Principles of Signals and Systems
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Lecture – 56
Discrete Time Fourier Transform: Definition, Inverse DTFT,.
Convergence, Relation between DTFT and z-Transform, DTFT of Common Signals

Hello welcome to another module in this massive open online course. So, far we are looking at the Fourier analysis of discretetime signals; particularly periodic discrete time signals and we have looked at the Fourier series representation of such signals.

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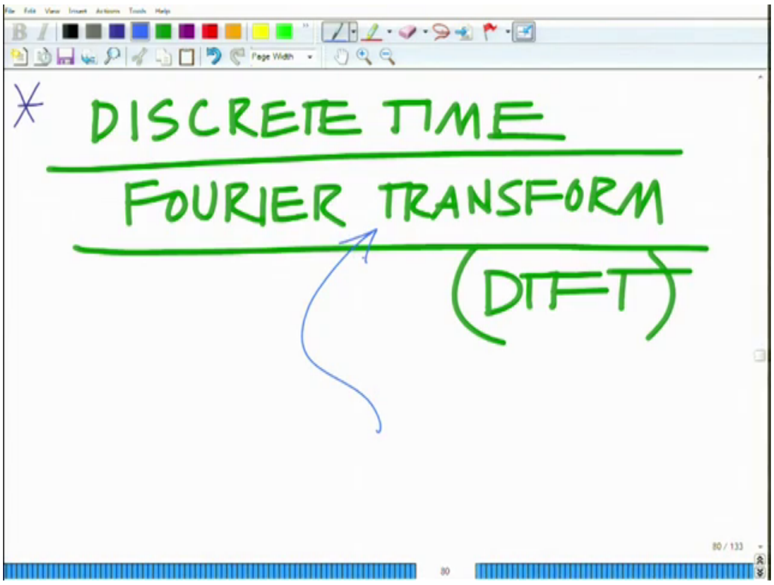

In this module we will start looking at the Fourier analysis for discrete time aperiodic signals through the discrete time Fourier transform. So, we will start talking about or we will discuss the discrete time Fourier transform and its various properties ok. So, want to start looking at the discrete time Fourier or basically what is also abbreviated as the DTFT the discrete time Fourier transform and as I already said this is basically used for the free analysis of discrete time aperiodic signals.

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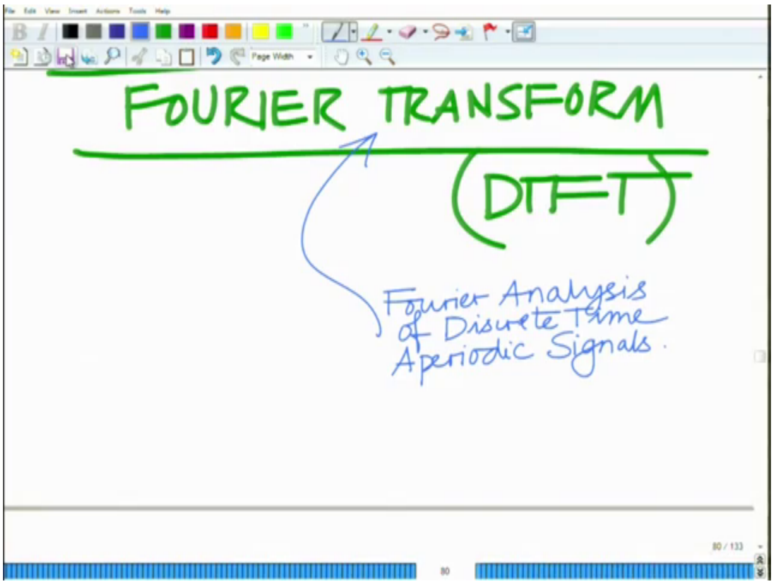

The Fourier analysis of discrete time the Fourier analysis of discrete time a periodic signals.

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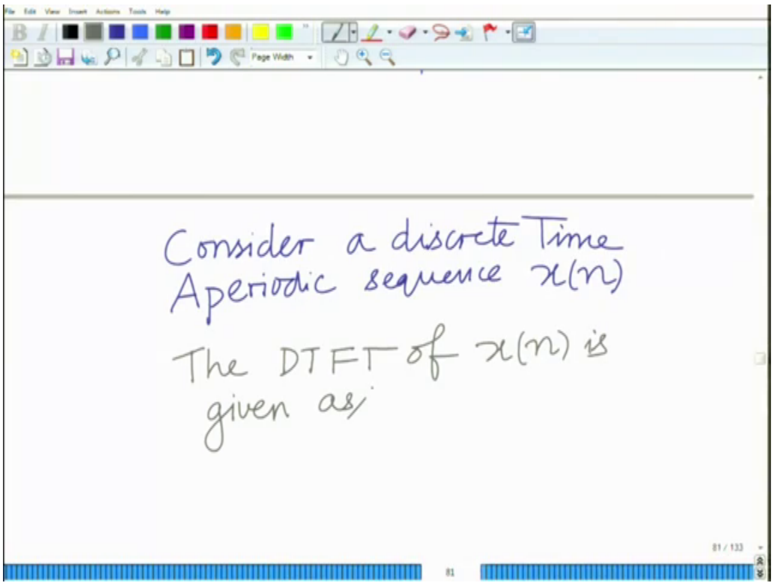

And so, we consider a discrete time sequence x n consider a discrete time aperiodic sequence alright. And the DTFT of this is given as the discrete time Fourier transform of x n is given as it is defined as X of omega equals summation n equals minus infinity to infinity x n e raised to minus j omega n.

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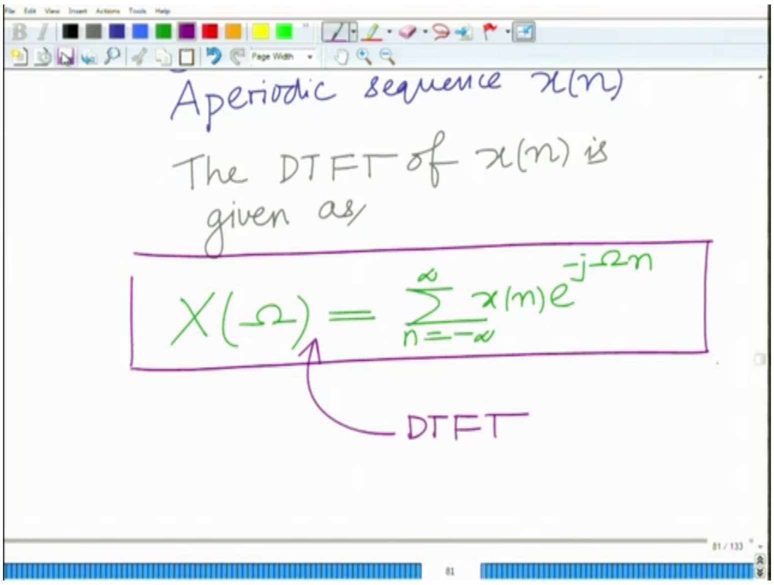

This is the expression for the DTFT the discrete time Fourier transform; so this is the expression for the DTFT ok. And further one can derive the inverse DTFT as follows.

