Principles of Signals and Systems
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Lecture - 55
Fourier Analysis of Discrete Time Signals – Duality, Parseval's Theorem

Hello, welcome to another module in this massive open online course. So, you are looking at the Fourier analysis for discrete time signals. In particular we are looking at the discrete Fourier series. So, let us continue our discussion on the discrete Fourier series.

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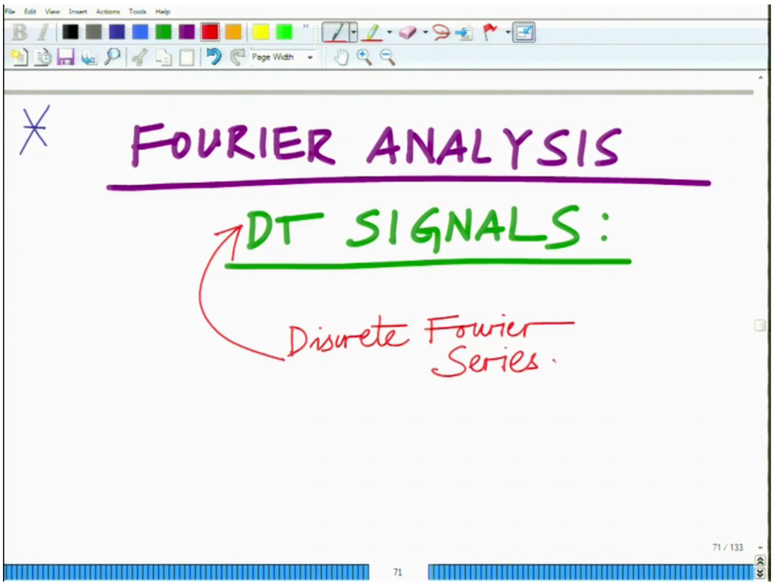

So, we are looking at the Fourier analysis of discrete time signals. In particular we are looking at what is known as the discrete Fourier series.

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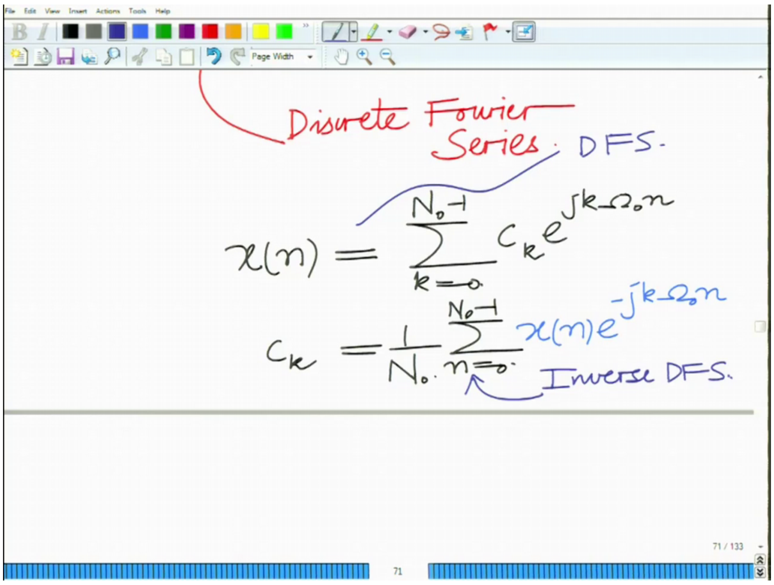

In particular we are looking at the discrete Fourier series where a signal x of n can be expressed as periodic signal periodic discrete time signal, can we express this x n can be expressed as summation k equal to 0 period N naught that is a periodic discrete time signal with period N naught can be expressed as summation k equal to 0 to N naught minus 1 C k e raised to j k omega naught N ok. And similarly the inverse discrete Fourier series that is the coefficients C k can be derived as 1 over N naught summation n equal to 0 to N naught minus 1 x N e raised to minus j k omega naught N.

So, this is your discrete Fourier series. This is your inverse discrete Fourier series that is to derive the coefficient C k. Let us now, look at another interesting aspect of this discrete Fourier series which is basically the duality property of the discrete Fourier series.

