factor of this is up sampling by a factor of 2; there up sampling by a factor of 2 ok.

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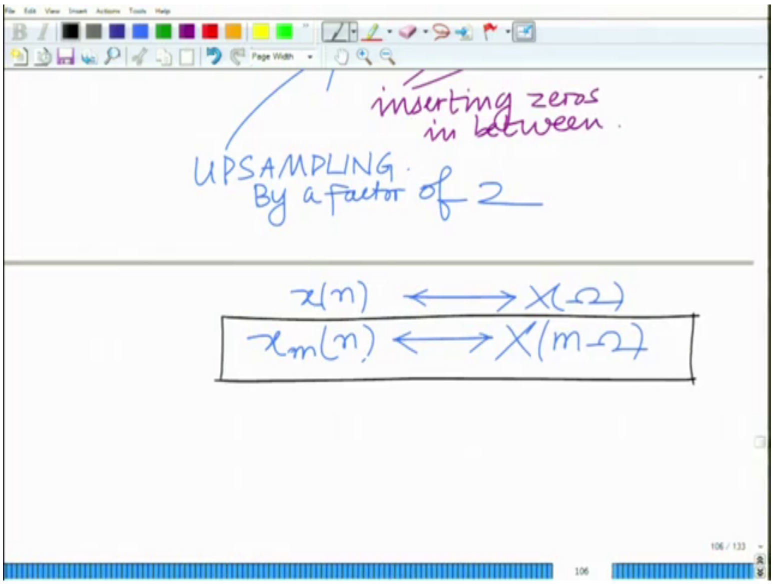

Now even you up sample a signal; that is x of n of n the resulting Fourier transform is x m of omega, that is if x of n has the Fourier transform X of omega, then x m of n has the Fourier transform X m omega that is in the frequency domain it shrinks shrinks by a factor of 2. So, we have x of 2 of omega; that means, with what was at 2 pi comes 2 pi ok. So, that can be seen as follows.

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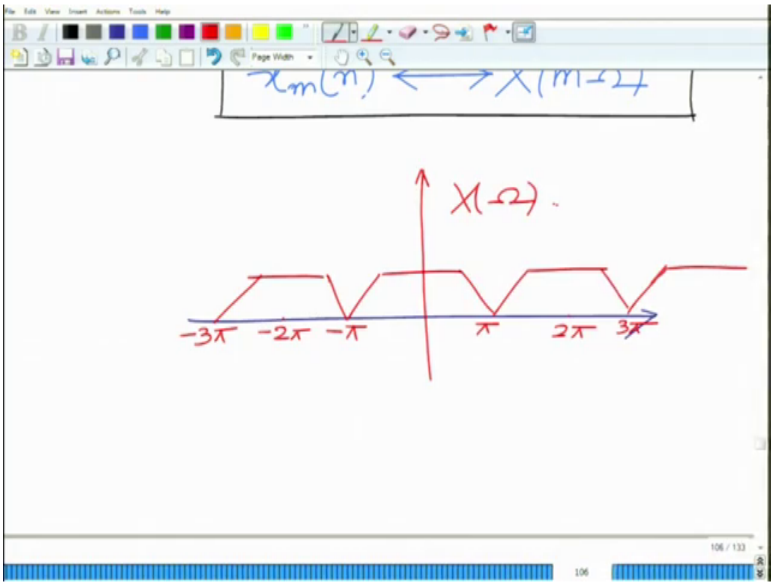

So, X of omega is periodic; let us try to plot a periodic let us plot a periodic DFDT first ok. So, this is let us say. So, this is minus pi 2 pi, this is 2 pi 3 pi minus 2 pi minus 3 pi. So, this is X of omega.

