Principles of Signals and Systems
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Lecture - 47
Fourier Analysis Examples – Fourier Transform of Gaussian Pulse, Fourier Transform Method to find Output of LTI Systems Described by Differential Equations

Hello welcome to another module in this massive open online course. So, we are looking at example problems to understand the applications and theory behind the Fourier transform especially with relevance to its analysis of signals and systems alright.

So, let us continue this let us look at other problems. So, in this module specifically we are going to look at the Fourier transform of an important signal that is the Gaussian pulse we are going to start with that ok. So, let us start by looking at the Fourier transform then we are looking at example problems and more specifically now let us start a fresh example number 12.

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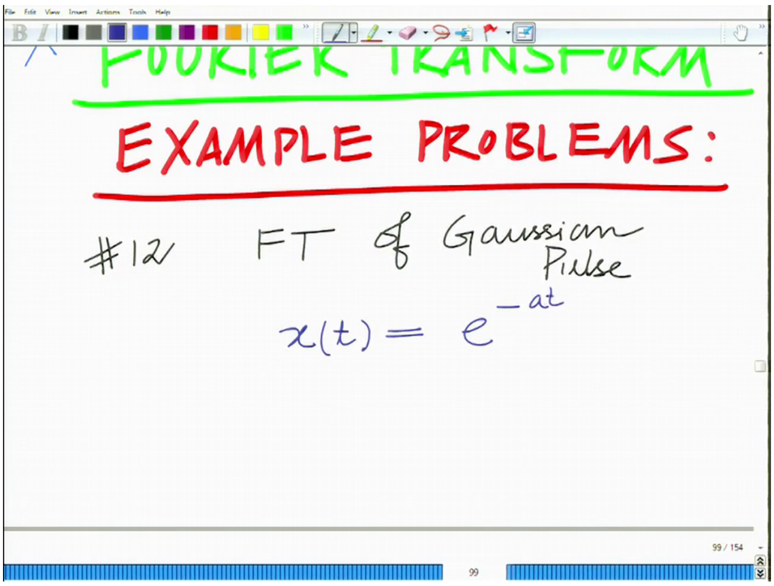

If I remember correctly let me just check it the previous one was 11 yes example number 12. So, we want to look at the Fourier transform the FT of a Gaussian pulse FT of Gaussian pulse. And the by the Gaussian pulse we mean a signal x t which is of the form e raised to minus at square this is a Gaussian pulse ok.

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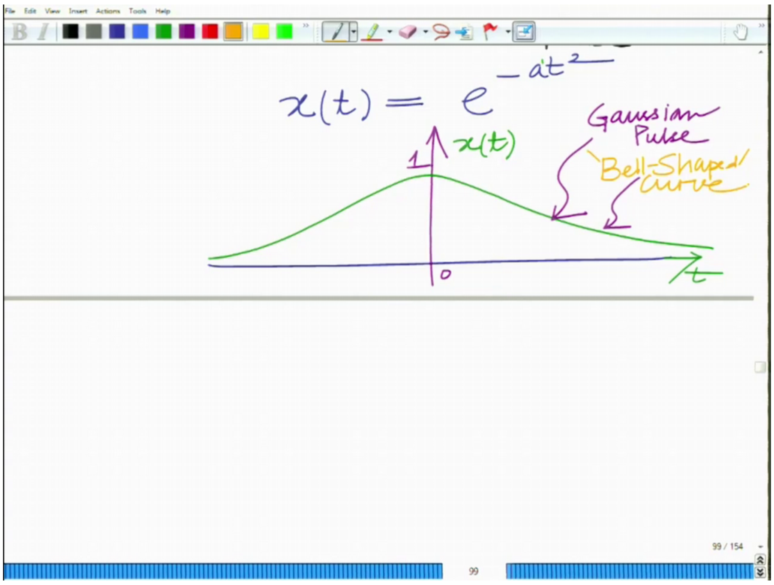

And this looks as follows e raised to minus at square you can see as t tends to both infinity and t tends to minus infinity this tends to 0 because this is e raised to minus at square and it is a bell shaped curve that looks like this ok.

