Principles of Signals and Systems
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Lecture - 52
Sampling: Spectrum of Sampled Signal, Nyquist Criterion

Hello. Welcome to another module in this massive open online course. So, we are looking at sampling and considering impulse train based sampling which is the most common form of sampling a continue and in continuous time analog signal to convert it into a discrete time signal ok.

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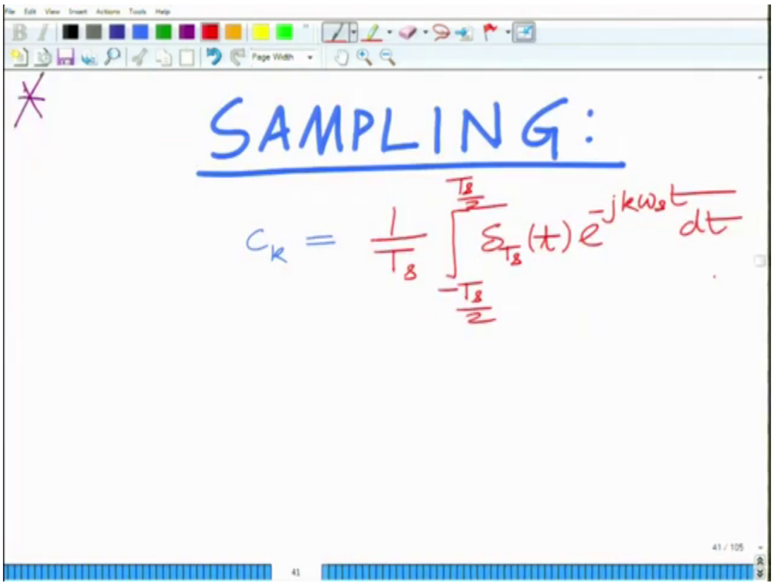

So, what we are looking at is we are looking at the principle or the theory behind sampling and try to trying to understand it is various aspects or various properties of sampling using the Fourier transform and what we are using is the impulse train correct to sample the signal.

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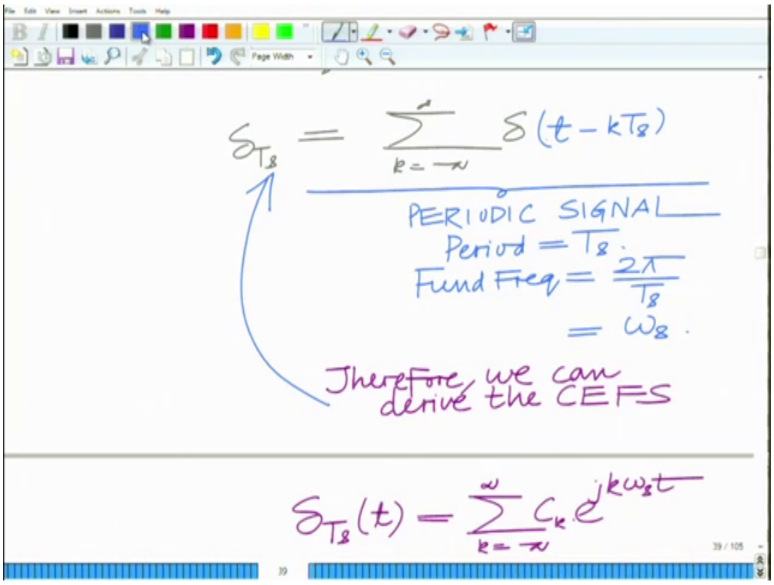

And we said the impulse train is a periodic signal delta Ts, summation delta t minus k Ts k going from minus infinity to infinity Ts, this quantity is known as the sampling duration of the sampling interval 1 over Ts which is fs is known as the sampling frequency, 2 pi over Ts that is omega s is the angular frequency of sampling in radians per second ok.

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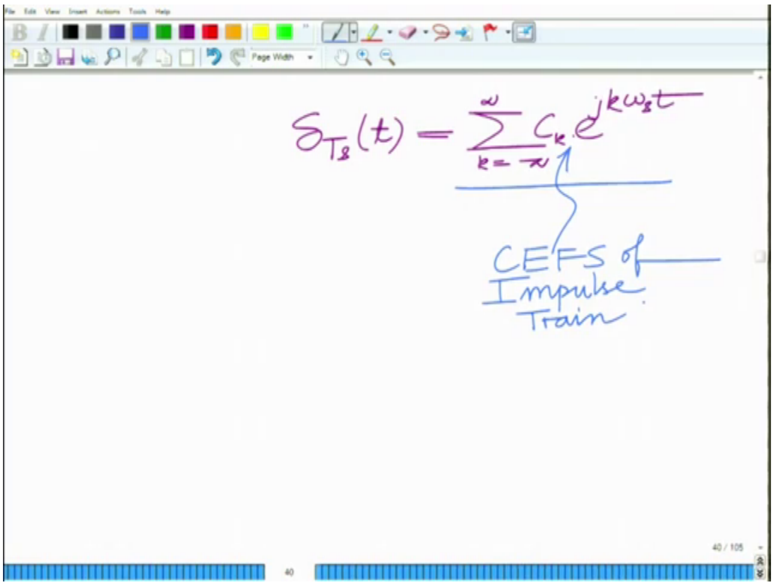