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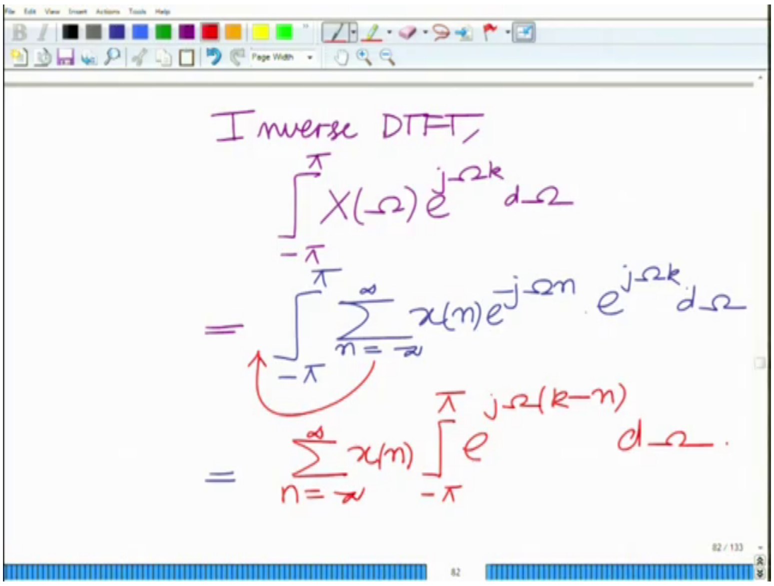

To find the inverse DTFT we start with integral over pi let us say minus pi to pi of X of omega e raised to j omega k d omega. I can simplify this as integral minus pi to pi substitute the expression of X of omega that is n equals minus infinity to infinity; x n e raised to minus j omega n, d omega e raised to minus j omega n into e raised to j omega k d omega.

Now interchanging the integral and the summation as usual we have bringing the summation outside this will be summation k n equals minus infinity to infinity minus infinity to infinity. And x n will also come outside since it depends only on n integral minus pi to pi e raised to j omega; j omega k minus n d omega and now you can see this is a complex exponential this e raised to j omega n.

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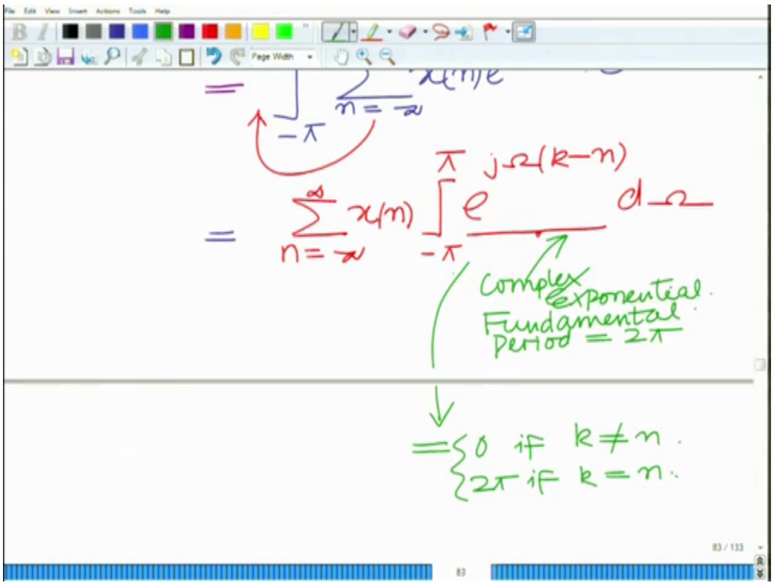

This is a complex exponential and the e raise to j omega the fundamental period is period equals 2 pi of e raised to j omega. So, e raised to j omega k minus n is basically a harmonic correct harmonic there is a multiple of the fundamental frequency which is 1 radian per second.

So, if you integrate it over any fundamental period that is over minus pi to pi the integral is going to that is you take in any harmonic of omega integrated of over minus pi to pi correct its integral is going to be 0 except when k is equal to n in which case e raised to j omega k minus n is 1 when k equals n; so, this is equal to. So, if; so, in this case; so, this integral this is equal to 0, if k is not equal to n if k is not equal to and this is equal to basically 2 pi if k is equal to n.

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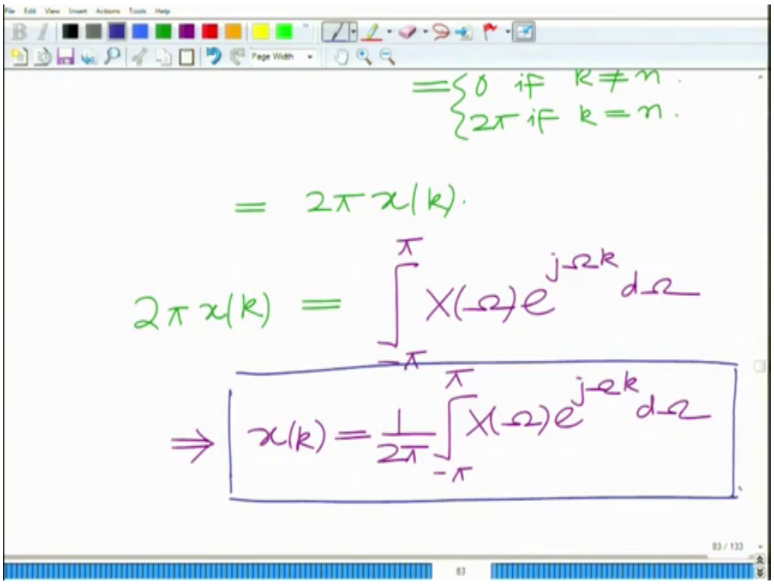

So, therefore, this only survives the integral only survives where k equal to n and for that it is two pi. So, what you get is 2 pi x k. So, we have basically to extract this element x k we have 2 pi x k equals integral minus pi over pi X of omega e raised to j omega k d omega which implies that x of k that is the inverse Fourier trans inverse discrete Fourier transform is given as one over 2 pi minus pi to pi; X of omega e raised to j omega k d omega this is basically the expression for the inverse discrete Fourier transform ok.