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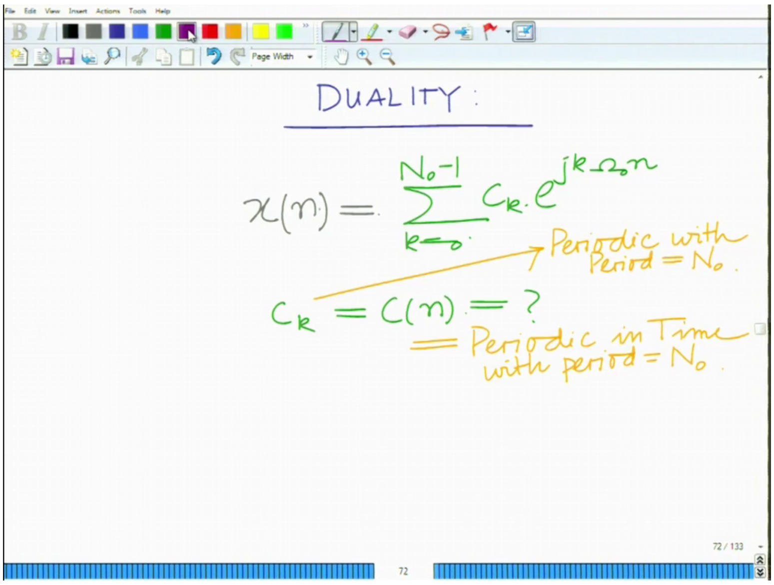

So, duality, so remember we have x k as we already said the discrete Fourier series is x n equals summation k equals N naught minus 1 C k e raised to j k omega n. What can we say about the sequence with C k equals C n, that is what can we say about this sequence that is when, remember duality is about when you have for instance when the Fourier transform when a signal small x of t as a Fourier transform capital X of x, X of f what happens or capital X of omega. What is the Fourier transform of the signal capital X of t that is what that is the frequency domain becomes a time domain what is the corresponding Fourier transform of the signal.

Similarly if the sequence C k becomes a time domain signal. Now, remember we already said C k is periodic remember that is what we have derived in the previous module C k is periodic with period equals N naught implies that the time domain signal, C n is also periodic in time with again period equals N naught.

Now, further if you look at this you have x n equals summation C k e raised to j k omega naught n.

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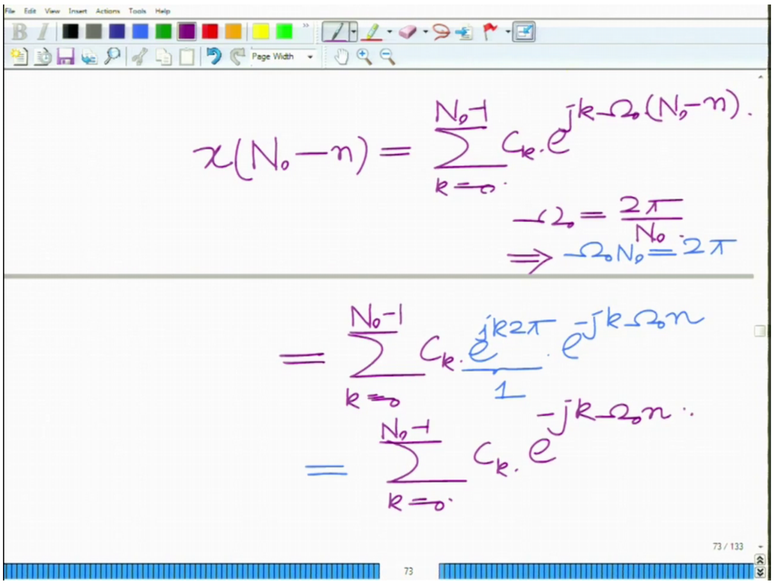

If you consider x of N naught minus n then that is summation k equal to 0 to N naught minus 1 C k e raised to j k omega naught N naught minus n which is equal to summation k equal to 0 to N naught minus 1 C k e raised to j k omega naught N naught naught omega naught recall is 2 pi over N naught which implies omega naught N naught is equal to omega naught N naught equals 2 pi. So, this reduces to C k e raised to j k omega naught N naught or e raised j k 2 pi into e raised to minus j k omega naught n. And we know e raised to j k omega 2 pi e raised to j k 2 pi is 1 for any integer k. So, therefore, this quantity here is 1 and what we are left with is we are left with this quantity k equal to 0 to N naught minus 1 C k erased to minus j k omega naught n.