So, this is t this is x of t the signal and this is a bell shaped curve, this is a Gaussian pulse. In fact, at t equal to 0 you can see this is e raised to minus 0 the e raised to 0 that is 1 and that is the peak value and then it decays as t goes to both that is as t increases from 0 or t decreases from 0 and the negative side basically decays to 0 and this has a bell shaped curve ok.

So, this is a also frequently referred to conveniently as simply a bell shaped and this is a very important class belongs to a very important class of signals it is used in both signal processing also in communication for instance in communication this Gaussian pulse is now used in Gaussian shift keying all Gaussian minimum shift keying all right which forms the basis of the GSM a digital cellular standard all right. So, this pulse shape is used in the GSM standard ok.

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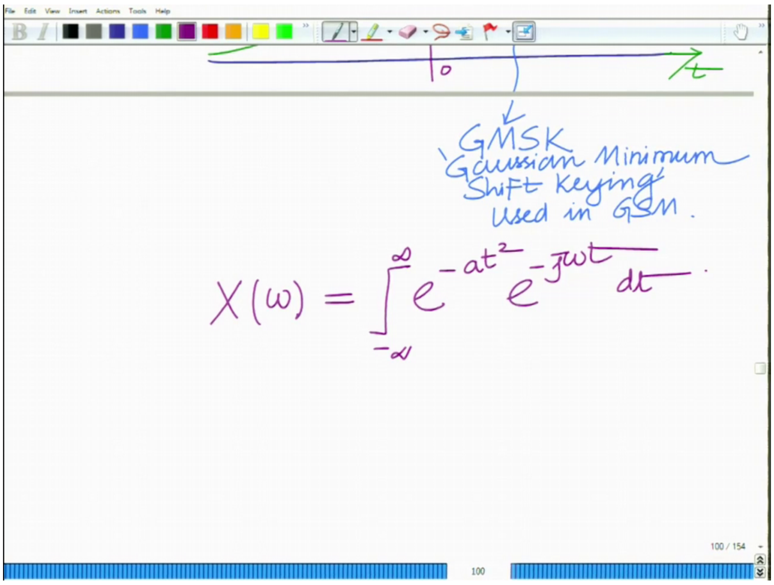

So, this is termed as Gaussian minimum shape this is termed as the Gaussian minimum shift keying scheme GSMK ok. So, this is your Gaussian minimum shift key ok. So, this is used in this is used in the GSM standard ok; now the Fourier transform of the Gaussian pulse that is X of omega is integral minus infinity to infinity x t that is e raised to minus t square e raised to minus j omega t d t.

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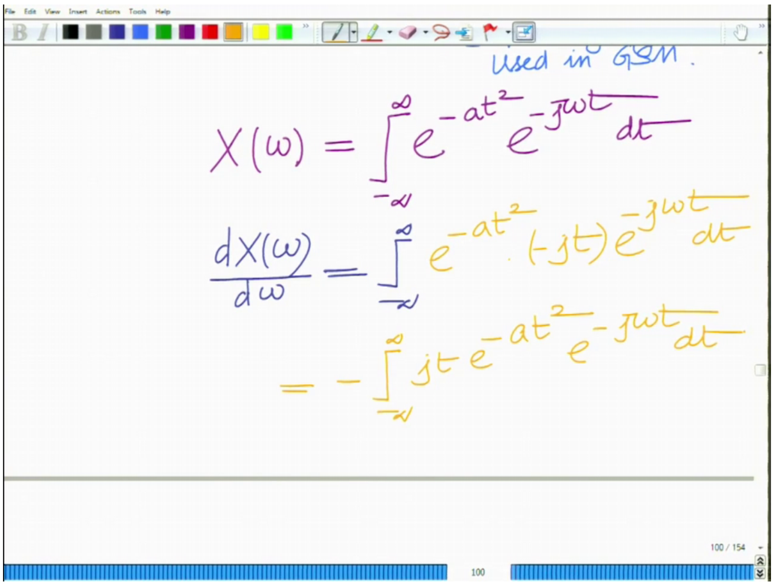

Now, there is no straightforward way to derive this. So, what we are going to do is I am going to show you a sort of a different way to derive this unless something that involves a lot more thought than just another evaluating the integral in the straightforward fashion. So, first we differentiate this; so, we have d X omega d omega that is integral minus infinity to infinity.