And we are trying to find the coefficient Ck of the complex exponential Fourier series of the impulse train and the Cfo of the impulse train the coefficient can be found as follows, remember Ck this coefficient equals well over 1 period 1 over Ts the fundamental period integral over any period.

So, we can choose minus Ts over 2 to Ts over 2 delta Ts t or delta t in 1 period it is simply delta t delta Ts t let me just write the delta Ts t e power minus j k omega s t dt, this expression for the coefficient Ck of the complex exponential Fourier series.

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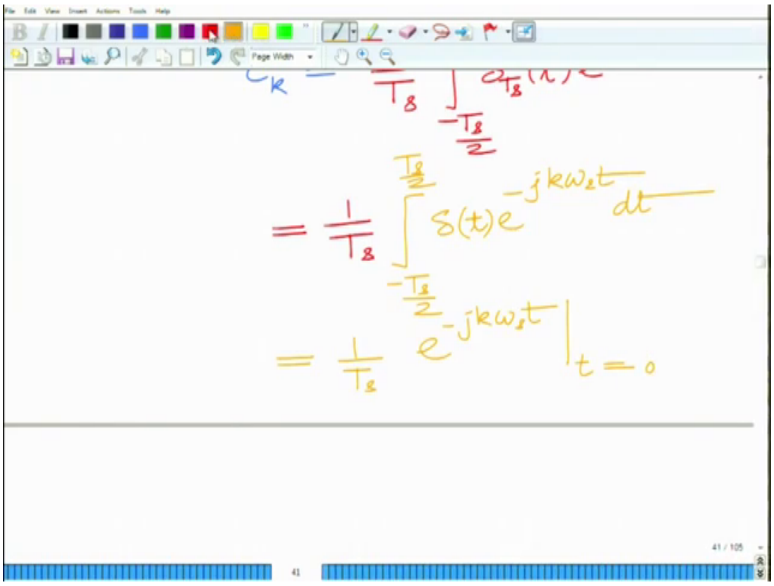

But in the interval minus Ts by 2 2 Ts over 2 this simply delta t, so this is minus Ts over 2 to Ts over 2 delta te raised to minus j k omega s t dt, which is basically if you look at this 1 over Ts e raise to minus because integral with delta t is simply e raise to minus jk omega s t evaluated at t equal to 0 and therefore this is simply equal to 1 over Ts.

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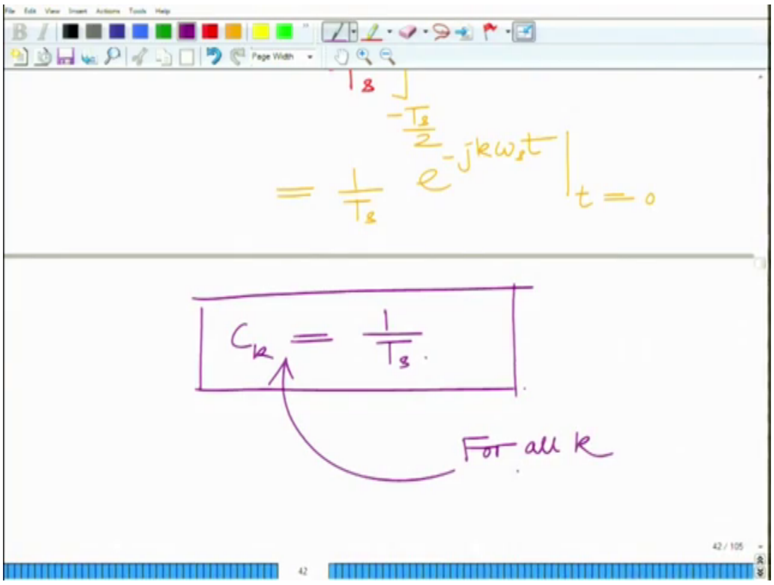

So, if you look at this basically all the coefficients ck this is for all k ok. So, all the coefficients Ck are basically equal to 1 over Ts and therefore the complex explanations Fourier series is given as summation ck e raise to minus summation Ck e raised to j k omega s t all the coefficients ck 1 over Ts. So, this will simply be 1 over Ts summation e raised to jk omega s t that is a complex exponential Fourier series. So, your delta Ts t which is the impulse train with sampling duration t that is simply 1 over Ts summation k equal to minus infinity to infinity e raised to j k omega s t, this is the complex exponential Fourier series of the impulse train.