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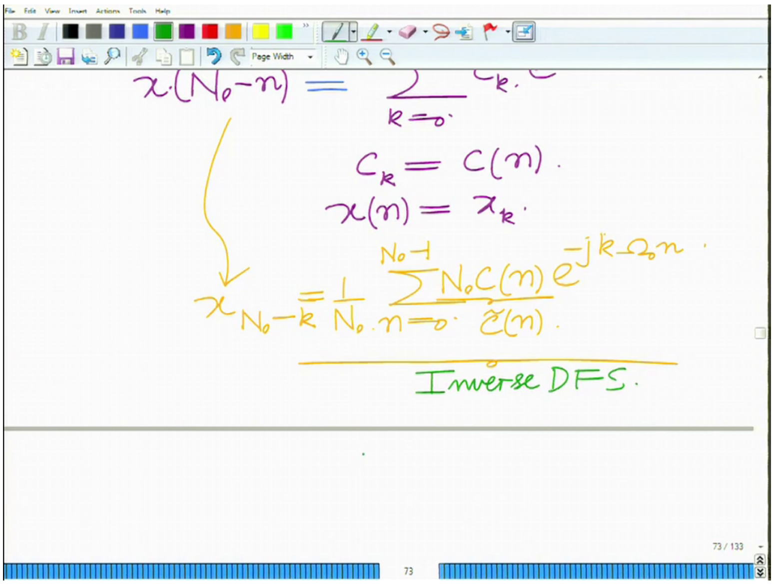

So, now, x of N naught minus n correct x of N naught minus n is C k e raised to minus j omega naught n. Now, we make C k equal to C n and we make x n equals x k that is the previous x n equals x k then what we have from this statement is basically that x of N naught minus k equals remember k becomes n k equal to 0 summation n equal to 0 to N naught minus 1 C n e raise to minus j k omega n.

Now, if you look at this summation C n e raise to minus j k omega naught n this is similar to this summation n equal to 0 to N naught minus 1 x n e raise to minus j k omega naught n except for this factor one over N naught. So, what I am going to do is I am going to multiply and divide by this factor N naught. So, now, you have 1 over N naught times N naught and now, you can see if I call this as this C tilde of n. Now, you can see that x of that is C tilde of n that is this is submit this is basically identical to the inverse discrete Fourier series. This is the identica equal identical to the inverse discrete Fourier series where you are basically coefficient x N naught minus k that is C tilde of n has basically the discrete Fourier series coefficients x N naught minus k which is equal to x of minus k because remember x is a periodic sequence x of minus k ok.

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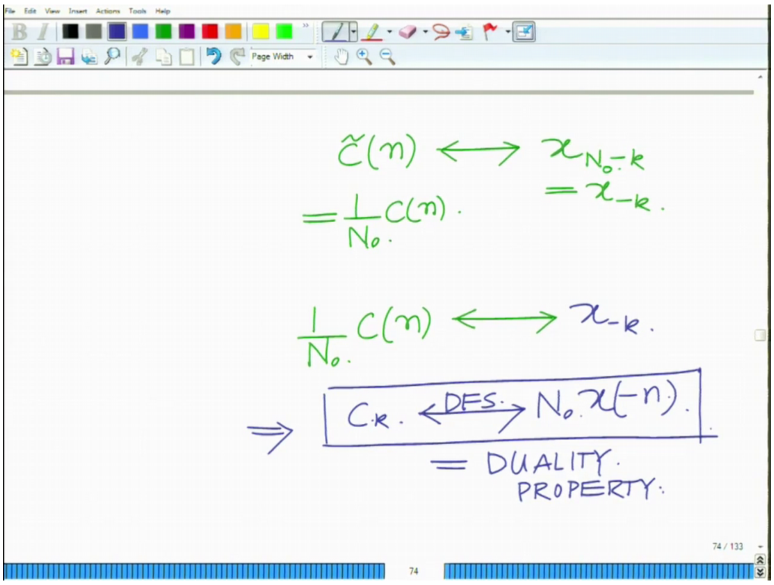