So, if you differentiate it now I take the differentiation sign in differentiation inside. So, that is the different derivative with respect to minus j omega. So, what that gives me is minus. So, e power minus at square you can clearly see this does not depend on omega. So, differentiating e raised to minus j omega t that yields minus j t differentiating with respect to omega minus j omega t and the integral with respect to d t; So, which you can now bring outside the negative sign.

So, that will be integral that is equal to minus integral minus infinity to infinity jt e raised to minus at square e raised to minus j omega t d t ok.

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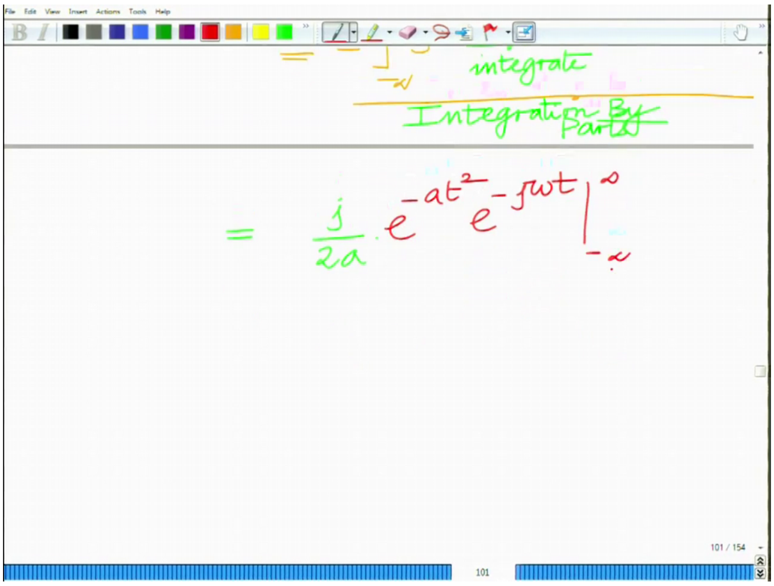

Now, what we are going to do is we are going to we are basically going to carry out this integration by parts. So, we are going to integrate use integration by parts we are going to use integration by parts and when we use integration by parts; what we obtain is we obtain this first I am going to integrate this.

So, multiply and divide by 2 a. So, this is minus j. So, I have to multiply and divide by minus 2 a; so, that gives me a plus sign. So, this becomes j over 2 a. So, 2 a e raised to minus at square that becomes in e raised to minus at square e raised to minus j omega t evaluated between the limits minus infinity to infinity minus j over 2 a now you have to differentiate it.

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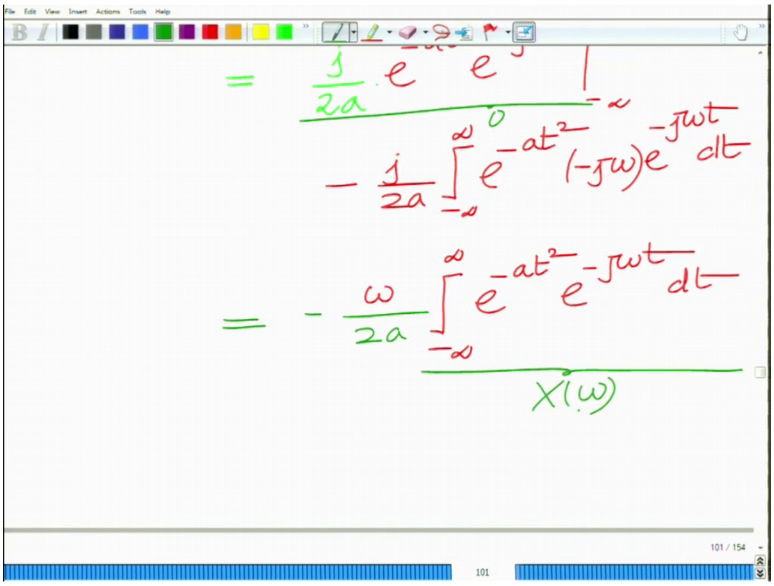

Now, you have to differentiate the other terms. So, e raised to minus at square differentiate the other term with respect to t. So, that gives you minus j omega e raised to minus j omega t d t. So, when you differentiate this with this e raised to minus j omega t with respect to t what you have is e minus j omega e raised to minus j omega t and the integral with respect to d t is there.