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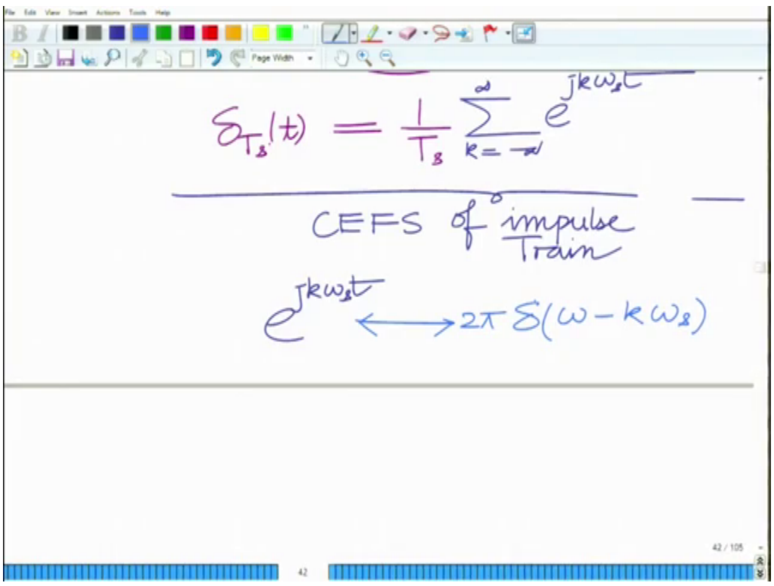

This is the complex exponential Fourier series of the impulse train, now taking the Fourier transform what we have is you now you recall that e raised to j k omega s t, this has a Fourier transform delta omega minus k omega sorry. In fact, 2 pi delta omega minus k omega s.

So, if you take the Fourier transform of the impulse train that will simply be summation 1 over Ts 2 pi delta omega minus k omega s because the Fourier transform each j each e raised to jk omega s t is simply 2 by delta omega minus k omega s ok.

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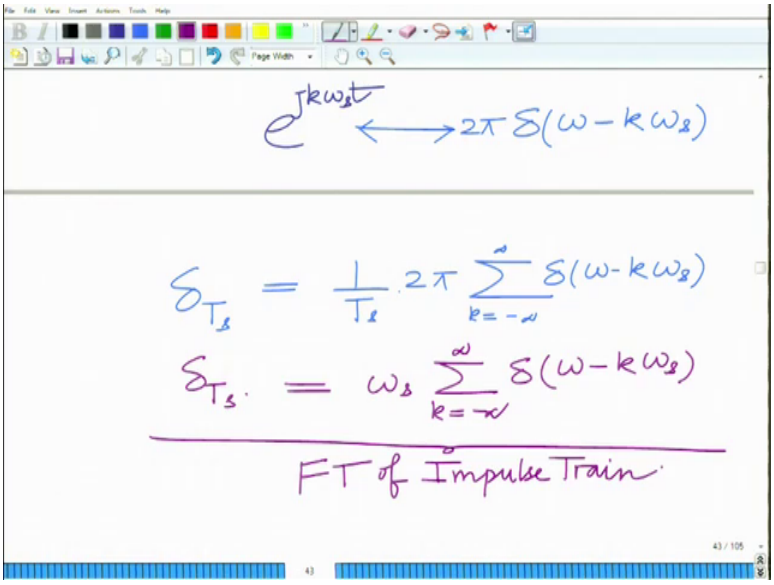

So therefore, if you take the Fourier transform of the impulse train, that will simply be 1 over Ts into 2 pi summation k equals minus infinity to infinity delta omega minus k omega s; which is equal to 2 pi over Ts is omega s which is k equals minus infinity to infinity 2 power Ts is omega s. So, this is omega s summation k equal to minus infinity to infinity delta omega minus k omega s. So, this is basically your Fourier transform of impulse train.