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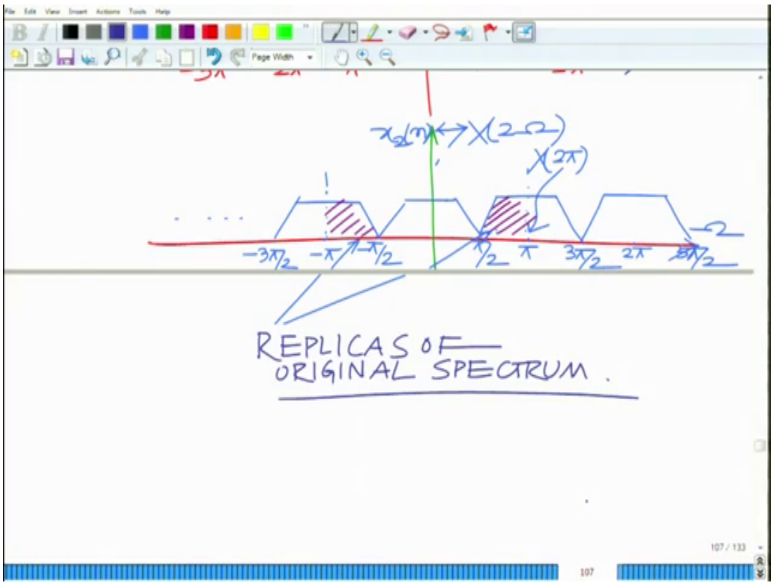

Now x 2 of n will have the Fourier transform which looks something like this. So, this is like. So, this is your x 2 of omega, which is the DFDT of x 2 of DFDT of x 2 of n. So, that will look something like.

So, you will have basically at; so this will be pi by minus pi by 2 to pi by 2 what is a 2 pi will come 2 pi. So, what is that? So, at 2 pi you will have what is that. So, you have something that looks like pi by 2. So, at 3 pi by 2. So, this is at pi 3 pi by 2 at 2 pi you will have what is 5 pi by and so, on and similarly here you will have something that is and so, on ok.

So, this is basically your X of 2 omega and remember this is X of basically 2 pi ok. So, what is at 2 pi will come to. So, this was spectrum basically shrinks and what happens is as a result of that you can see you have replicas of the same spectrum, if you look at the minus pi to pi spectrum you will have the replica. So, what this is basically if you can see this it is a replica of the previous copy of the previous signal ok.

So, you have replicas of the original spectrum n minus pi to pi and you can see that the number of such replicas is equal to the up sampling factor. So, if you have up sampled by a factor of 3; then you have 3 replicas 1 is the original and then you have three other replicas if you have up sampled by factor of 4 4 replicas one original and three other replicas. So, so that is the interesting aspect of this that is up sampling all right. Now for the when you down sample that is the opposite of up sample that is you scale down the signal of time domain it is slightly more difficult to derive the Fourier the DFDT of the resulting signal, because you can see that results in a loss of samples.

When you shrink the time domain signal unlike in a continuous time signal, when you shrink the time domain signal, for instance if you shrink it by the factor of 2 take only the even samples then which then it means basically the odd samples of the signal have to be reversed. So, there is if there is a certain loss that is occurring when you are down sampling and therefore, the DFDT is not it is not straightforward to derive the DFDT in such a scenario.

So, basically one has to realize that the result for scaling in the continuous time does not hold; in general for discrete time signal using the DFDT in the context of the DFDT that is the discrete time Fourier transform all right ok.