Now, if you look at this first term that is j over 2 a e raised to minus at square e raised to minus j omega t; now if you substitute the limits you can see e raised to minus at square when t equal to infinity as well as t equal to minus infinity this is 0. So, this term goes to 0. So, what we are left with and you can clearly see and we can bring the j outside. So, minus minus becomes plus j into j is minus 1. So, this is minus 1 over 2 a times.

In fact, you can also bring the omega outside because the omega does not depend on the integrate the integration of variable of integration that is t. So, this becomes minus omega over 2 a integral minus infinity to infinity integral minus infinity to infinity e raised to minus at square e raised to minus j omega t d t and if you recognize this; this is nothing, but X of omega that is the Fourier transform of e raised to minus at square.

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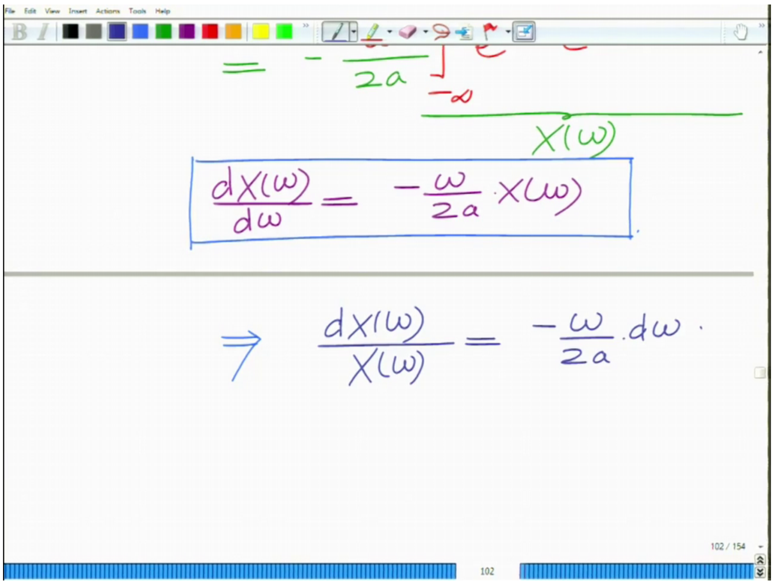

So, what we have showed is we have shown the first step that is something that is very essential for the rest of the proof that is the derivative of X of omega d X omega over d omega this is equal to minus omega over 2 a times x of times X of omega minus omega 2 a times X of omega ok. So, that is what we have that is the first thing that we have which means this implies that if you take here d X omega.

Now, you treat this as a differential equation dx omega, but X omega equals minus omega by 2 a d of omega ok. So, from this we are obtaining a differential equation that is d X omega by X omega equals minus omega or 2 a times d omega. So, this is a differential equation now we are going to solve this differential equation ok.

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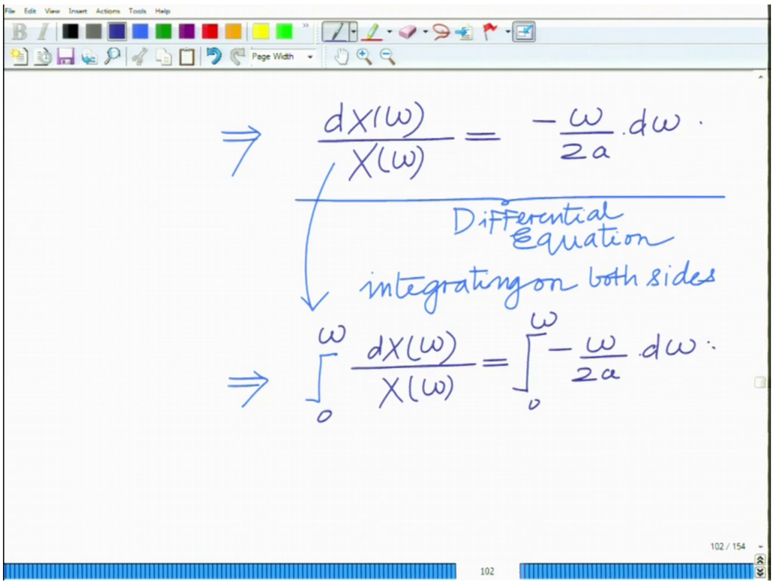

So, this is basically your differential equation; now integrating on both sides now integrating on both sides. So, integrating this on both sides what we have. So, this implies integral 0 to omega d X omega over d omega equals I am sorry d X omega over X omega equals integral 0 to omega minus omega over 2 a d omega.