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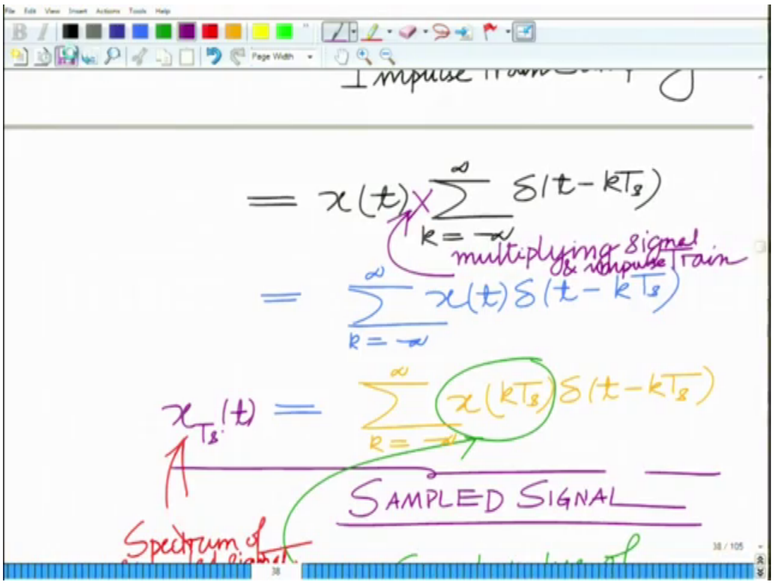

Now remember what are we doing if you recall what we are doing in the sampling process, what we are doing is a basically we are taking the symbol the signal x t and you are multiplying with the impulse train. So, this is basically multiplying the signal with the impulse train, we are multiplying the signal with the impulse train correct. So, when you multiply 2 signals in the time domain, in the frequency domain right the corresponding Fourier transforms are convolved.

So, it is given by the corresponding spectrum of the sampled signal, is therefore given by the convolution of the Fourier transform of the original signal xt with that of the impulse train that we have just derived ok. So, next step what we want to do is we want to derive the spectrum of the sample signal and now you can now you realize that is fairly simple to derive the spectrum of sampled signal.

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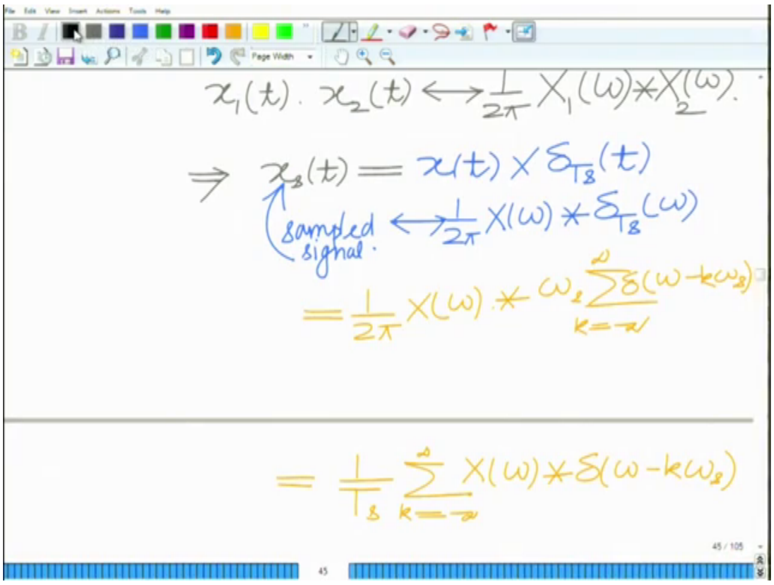

The spectrum of the sampled signal can be derived as follows. So, we have removing all the property x 1 t multiplied by x 2 t, the corresponding Fourier transform is 1 over 2 pi x 1 omega convolved with x 2 omega. Now, therefore now recall that our sampled signal x s t is the product of the original signal xt with the impulse train that is delta Ts t ok.

So, this is your sampled signal, which implies it is Fourier transform is going to be x of omega. In fact, there is there has to be a constant factor of 1 over 2 pi 1 over 2 pi X of omega convolved with the spectrum delta Ts, let us call this delta Ts of omega or whatever.

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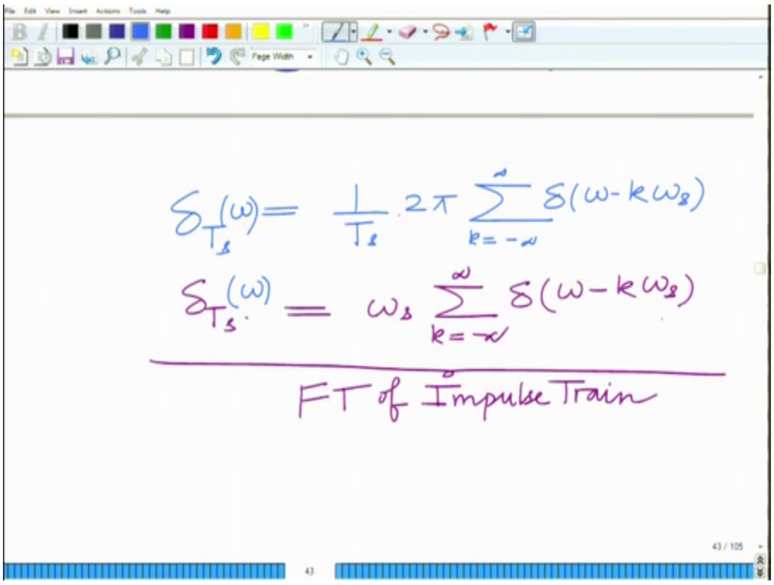