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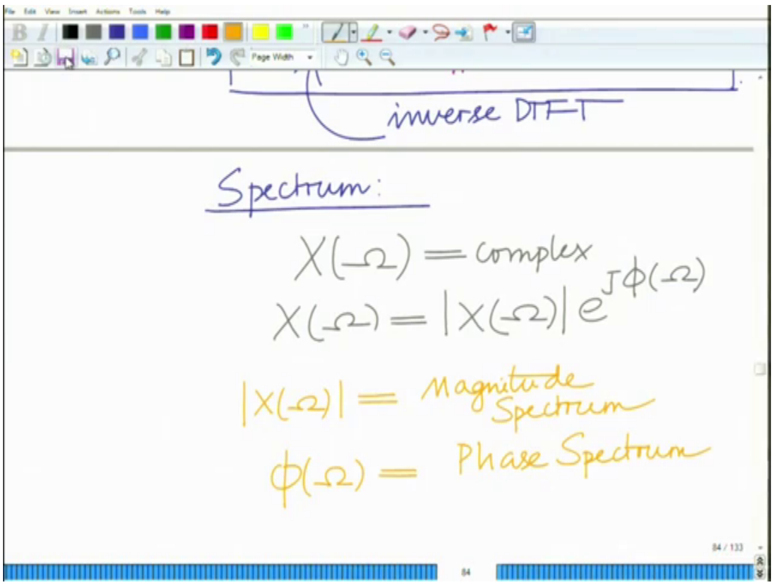

This is the expression for that inverse discrete Fourier transform alright. So, this basically gives us the expression for both the discrete time Fourier transform, as well as the inverse discrete time Fourier transform.

Alright and this is for a discrete time a periodic sequence x ok. Let us look at some of the properties of this DTFT let us first start with the spectrum now the spectrum X omega in general you can see that it is complex; the spectrum X omega is complex. And I can represent X of omega as magnitude X of omega into e raised to J phi of omega. This quantity magnitude X of omega this is the magnitude spectrum this gives us the magnitude spectrum. And the quantity phi of omega this is the phase spectrum; so, we have the magnitude spectrum, we have the phase spectrum.

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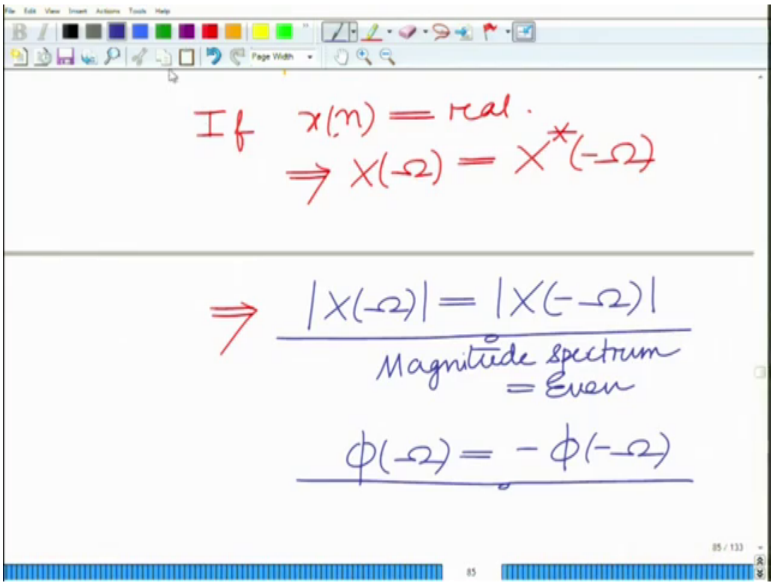

Now, further if x n is real if the sequence x n is real now this implies when x n is real we have X of omega equals X conjugate of minus omega similar to what we have seen for the Fourier transform also; X of omega is x conjugate of minus omega. This further implies that basically your magnitude of X of omega is magnitude X of minus omega because they are conjugate of each other. So, basically this means that the magnitude spectrum for real signals magnitude spectrum equals is even.

Further, if you look at the face spectrum now you can see that the phase of omega because your conjugate phase of omega is minus of phase of minus omega. So, the phase spectrum is odd the phase spectrum exhibits odd symmetry this is for a real signal ok.

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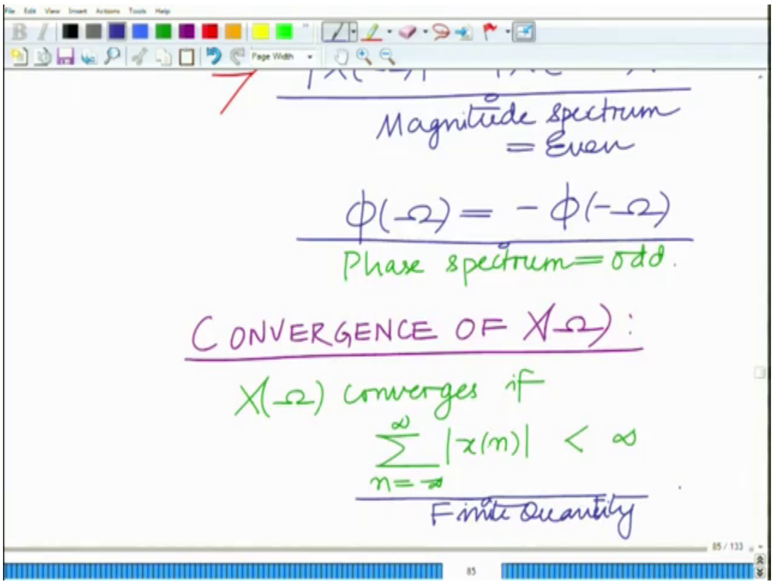

So, for a real signal as usual we have the magnitude spectrum of the DTFT that exhibits even symmetry and the phase spectrum of the DTFT that is that exhibits odd symmetry ok. And further a brief note about the convergence it is also interesting to look at the convergence.

Now the condition for convergence of X of omega X of omega converges; if that is a sufficient condition summation n equal to minus infinity minus infinity x n is less than infinity that is the sequences signal is that is this is a finite quantity implies the sequence is absolutely summable that is if you take the magnitude of each element sum from n equal to minus infinity to infinity that is a finite quantity which means sequence or signal is absolutely summable the signal is absolutely summable.