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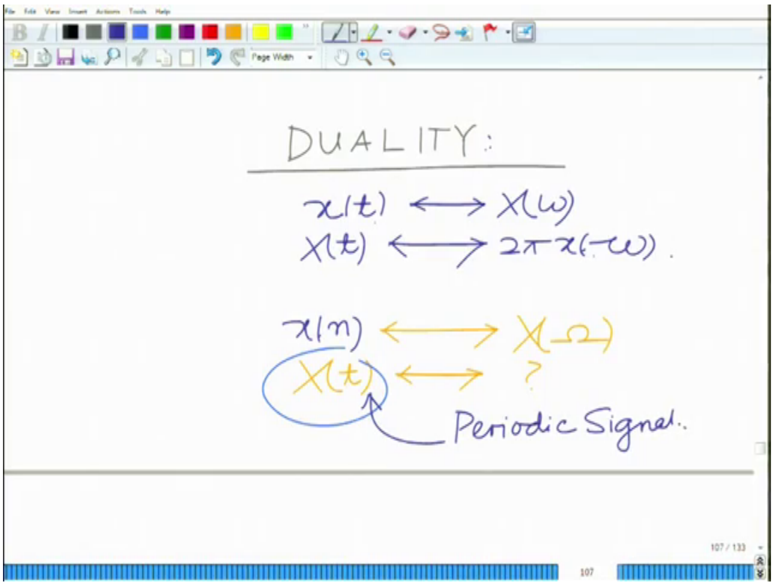

Again duality that is another interesting very interesting property what happens to duality; and this is also very interesting, because we see in the continuous time if you have x of t which has a Fourier transform X of omega, then we know that X of capital X of t has the Fourier transform 2 pi x of minus omega in the k in the context of a discrete time signal again it is slightly complex now first or so, let us say we again ask ourselves this question x of n has the DFDT X of omega what can we say about X of omega.

Well, first realize that x of t is a periodic signal that is capital X of t is a periodic signal and this is because capital X of omega is periodic.

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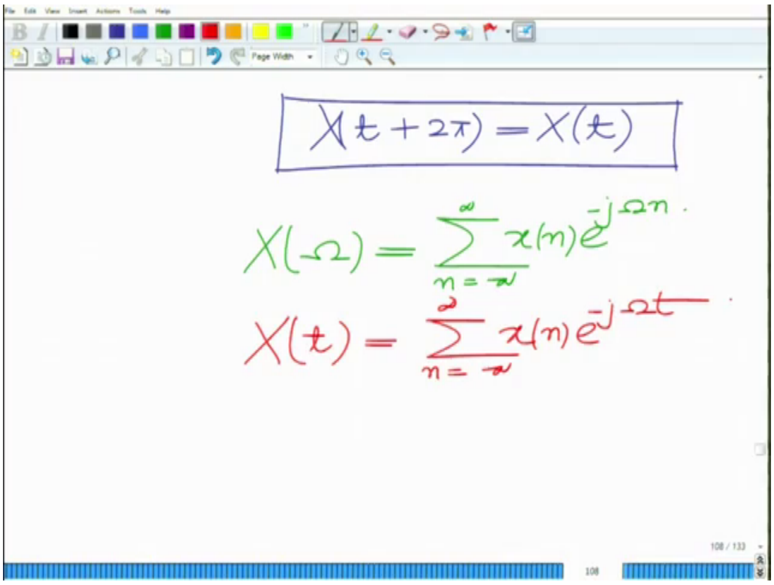

So, we have x of t plus 2 pi equals x of t. So, this is the fundament. So, this is a periodic signal. Naturally when we have a periodic signal we cannot use the continuous time Fourier transform, but we have to look at the complex exponential Fourier series and that is going to be related to the period the, that is going to be related to the discrete time signal.

How is that related that is not very difficult to see; if you look at X of omega we have X of omega equals summation n equals minus infinity to infinity x n e raised to minus j omega n which means, now if you replace omega by t we have x of t equals summation n equals minus infinity to infinity x of n e raised to minus j omega t and if you replace m equals minus n if you replace minus n or if you replace minus n by m then you have m equals Minus infinity to infinity x of minus m e raised to j im sorry this will still be e raised to j sorry this will be e raised to minus j n t. So, this will be raised to j plus n t or e raised to j m t in fact, ok.