So, convolved with the spectrum of the impulse train that is delta Ts omega, which is basically equal to 1 over 2 pi X of omega convolved with omega s summation k equal to minus infinity to infinity, X of omega minus k omega s and I am sorry delta omega summation k equal to minus infinity to infinity delta omega minus k omega s; now which is basically omega s or 2 pi now omega s or 2 pi remember that is fs which is 1 over Ts.

So, this is 1 over Ts taking this since the convolution operator is associative taking the convolution operator inside, xn convolved x omega convolved with delta omega minus k omega s, which is equal to now X of omega convolved with delta omega minus k omega s is simply when you convolve with shifted impulse by k omega, it is simply shifting the function that is X of omega shifted to omega s that is we get X of omega minus k omega s.

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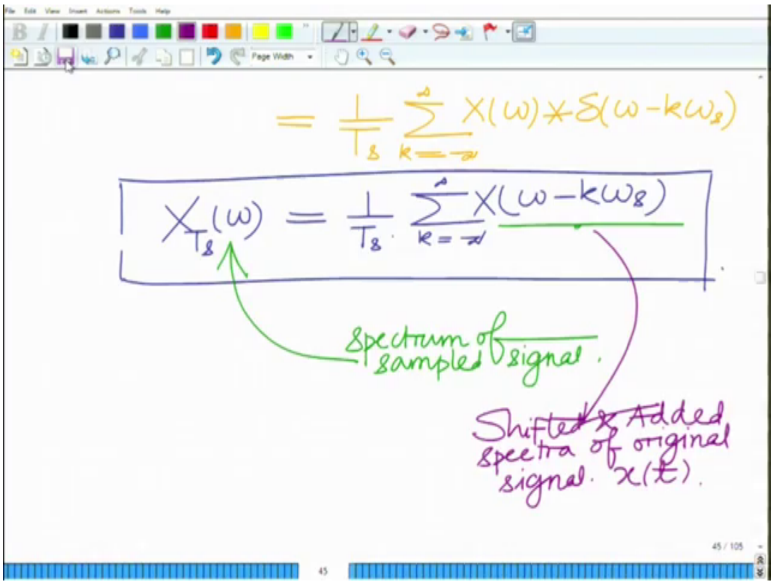

So, this will be basically 1 over Ts summation k equal to minus infinity to infinity X of omega minus k omega s, this is the spectrum of the this is the spectrum of the sampled signal spectrum of the sampled signal signal sampled by the impulse train.

There is a spectrum of the sampled signal and you can see this basically shifting and adding. So, what you are doing is X of omega you are shifting it by each integral multiple of omega s, that is you are shifting it by omegas that is s of omega minus omega s shifting it by 2 omega s that is X of omega minus 2 omega s. Similarly you are shifting it on the negative side that is X of omega plus 2 omega X of omega plus omega so on. And you are basically adding all these shifted copies of the original spectrum externally.

So, basically the summation or some shifted and added spectra of the original message signal x the some of the shifted and so these are replicas right. These are replicas of the original spectrum s omega shifted by every integral multiple of the sampling frequency omega s and then all these shifted replicas are basically added ok.

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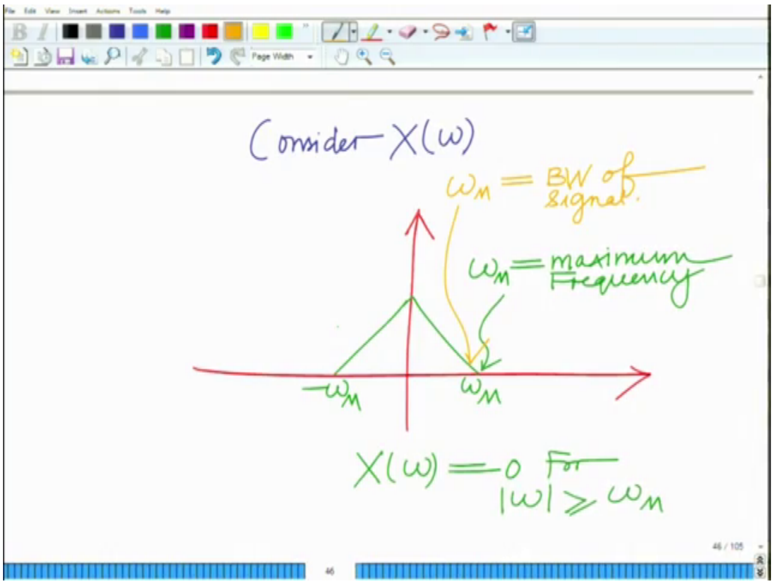