So, basically if you look at that you can say C tilde n or basically if C tilde n equals remember this is one over N naught C of n. So, 1 over N naught C of n has basically the discrete Fourier series x of minus k or going back to the original notation C of n or C of k we take the N naught to the other side is basically has the discrete Fourier series representation that is N naught x of minus k. This is the duality property. Or x of this is basically x of or C of n is basically as the discrete Fourier series coefficients x of minus N naught times x of minus and that is if C k's become a time domain signal then the corresponding discrete Fourier series coefficients on the discrete Fourier series are given by N naught times x of minus n. So, this basically is the duality property.

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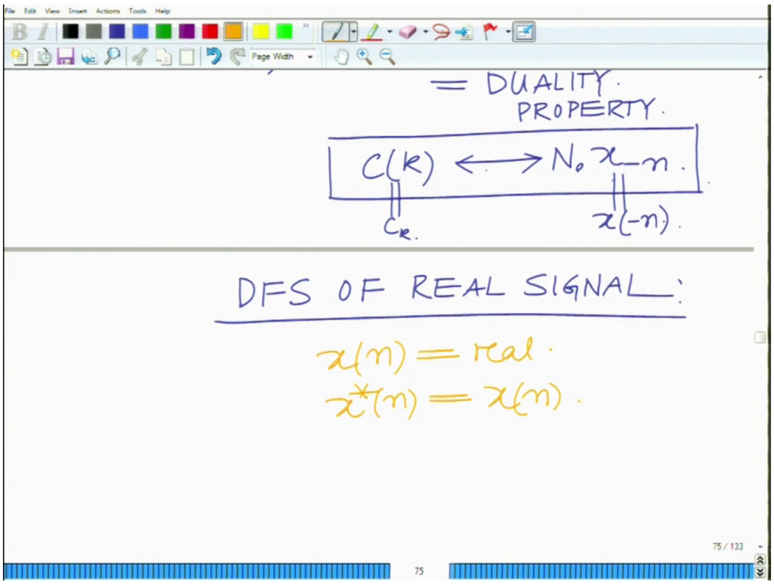

Or you can also write it since we are denoting time domain by the brackets you can also write C of k in the time domain becomes N naught x of minus and depending on which notation you are comfortable with you can appropriately write this thing ok. Both this thing means one in the same C of n equals C of k, x of minus n equals x subscript minus n equals x of minus n ok. So, that is basically the duality property for the discrete Fourier series.

Now, what happens when we consider a real signal? Let us consider the property of a DFS the discrete Fourier series of a for a real signal x n when x n is a real signal when x n is real which implies x conjugate of n equals x n because x n is a real signal.

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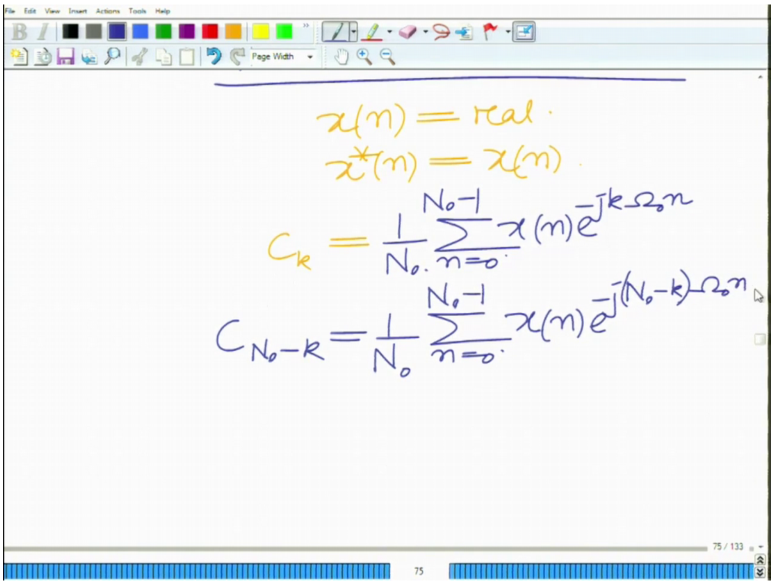

Then we have remember from the discrete Fourier series or from the inverse discrete Fourier series remember a C k is 1 over N naught summation n equal to 0 to N naught minus 1 x n e raise to minus j k omega naught n.