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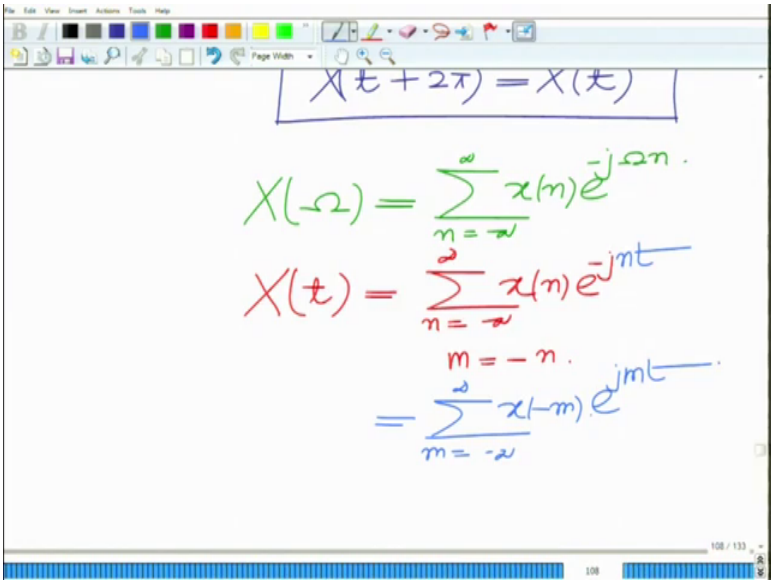

And now you can see here we have the fundamental frequency omega naught.

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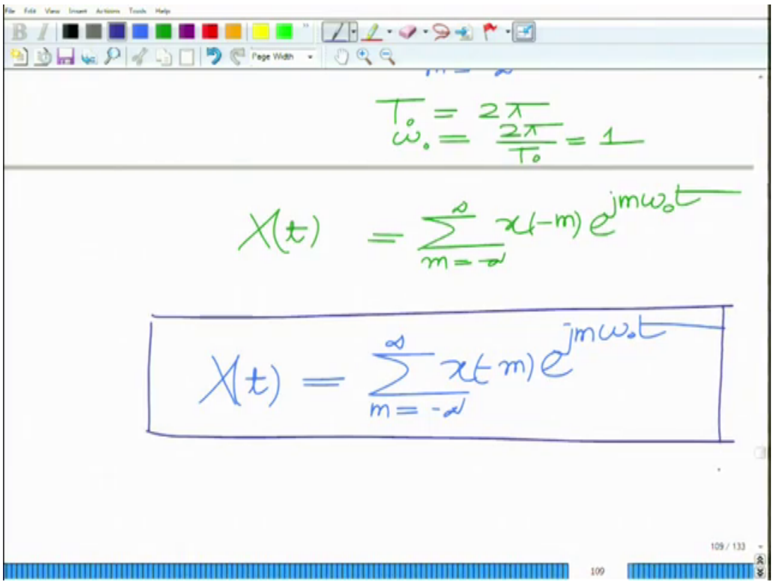

So, we have the time period t naught equals 2 pi the fundamental time period, which means the fundamental frequency omega not equals 2 pi by t naught this is equal to 1. So, I can denote omega naught by 1 and I can write this as summation m equals minus infinity to infinity x of minus m e raised to j omega naught or e raised to j m omega naught t where omega naught is equal to 1 and; that is the fundamental frequency and now you can see we have the discrete time, we have the complex exponential Fourier series for X of t. So, I can write it as just going to write it clearly; e raised to j omega m naught t and what this is; is this is a now you can see this is the complex exponential Fourier series, because you are expressing X t as a linear combination infinite number of complex exponentials at the fundamental frequency omega naught and it is harmonics.