Now, consider now let me describe this pictorially describe this process pictorially, consider X of omega; X of omega given as shown below and we have let us say we have a maximum a spectrum that looks something like this is omega m. So, X of omega equal to 0 for omega magnitude of omega greater than equal to omega m, that is either omega greater than equal to omega m or minus omega less than or equal to minus omega m ok.

So, omega is so the spectrum is 0. So, what we are saying is this omega m this is equal to the maximum frequency. So, this is the maximum frequency of the signal omega that is X of omega is 0 when omega is greater than omega m as well as when omega is less than minus omega m. So, X of omega is restricted to this band to omega m to omega m. So, you can say omega m is the bandwidth of the signal ok, basically omega m is the bandwidth of the signal the signal is 0 outside of this bandwidth.

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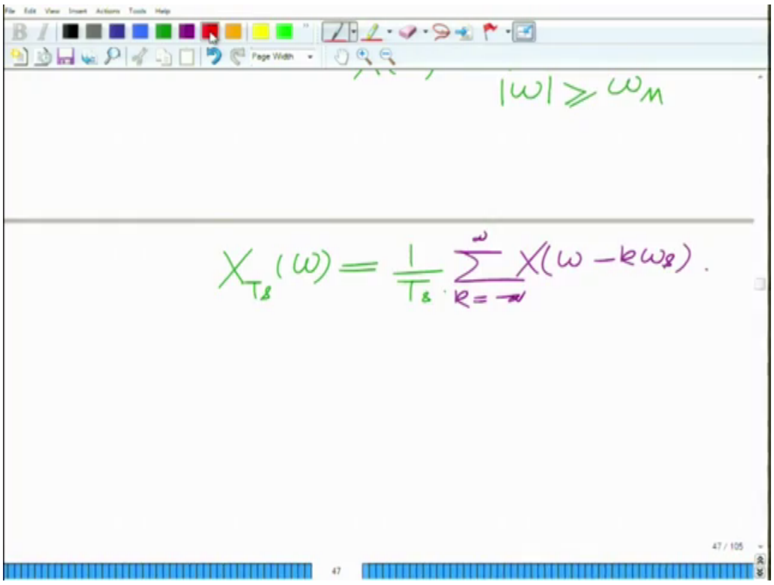

Now if you consider this signal which is x of the spectrum of the sampled signal, which is basically scaled by 1 over 2 s let us say the original signal is scaled unity. So, basically this is 1 over T s summation k equals minus infinity to infinity x of omega minus k omega s that as we said consists of shifted replicas of the original Shifted replicas of the original signal.

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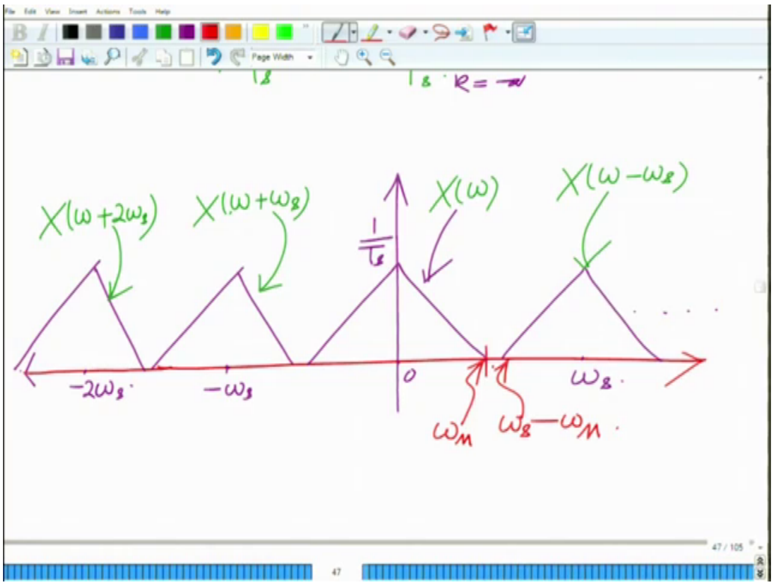

So, you have one, you have another that is shifted you have another that is shifted and you have another that is shifted by, you have another that is shifted by 2 time or shifted to minus 2 omega s. So, this is shifted to omega s this is 0 this is minus omega s minus 2 omega. So, this is your original copy each is scaled by 1 or Ts this is your original copy that is X of omega this is X of omega minus omega s and then you will have several copies over here.