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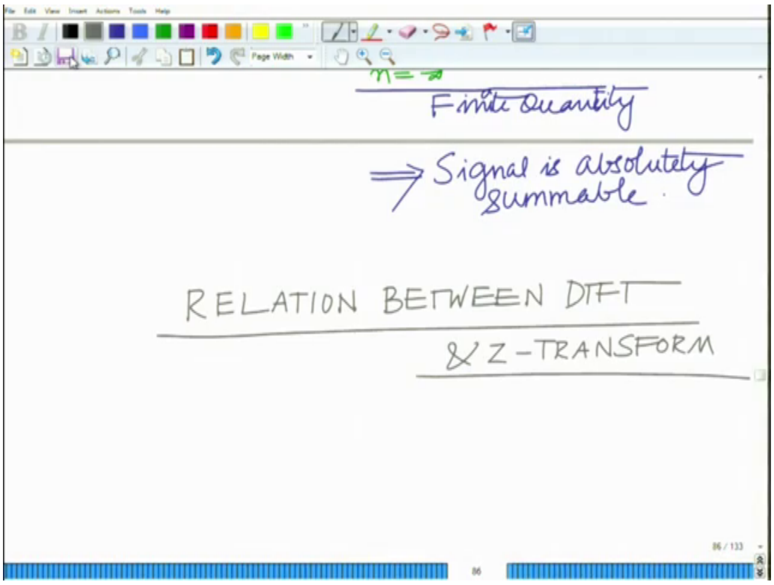

So, the absolute sum of the signal exists that is sum of the magnitudes of all terms from minus infinity to infinity then the DTFT is guaranteed to exist alright.

Let us look at the relationship between the Fourier transform and the Z transform and of course, one other thing that you can realize that is fine. So, let us look at the relation between the Fourier transform and the Z transform or the DTFT relation between the DTFT and Z transform. Now you can see that the Fourier DTFT is given as summation n equals minus infinity to infinity x of n; e raised to minus j omega n. And the Z transform x of Z equals n equals minus infinity to infinity x of n Z raised to minus n.

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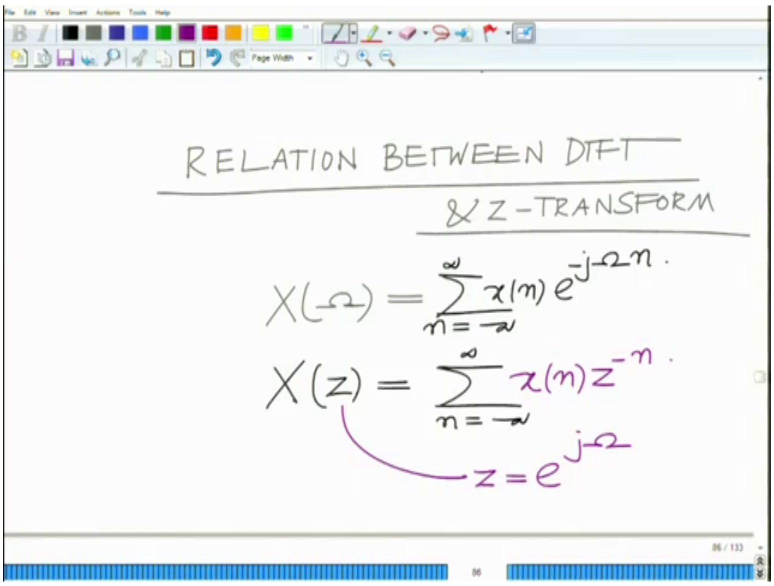

This implies you can see that if I substitute Z equals e raised to minus j omega in the Z transform that is substitute not e raised to minus j omega. In fact, if I substitute Z equals e raised to j omega correct; if I substitute Z as to Z equals e raised to j omega then the Z transform becomes the Fourier transform.

But remember for the Z transform to converge Z equal to e raised to j omega has to be in the ROC of the Z transform only then can I substitute Z equal to e raised to j omega in the Z transform to derive the DTFT ok. So, that is an important; it is an interesting observation, but so, very important to realize that exists only if e raised to j omega is in the ROC that is the region of convergence of the Z transform.

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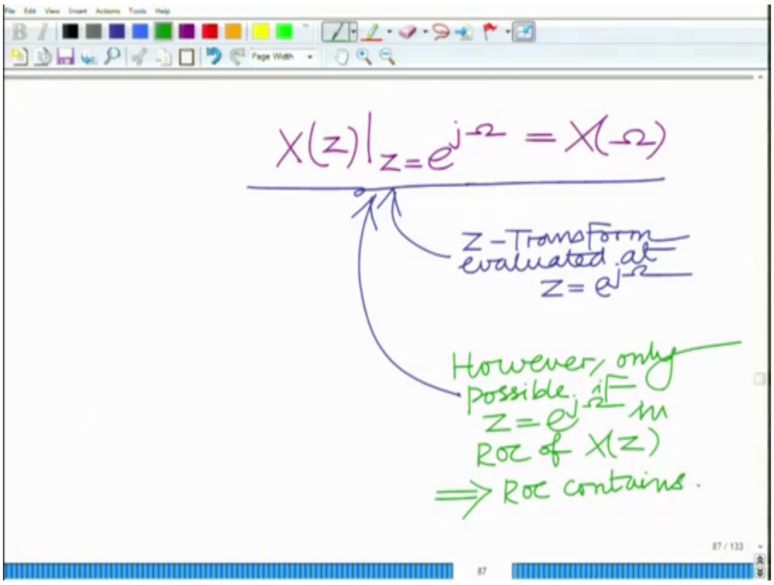

So, the Z transform evaluated at Z equal to e raised to j omega that gives the DTFT; however, this is basically your Z transform; however, is only true only possible if e raised to j omega in ROC of X Z implies ROC contains the implies.

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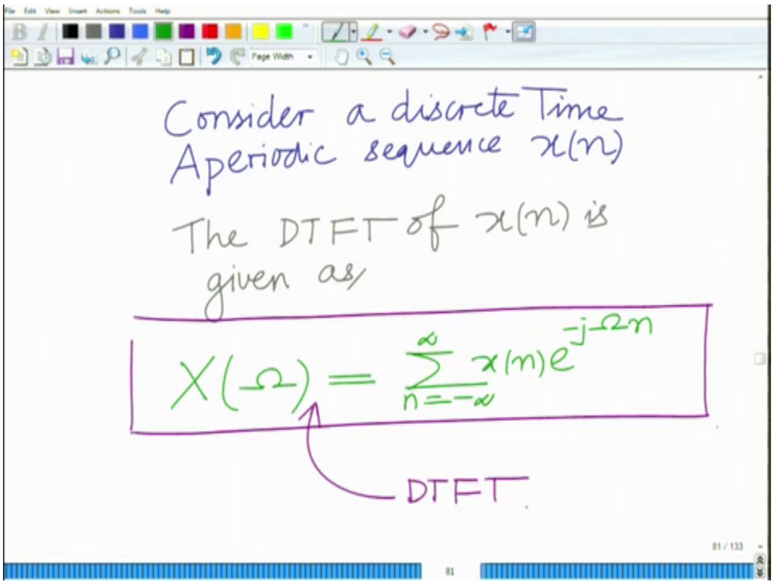

That basically your ROC has to contain the, implies ROC contains the unit circle.