Now, C of N naught minus k this is equal to 1 over N naught n equal to 0 to N naught minus 1 x n e raise to minus j k omega naught or e raised to k is by N naught by minus k. So, it is 2 minus g N naught minus k omega naught n which is once again you can substitute this as 1 over N naught summation n equal to 0 to n minus 1 x n e raise to minus j N naught omega naught is again 2 pi e raise to minus j 2 pi n into e raise to minus j omega naught n.

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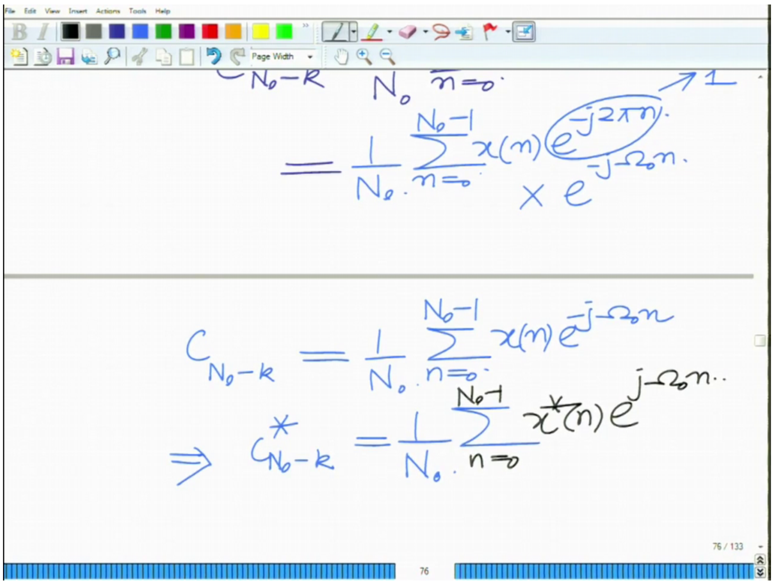

Now, this quantity is 1 e raise to minus j 2 pi n. So, this reduces to 1 over N naught summation n equal to 0 to N naught minus 1 x n e raise to minus j omega naught n this is C of N naught minus k, which means C of N naught minus k conjugate equals 1 over N naught summation n equal to 0 to N naught minus 1 x conjugate n e raised to minus j omega naught in conjugate that is it is 2 j omega naught.

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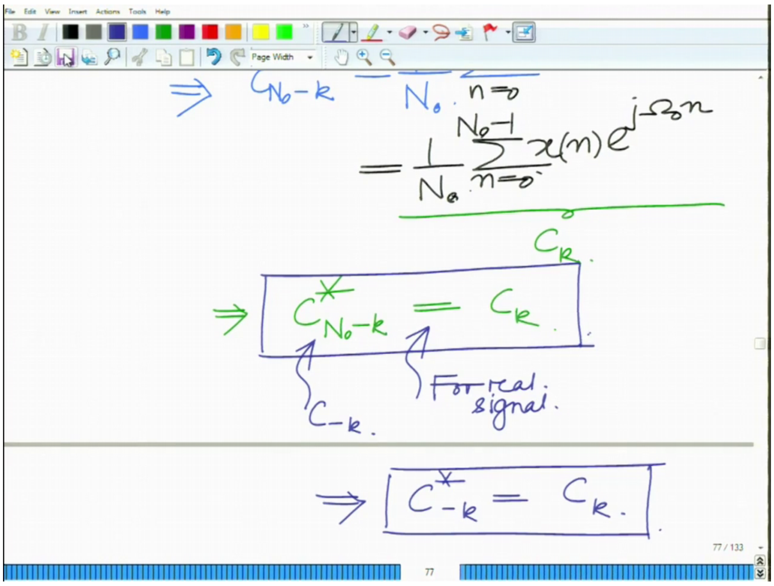