Similarly this is your X of omega plus omega s this is X of omega plus 2 omega s and so on and now, if you look at it something interesting happens if you look at this point, this is omega m and this point of the shifted 1 is omega s minus omega m. Now you look at this gap if omega s minus omega m is greater than or equal to omega m then you will have this gap correct.

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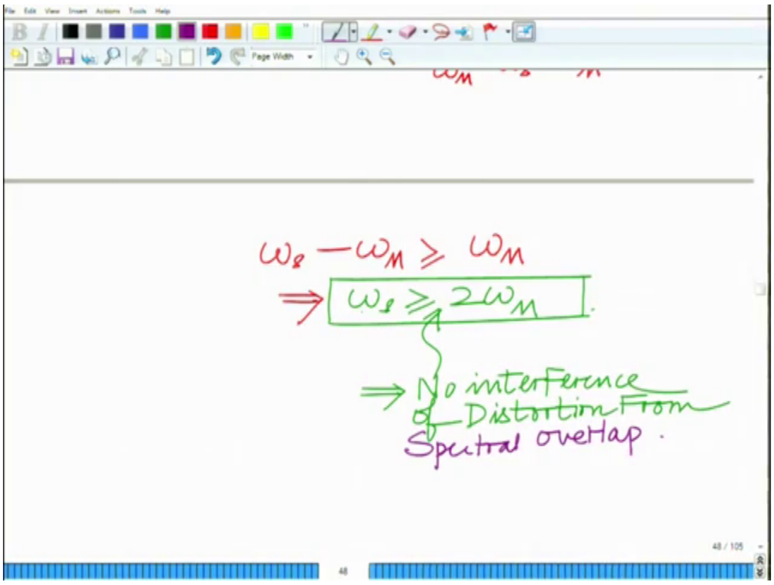

Which implies that these 2 spectral copies corresponding to X of omega and X of omega minus omega is these 2 do not overlap. As a result they will not interfere with each other; that will happen when there is a gap. And for there to be this gap or this guard band between the original spectrum and it is xi various shifted copies, this condition has to be satisfied that is omega s minus omega the maximum frequency has to be greater than or equal to omega. Which implies basically now the condition that omega s greater than or equal to 2 omega and this quantity omega is a sampling frequency omega m is the maximum frequency.

So, if this is satisfied this implies there is no interference from the spectral overlap, no interference or distortion from spectral overlap. However, if this condition is not satisfied that is if omega s minus omega m is or omega m is greater than omega s minus omega m, if this condition is not a satisfied and then you can see what happens.

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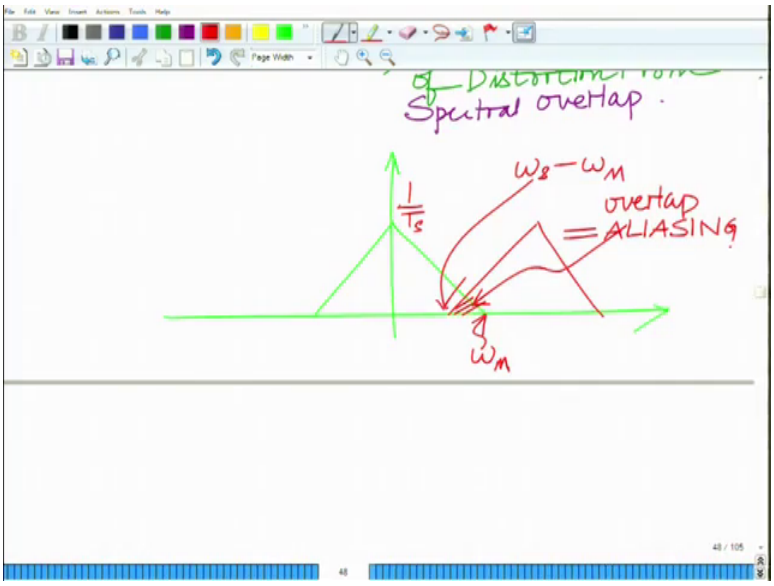

If this condition is not satisfied, then you will have your original copy that is your 1 over Ts and then you will have another copy. So, this is your omega m and this is your omega s minus omega m which is less than omega. So, these quantities where there is overlap and that causes distortion which is termed as aliasing. So, what happens in this scenario is that omega s minus omega m is less than omega m, implies omega s is less than twice omega m implies overlap which is basically equal to aliasing this is the distortion.