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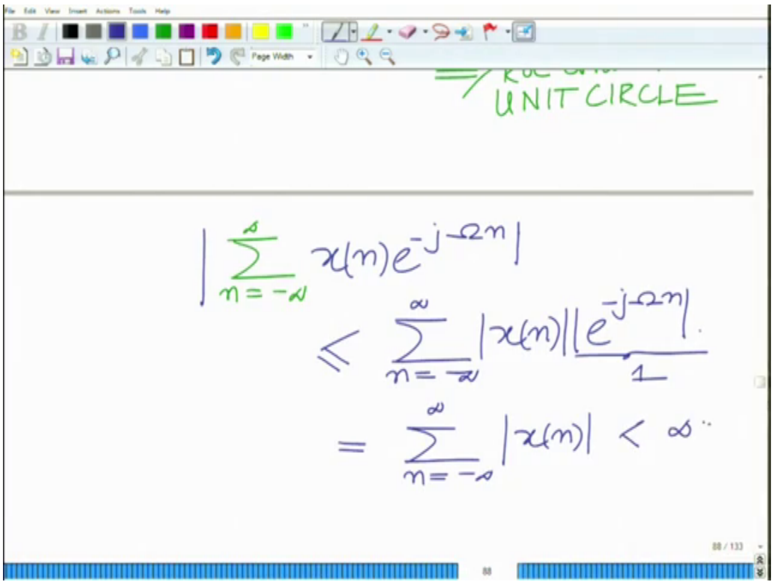

So, if the ROC contains the unit circle if the ROC contains the unit circle; then and only then can the DTFT of we obtained from the Z transform by the substitution of by replacing or by substituting Z equal to e raised to j omega ok.

And you can see that the ROC contains the unit a unit circle if the sequence is absolutely summable. For instance if you look at summation n equal to minus infinity to infinity x n e raised to j omega n; the magnitude of this is less than or equal to magnitude of sum is less than or equal to sum of magnitude e x n magnitude e raised to minus j omega, but magnitude raised to minus j omega and this is 1.

This is summation n equals minus infinity to infinity magnitude of which is less than infinity if sequences absolutely summable.

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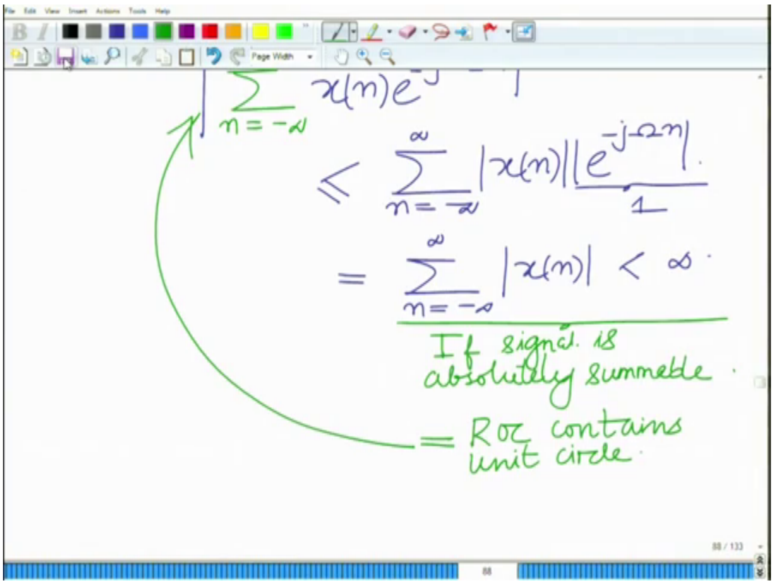

If signal is; if the signal is absolutely summable, then this is less than infinity implies implies ROC implies; the ROC contains the unit circle if because for every e raised to j omega that is where for every Z equal to e raised to j omega the sum the magnitude of this sum magnitude sum n equal to minus infinity to infinity x n e raised to minus j omega n that is the magnitude is finite.

So, basically the quantity is a finite quantity alright and therefore, the ROC contains the unit circle ok. And not generally true this is not generally true and therefore, this the substitution of Z equal to e raised to j omega to yield the DTFT from the obtain the DTFT from the Z transform does not generally hold true unless your sequence x n does not generally hold true if the sequence x x n is not absolutely summable.

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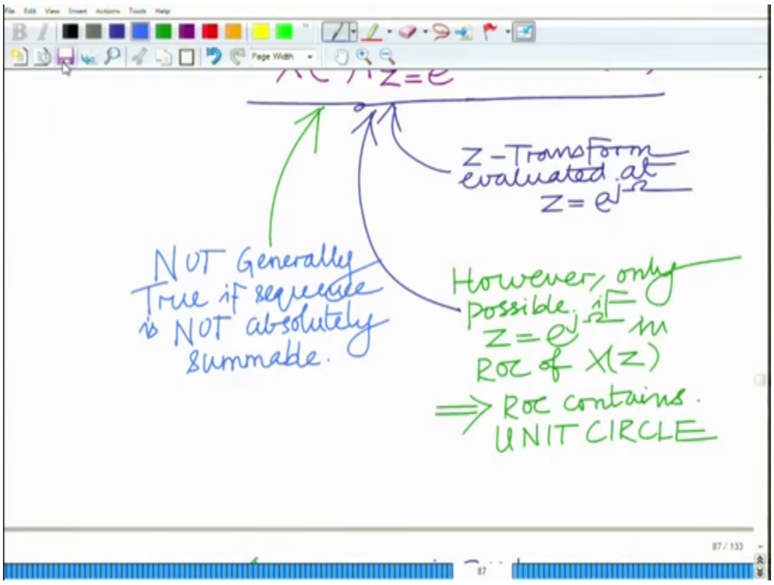

So, this is not generally true if sequence is not if the sequence is not absolutely summable ok. Now let us look at the DTFT of some common signals.