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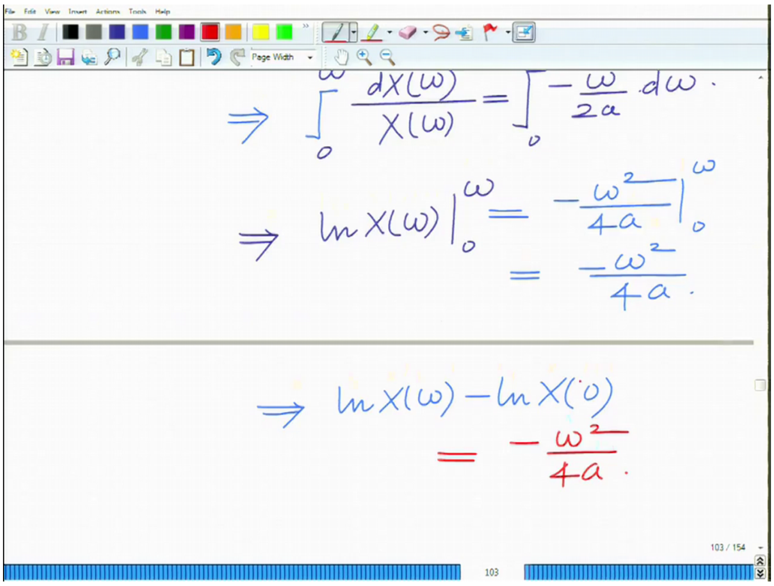

And therefore, this implies now this is log natural X of omega evaluated between the limits 0 to omega d X omega X omega integral is log of X omega. And this integral we know minus omega or 2 a d omega; the integral of this is minus omega square by 2 into 2 a minus omega square over 4 a evaluated between the limits 0 to omega which is minus omega square by 4 a; which implies that log X of omega minus log X of 0 that is equal to your minus omega square by 4 a.

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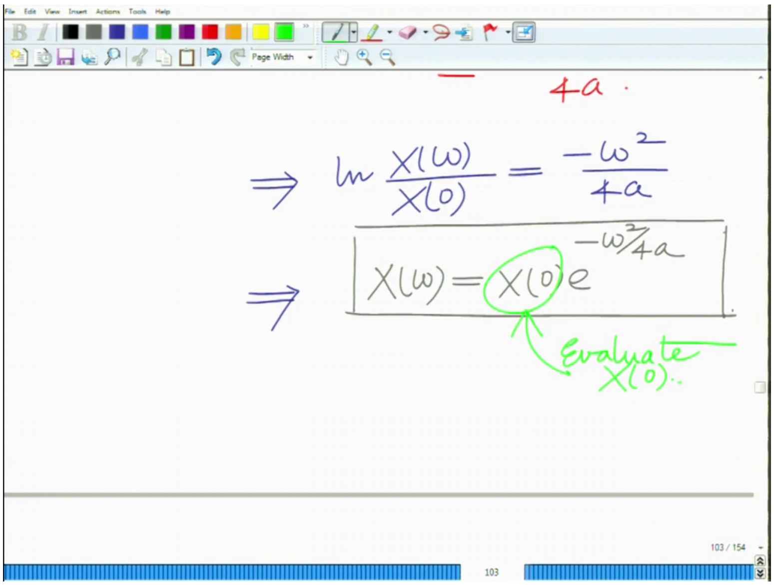

Which implies log a minus log b is log a over b which implies log X of omega divided by X 0 is minus omega square or 4 a and finally, from the original differential equation it follows that X of omega and this is important X of omega equals X of 0 into e raised to minus omega square over 4 a ok.

This is basically the next result that you have; so, we have derived accept this constant X of omega. So, we have X of omega is X of 0 into e raised to minus omega square over 4 a. Now, all that is remaining in this Fourier transform is to evaluate what this constant x of 0 which is nothing, but the Fourier transform value at omega equal to 0. So, we have to evaluate; so, it remains to evaluate it remains to evaluate X of 0.