But x conjugate of n equals x n, so, for a real signal this is simply 1 over N naught summation n equal to 0 to N naught minus 1 x n e raised to j omega naught n, and if you look at this this is simply your C of k. So, C conjugate N naught minus k, so C conjugate N naught minus k equals C of k, so this is the property for a x this is the property for a real signal there is C conjugate of. And remember C of n, C of this is a periodic sequence or C conjugacy of N naught minus k is basically C of minus k. So, you can also write this as C conjugate of minus k equals C of k again this is for a periodic signal ok.

So, basically what that means, is that is C cross over it has the magnitude has an even symmetry. So, the C conjugate of minus k is basically C of k ok, alright and C conjugate of basically N naught minus k correct C conjugate of N naught C conjugate of N naught minus k is basically the same thing as C conjugate of minus k because its periodic with N naught all right. And finally, let us also look at one final property that is the Parseval's relation which is of course, an important.

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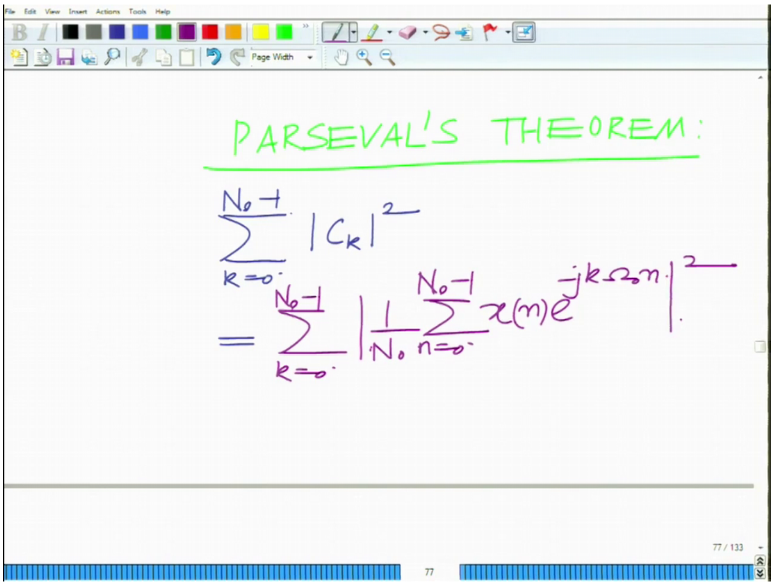

Parseval's theorem for the discrete Fourier series this is the Parseval's theorem energy of the DFS coefficient summation k equal to 0 to N naught minus 1 magnitude C k square.

Now, this is if I look at a corresponding time domain that is summation k equal to 0 to n minus 1 N naught minus 1. Now, I am going to replace the expression of the C k's by the inverse discrete Fourier series 1 over N naught summation n equal to 0 to N naught minus 1 x n e raise to minus j omega naught e raised to minus j k omega naught n whole square which is equal to summation k equal to 0 to N naught minus 1, 1 over N naught square. I am going to write this as the magnitude square is the quantity times its complex conjugate correct magnitude a square is a times a conjugate. So, I would write as the summation into its complex conjugate.