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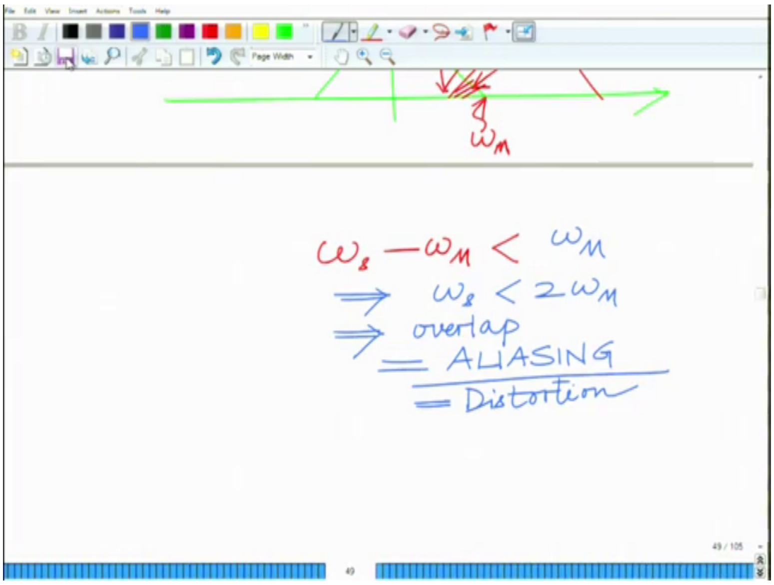

So, this basically in this spectral copies in the overlap superimpose over each other that leads to distortion of the original spectrum, then you cannot recover the original message spectrum by any kind of filtering operation. Hence this is termed as aliasing this distortion which is resulting from under sampling, remember you will have no distortion if omega s is greater than or equal to twice omega omega s is less than twice omega m you will have this distortion which is termed as aliasing and this criteria for no aliasing this is termed as the Nyquist sampling theorem or the Nyquist criterion for no aliasing ok, so for low aliasing. So, we need omega s greater than or equal to 2 omega.

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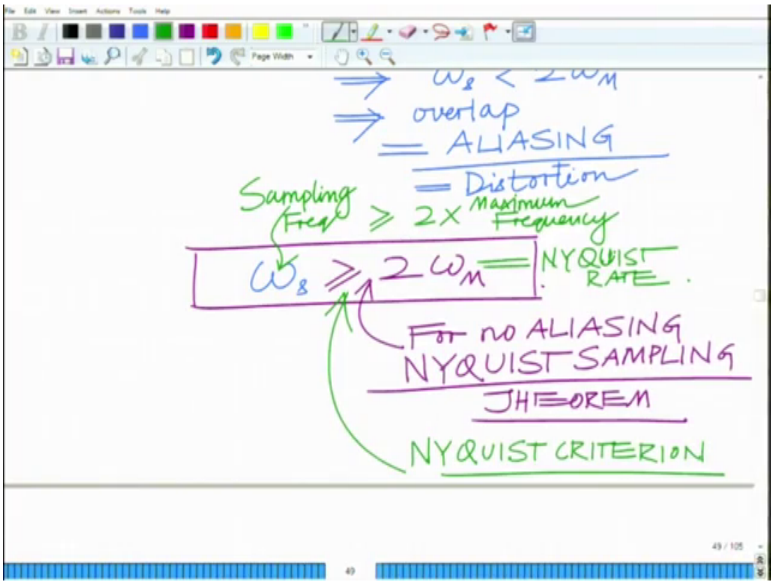

This is the criterion that we need for no distortion and this is done as a Nyquist sampling theorem, Nyquist sampling theorem or the Nyquist criterion this termed a Nyquist sampling theorem or this termed as the Nyquist criterion ok.

So, for no overlap so for no distortion no spectral distortion we need to have this sampling frequency greater than or equal to twice into the maximum frequency of the. So, the sampling frequency has to be greater than or equal to the sampling frequency. So, to avoid aliasing and this is also termed as the Nyquist rate this 2 omega m this also termed as the Nyquist, the Nyquist rate or basically the minimum rate at which the signal has to be sampled.

So, that there is no distortion this is a very important property 1 of the key properties all right, which one of the key properties which has to be kept in mind which has to be borne in mind; when converting a signal from the analog to the discrete time or the continuous time to the discrete time domain why our sampling. So, if the sampling frequency all right to avoid aliasing the sampling frequency omega s has to satisfy this key criterion.

So, this minimum threshold that omega s must be greater than twice the maximum frequency of this signal, if it is less than twice the maximum frequency then there is going to be distortion this is termed as aliasing. And therefore, this quantity 2 omega is termed as the Nyquist the Nyquist rate of sampling a signal. And this is the minimum rate of sampling a signal bad limited signal to avoid aliasing distortion.

So, I will stop here and continue in the subsequent modules.

Thank you very much.