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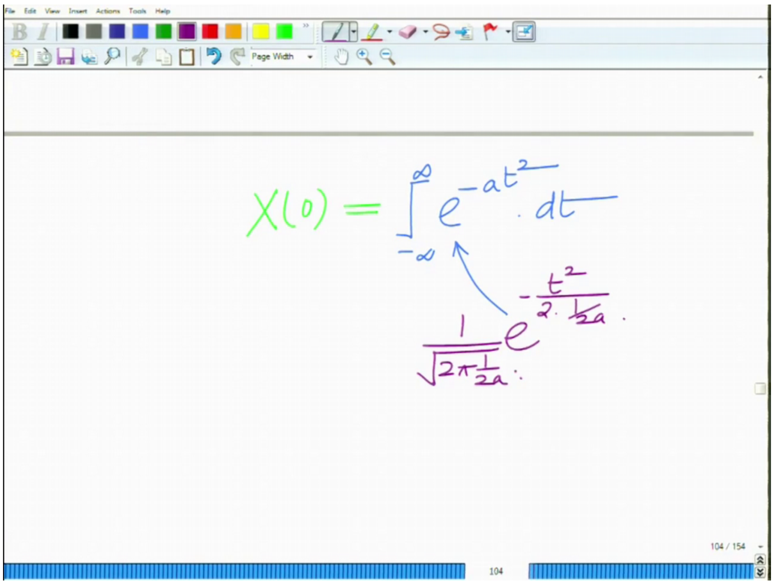

Now, recall that X of 0 is the Fourier transform value at 0 which is nothing but if you look at this minus infinity to infinity x of t that is e raised to minus at square; e raised to minus j omega t, but at omega equal to 0 e raised to minus j omega t is simply 1. So, therefore, this simply reduces to e raised to integrally raised to minus a t square d t. Now, this is difficult to evaluate except what we can do is we can realize something interest, there are many ways to evaluate this what we can realize that e raised to minus a t square also represents a Gaussian probability density function not exactly.

But we can consider this with some appropriate scaling to represent a Gaussian probability density function ok. So, this you can see I can write this as this I can write this as e raised to minus t square by 2 times 1 over 2 a ok.

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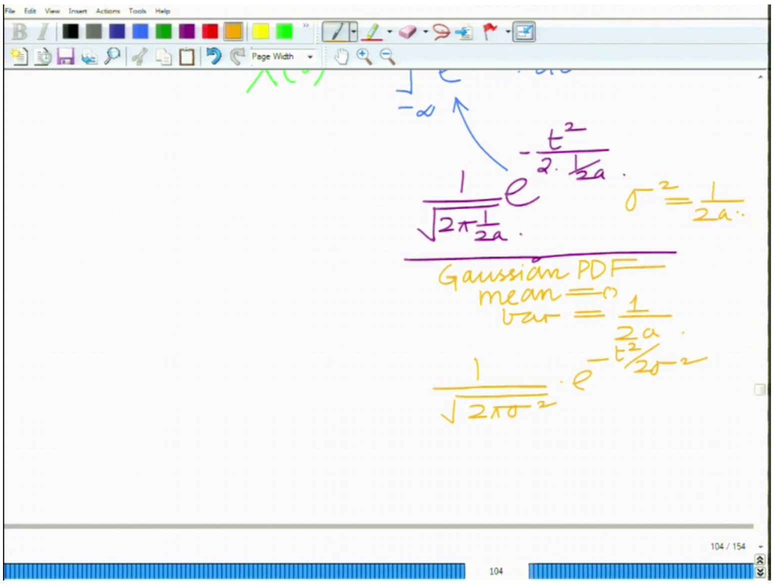

And now, if I multiply this by square root of 2 pi 1 over 2 a; So, this represents a Gaussian probability density function Gaussian PDF mean equal to with mean equal to 0 ok, variance is equal to 1 over 2 a ok.

Because the expression for the Gaussian probability density function with mean 0 and variance sigma square is given as e raised to minus a 1 over square root of 2 pi sigma square, e raised to minus t square by 2 sigma square here you can see sigma square equals 1 over 2 a.