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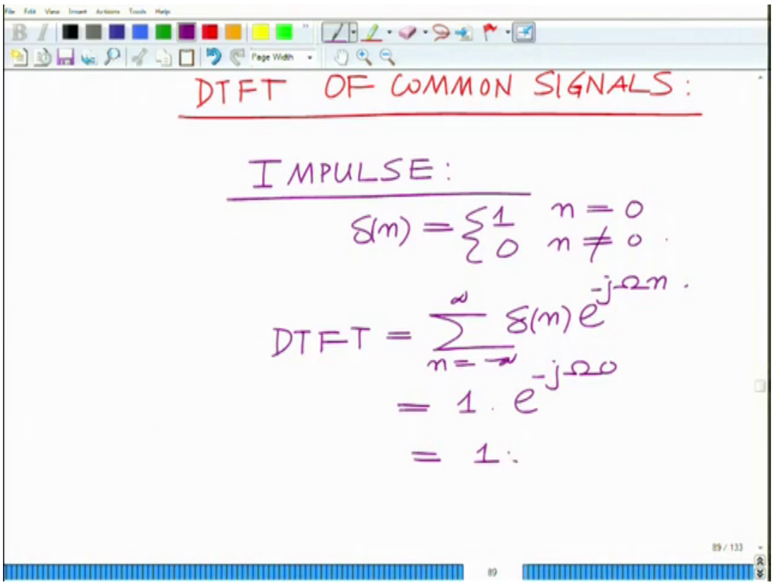

So, first we start with the impulse the discrete time impulse delta n equals to 1 for n equal to 0; this is 0 for n not equal to 0 this is very simple function the Z transform or the DTFT that is simply n equals minus infinity to infinity; delta n e raised to minus j omega and this is only nonzero only for n equal to 0 at which it is 1 e raised to minus j omega 0 which is 1.

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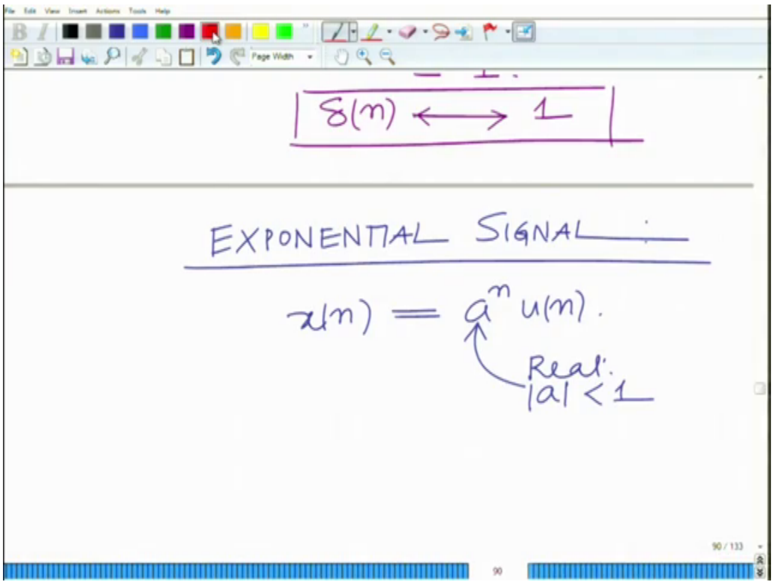

So, delta n has DTFT that is the unit the unit impulse is DTFT which is unity ok. And let us look at the causal exponential signal; now when we look at the exponential signal, we consider a decaying exponential that is a is real first. Let us consider a simple scenario where is real and also magnitude of a is less than 1 which means that basically it is decaying as n is increasing. And finally, as n tends to infinity the signal tends to 0 as magnitude a is less than 1.

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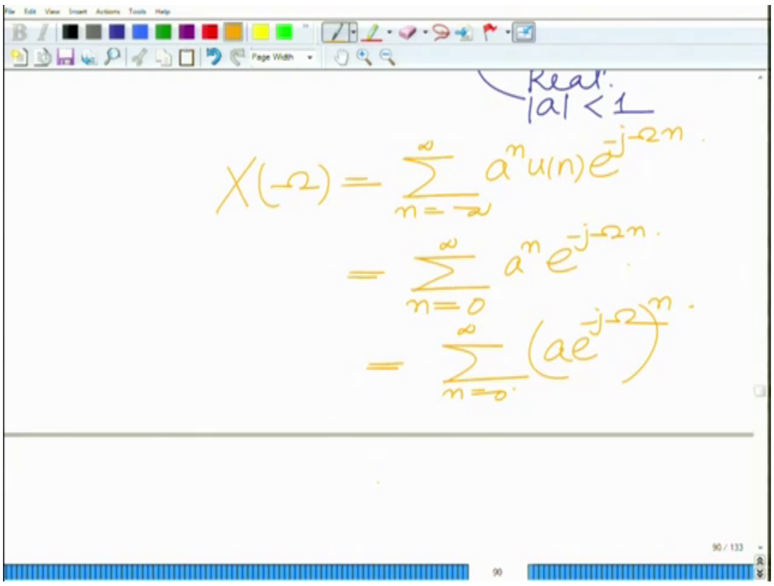

Now, for this we have basically your DTFT is X of omega equals n equals minus infinity to infinity; a n e raised to minus j omega and u n is nonzero only for n greater than equal to 0 and for n greater than equal to 0 it is 1; this simply reduces to n equals 0 to infinity a n sorry a n e raised to minus j omega n which is sum n equal to 0 to infinity a e raised to minus j omega n which is 1 over 1 minus a e raised to minus j omega.

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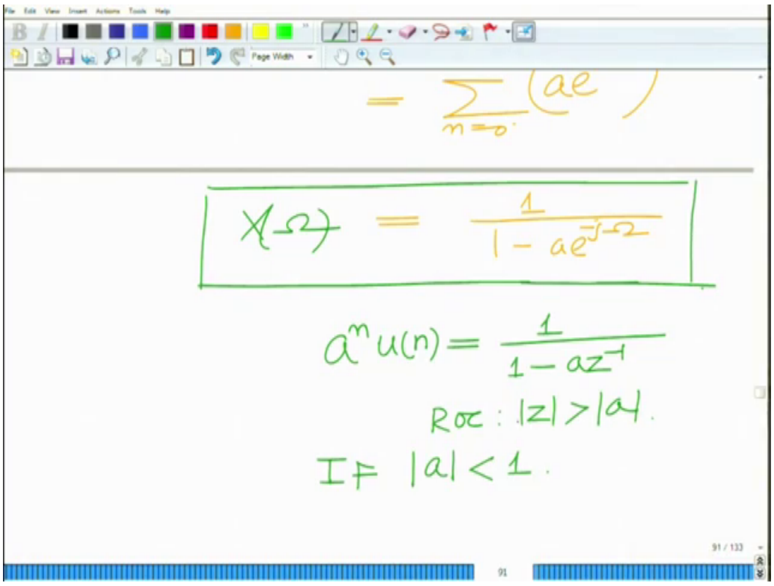

So, X of omega you can see is given by this expression 1 over 1 minus a e raised to minus j omega. And further you can see the Z transform of this is also given as a n u n called 1 over 1 minus a Z inverse and the ROC remember we already derived ROC is magnitude Z greater than magnitude a. Now if magnitude a is greater than 1 or I am sorry if magnitude a is less than 1.