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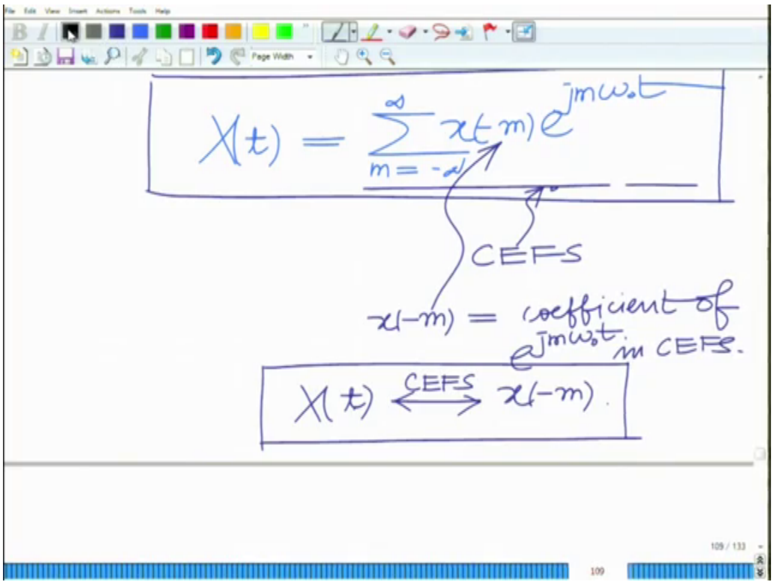

So, this is the complex exponential Fourier series and these are the x minus m is the coefficient of e power j m omega not t in the complex exponential Fourier series. So, basically you can say X of t has the complex exponential Fourier series given by x of minus m that is the duality result for the DFDT; because the asymmetric nature of the DFDT, when you take the resulting signal in time that is a periodic signal ok.

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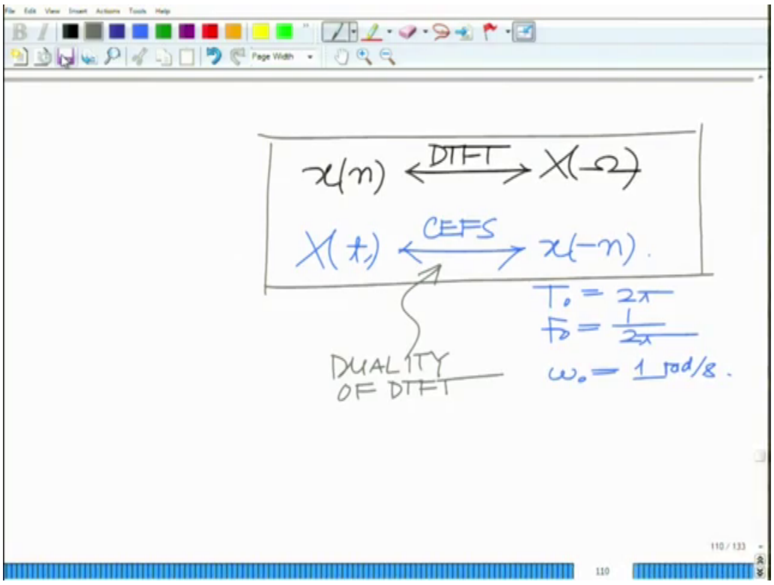

And in fact, so the duality result the complete result is x of n has here the top equivalence is in terms of the DFDT and the bottom; that I am going to write now; if you look at the signal in time it is naturally going to be a continuous time signal, because X of omega is a continuous time signal, but it is going to be a periodic signal X of t and this equivalent is in terms of the complex exponential Fourier series x of minus n with the fundamental period t naught equals 2 pi fundamental frequency linear frequency f naught equals 1 over 2 pi angular frequency omega naught equals 2 pi f not equals 1 radian per second and this is the result this is basically the result for this is basically the result for the duality, duality of the DFDT duality of DFDT alright.

So, basically that completes several properties of the DFDT starting from the simplest property linearity time shifting modulation etcetera and finally, we have looked at up sampling of a discrete time signal and also the duality property for the DFDT that is the duality property for the DFDT of a discrete type aperiodic signal alright. So, we will stop here and look at other aspects in the subsequent modules.

Thank you very much.