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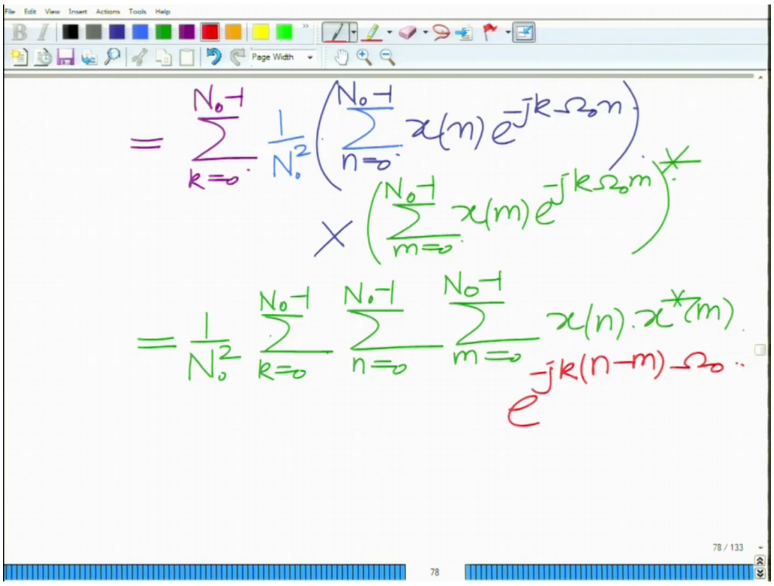

So, this summation n equal to 0 to N naught minus 1 x n e raise to minus j k omega naught n its complex conjugate I am just going to change the index and I am going to rewrite this as summation m equal to 0 to n minus 1 x n e raise to minus j k omega naught m complex conjugate, ok. Now, if I expand this I can bring the 1 over 1 over N naught square outside since its constant summation k equal to 0 to N naught minus 1 summation n equal to 0. So, there are going to be three summation n equal to 0 to N naught minus 1 m equal to 0 to N naught minus 1 times x n into x m conjugate. So, times x n into x m conjugate into e raised to minus j k all right, raised to minus j k n minus m omega naught which is equal to now, I interchange the summation.

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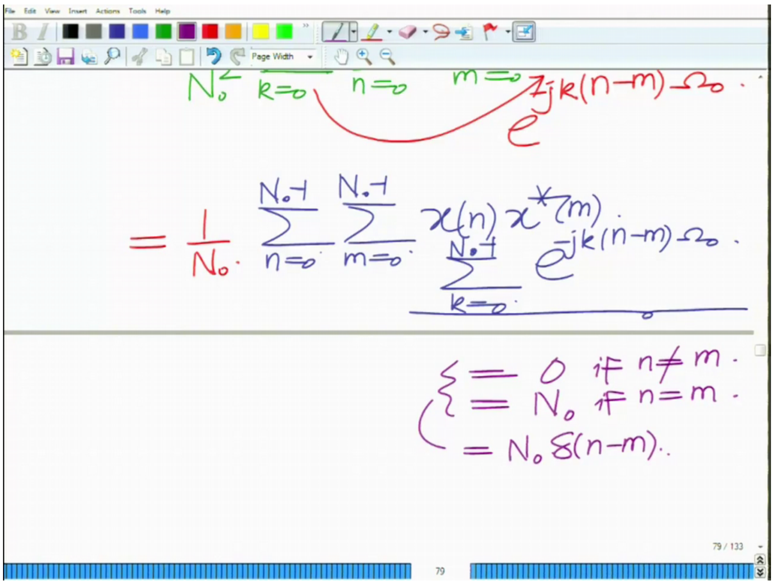

So, I will bring the summation with respect to k inside. So, that will give me summation n equal to 0 to N naught minus 1 summation m equal to 0 to N naught minus 1 summation k equal to, so now, this will be since x n and x m do not depend on k. So, I will bring the x n and x m as well outside and the last summation is summation with respect to k equal to 0 to N naught minus 1 e raised to minus j k n minus.

Now, this summation you can clearly see summation k equal to 0 to n minus 1 e raise to minus j k n minus m mu naught this is equal to remember is to minus j k n minus m omega naught over one fundamental period that is omega naught. The summation of any complex exponential which has fundamental period omega naught or some other fundamental period N naught over N naught is basically 0 if n is not equal to 0 is n is not equal to m and if it n is equal to m then simply it is simply summation 1 which is n naught.

So, this is equal to 0 N naught equal to m equal to N naught if n equal to m. So, effectively this is basically equal to N naught delta n minus m that is if n equal to m this is delta 0 which is 1. So, this N naught into 1 otherwise if n is not equal to m then this is delta n minus m which is 0.