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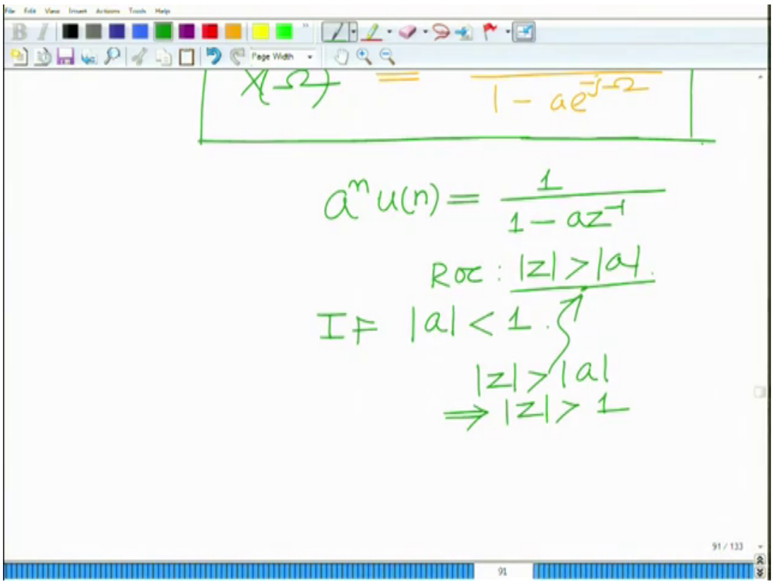

Now magnitude of Z greater than magnitude of a. implies magnitude of a that is ROC is greater than basically the unit circle what this means is. So, we have the magnitude of Z is greater than so, ROC includes everything that is magnitude of c. So, magnitude of a is less than 1 correct; so, ROC includes the unit circle and this can be seen as follows.

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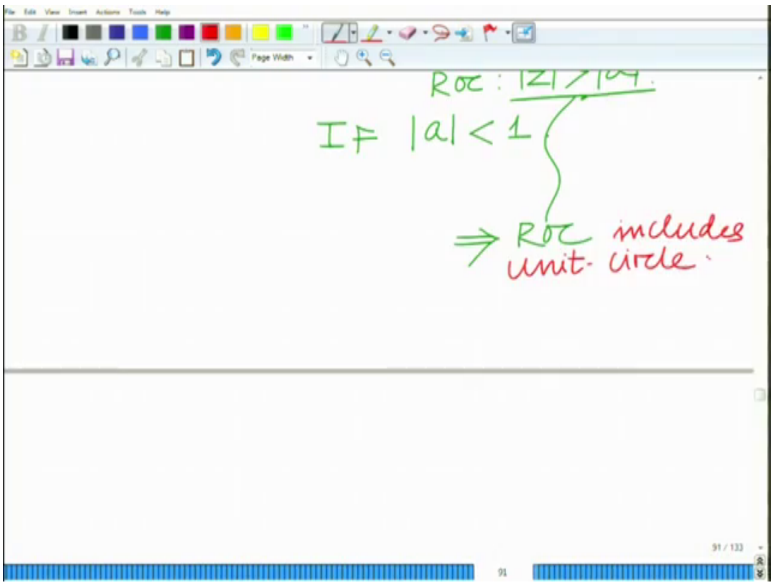

This implies ROC includes this can be seen as follows.

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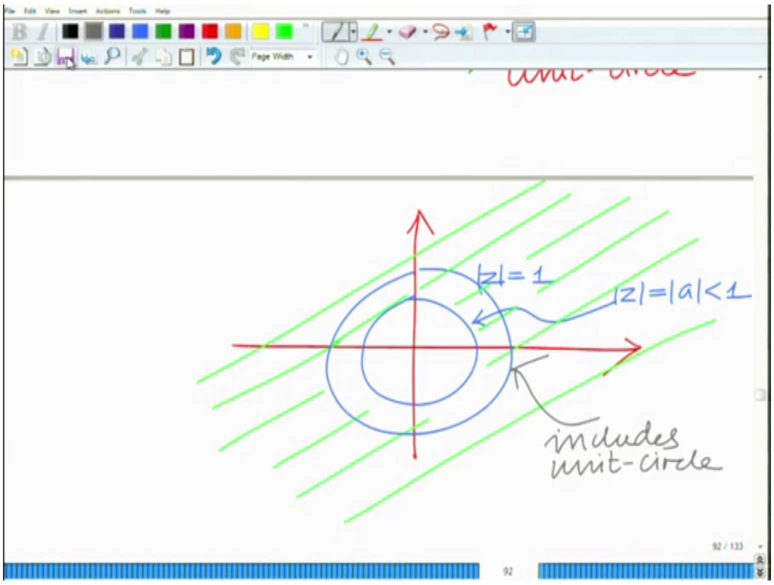

For instance, you have your this thing and you have your circle. So, this is your magnitude Z equals 1, this is magnitude Z equals magnitude of a less than 1. So, your ROC is basically magnitude of Z greater than magnitude of a, which is basically everything outside the circle which is everything outside everything outside the circle which means everything out outside a circle.

And you can see it includes the this includes the; so, it is everything outside the circle. So, you can see this includes the unit circle; so if. So, if magnitude a is less than 1 correct then the ROC of a raised to n u n the ROC of that signal basically includes the unit circle. And therefore, now if you replace the Z transform that is if I look at the Z transform.

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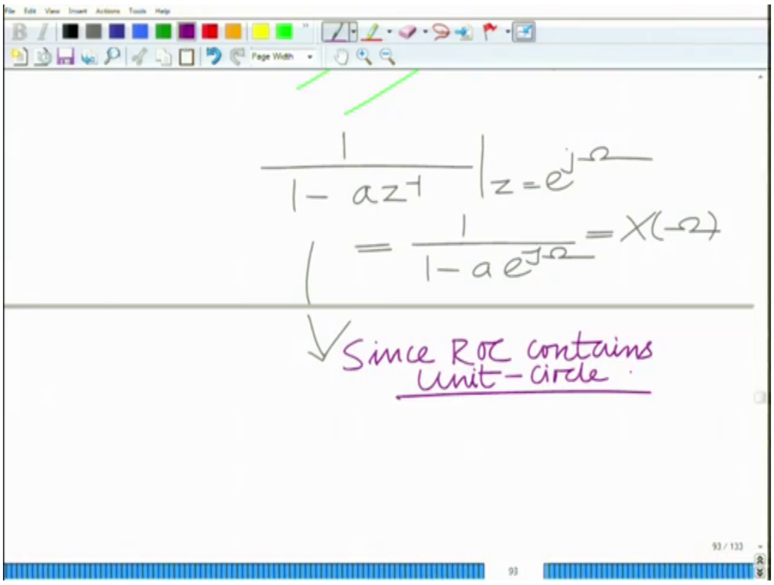

1 minus a and if I replace Z by e raised to j omega I obtain 1 minus a raised to j omega 1 over 1 minus a e raise to minus j omega which is DTFT. So, since ROC; so, this is possible since ROC contains since ROC contains the unit circle.