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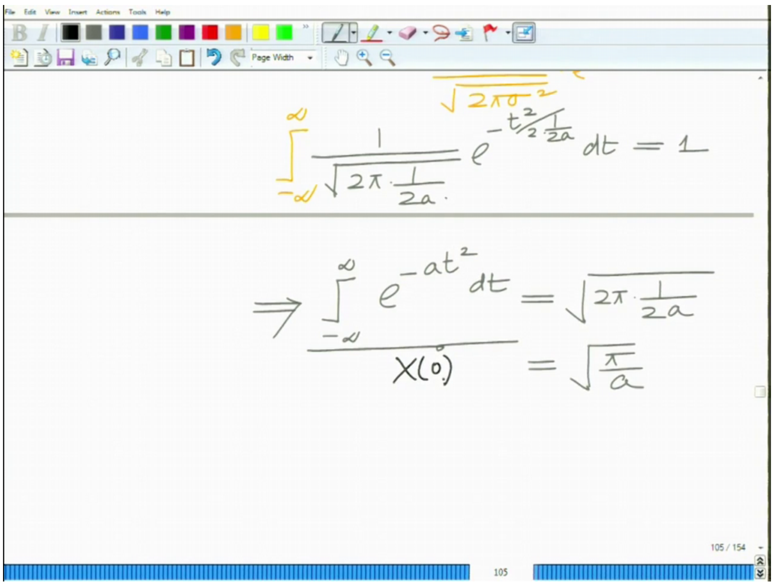

And we have integral minus infinity or minus infinity to infinity the Gaussian probability density function the integral is unity which implies that integral minus infinity to infinity 1 over square root of 2 pi 1 over 2 a e raised to minus t square by 2 into 1 or 2 a or in other words e raised to minus a t square d t is 1; this is the integral of the Gaussian probability density function which implies.

Now, taking the constant to the other side integral minus infinity to infinity e raised to minus a t square d t; this is equal to well square root 2 pi 1 over 2 a equal square root pi over a; this is square root pi over a and this you can see is nothing, but your X of 0 ok.

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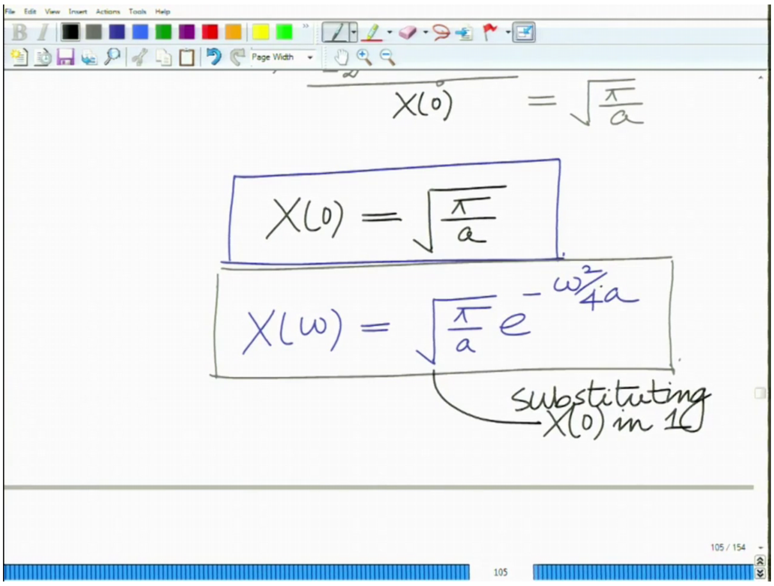

So, basically that shows us that X of 0 equals square root of X of 0 equals square root of pi over a ok.

And therefore, X of omega equals square root of pi over a e raised to minus omega square divided by 4 a and this is the Fourier transform of the Gaussian pulse ok. So, substituting; so, what we are doing is simply we are substituting the value of. So, if we call this as 1; so, what we are doing is substituting x 0 in 1. So, this we obtain by; so, this we obtain by substituting X 0 and 1.