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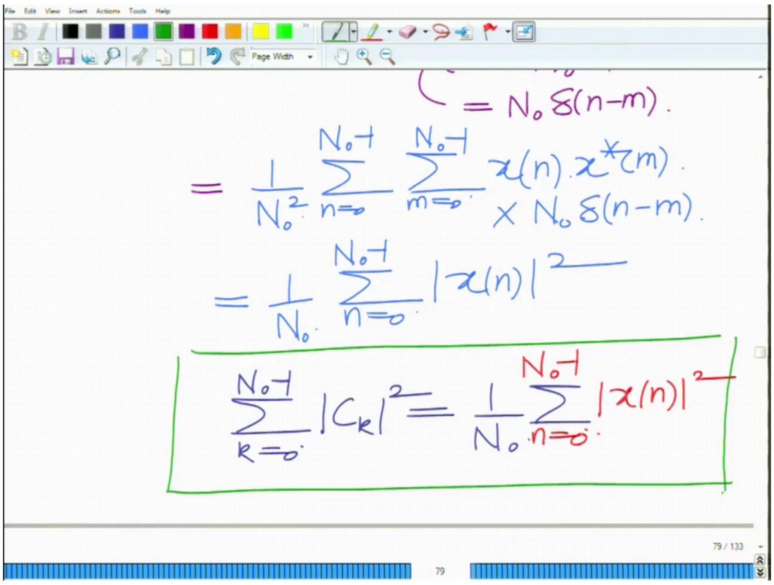

So, this is basically simply 1 over N naught summation n equal to 0 to n minus 1 or this is 1 over N naught square in fact, 1 over N naught square summation m equal to 0 to N naught minus 1 x n to x m conjugate into N naught delta n minus m.

So, the N naught over N naught square gives N naught summation n m the delta n minus m means only terms which n equal to m survive if n equals m then x n and x m conjugate is magnitude x m square. So, this is summation n equal to 0 to N naught magnitude x n square. So, that is the Parseval's relation. So, Parseval's relation tells us that basically, Parseval's relation tells us that for the discrete Fourier series summation k equal to 0 to N naught minus 1 magnitude C k square equals 1 over N naught summation k equal to 0 for summation n equal to 0 to N naught minus 1 magnitude x n square and this is your Parseval's relation. This is the par Parseval's relation for the discrete Fourier series alright.

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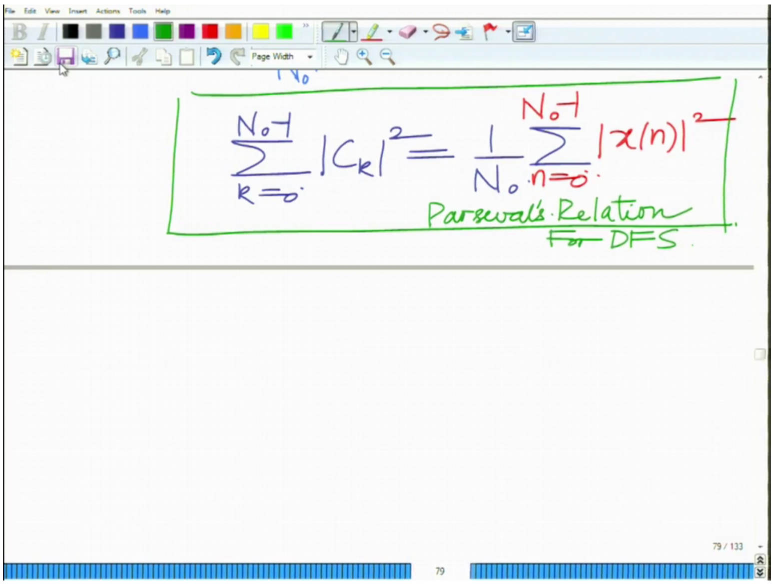

So, this basically completes our discussion of the discrete Fourier series alright. So, basically we have looked at the discrete Fourier series representation of a periodic discrete time signal with fundamental period N naught. We have looked at it several properties and also for instance such as the duality property if the Parseval's relation etcetera.

So, we will stop here, and start with another transform which is the discrete time Fourier transform for continuous for discrete type a periodic signals in a subsequent module.

Thank you very much.