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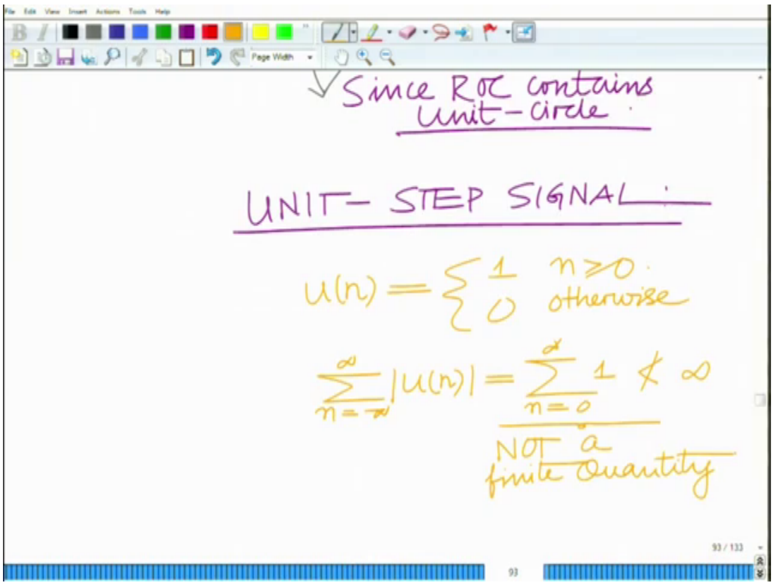

Now, on the other hand if you look at the unit step signal correct consider the. Now if you consider the unit step signal u n equals 1; n greater than equal to 0 and n equal to 0 otherwise. Now this sequence you can see is not absolutely summable n equals minus infinity to infinity; magnitude u n equals summation n equals 0 to infinity of 1, which is not a finite quantity this is not; this is not a finite quantity implies you cannot.

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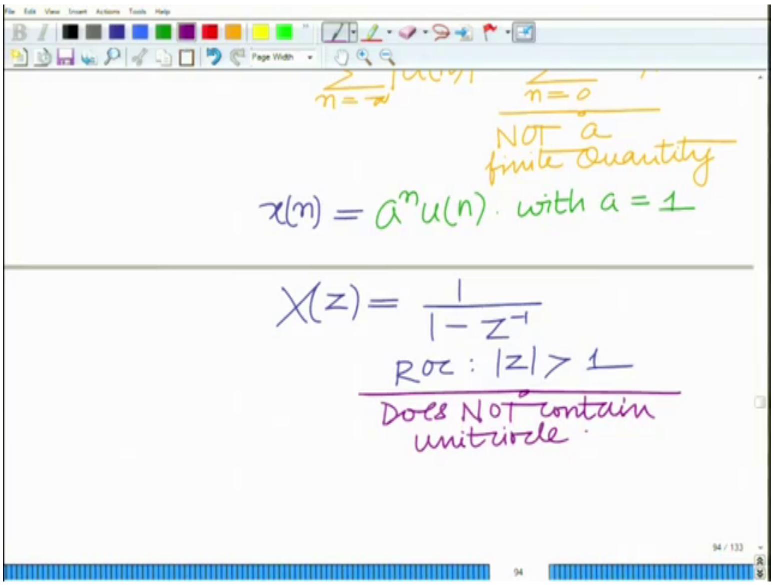

So, this is of the form a raised to n u n where a equal to 1 with a equals. So, magnitude of a is not less than, but rather magnitude of a equals 1 which means the ROC does not contain. So, ROC is magnitude of Z greater than magnitude of a alright. So, the ROC does not contain the unit circle in this case. So, for if I look at x n; X Z equals with a equal to 1; X Z will becomes 1 minus Z or 1 minus Z inverse and the ROC of this magnitude of Z greater than 1 does not which does not include unit circle.

Hence you cannot obtain the DTFT by simple substitution of Z equal to e raised to j omega.

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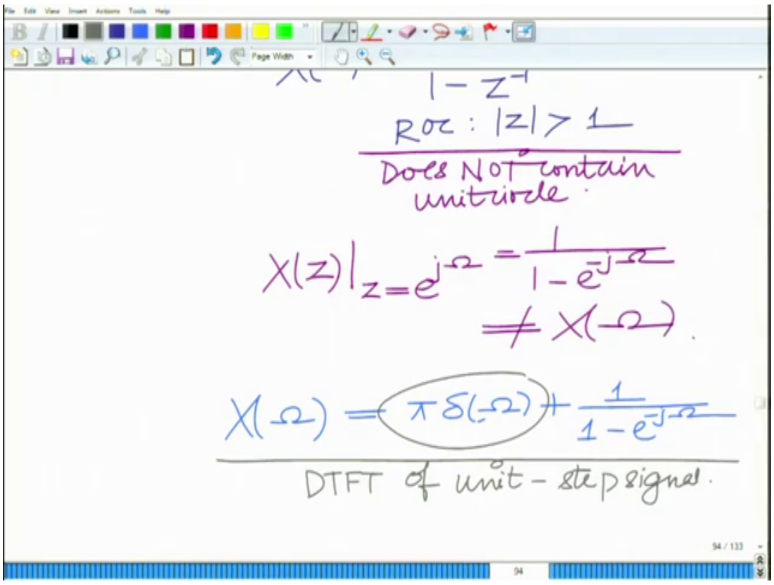

Hence if you take that is 1 over 1 minus e raised to minus j omega that is not equal to X omega. So, DTFT cannot be obtained by substitution offer the unit step signal cannot be obtained by simple substitution of Z equal to e raised to j omega in the Z transform because the ROC does not contain the unit circle.

In fact, you can show that X of omega for the unit step signal is given as something that is slightly different that is pi of delta omega plus 1 over 1 minus e raised to minus j omega this is the DTFT discreet time unit step signal ok. So, this is the DTFT of the discrete time unit step signal; that is it contains this additional term you can see this is pi times delta of omega, there is an impulse at omega equals 0 plus 1 over 1 minus e raised to minus j omega alright.

So, in this module we have looked at several aspects, we looked at the definition of the DTFT the inverse DTFT and also the convergence issues related to the DTFT and the DTFT of some of the commonly occurring or frequently occurring discrete time signals. So, we will stop here and continue in subsequent modules.

Thank you very much.