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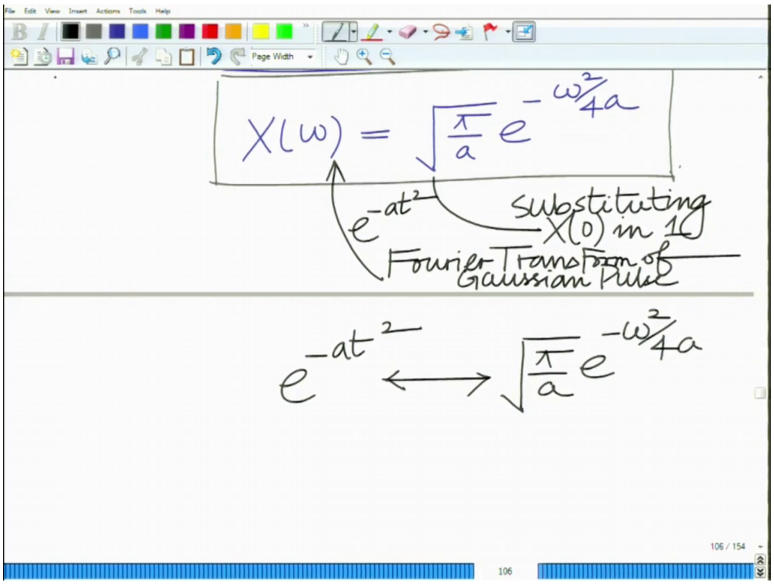

So, this is square root of pi over e raised to minus omega square over 4 a and this is the Fourier transform of the Gaussian pulse. So, this is the Fourier transform of the; so, the Fourier transform the Gaussian pulse remember e raised to minus at square the Fourier transform pulse e raised to minus at square is square root of pi over a e raised to minus omega square over 4.

And as I already told you the Gaussian pulse is a very important signal it has many applications. So, its and it is not very straightforward to derive the Fourier transform of the Gaussian pulse. So, it is important to keep this expression in mind ok; so, e raised to the signal e raised to minus a t square has a Fourier transform square root of pi over a e raise to minus omega square over 4 a ok. So, that is the Fourier transform of the Gaussian pulse.

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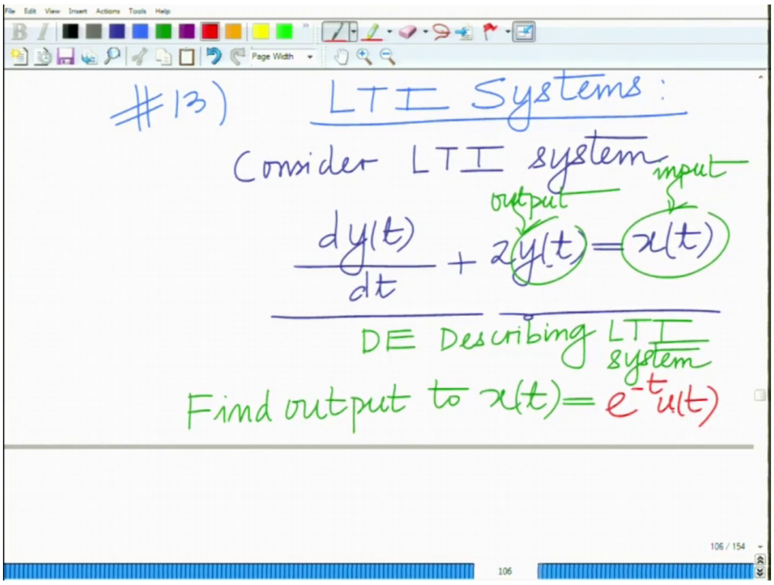

Let us now look at the next example number 13; we come to know LTI systems the applicability of the Fourier transform in the analysis of the properties and the behavior of LTI systems.

So, consider now we have; so, consider the LTI system with output that given by the linear differential equation or the differential equation the constant coefficient what we also call the constant coefficient the differential equation dy by dt plus twice yt equal to xt ok. So, this is the differential equation that is the DE that describes the DE that describes the LTI system ok.

And what we want to do is we want to find the output to a given. So, xt equal to; so, again this xt you can see this is the input and this yt this is the this is the output. And what we need to do for this lt; so, xt is the input for this LTI system yt is the output for this LTI system its described by this constant coefficient differential equation and we have to find the expression for the out of the output for a given input signal ok.

So, that is what we want to do what you want to do find the output to given signal x t equals e raised to minus t. So, this is a simple signal e raised to minus t u t now first thing that we can do is we can consider the Fourier transform of the differential equation. So, we need to find the output to the given signal that is xt equals e raised to minus t u t.

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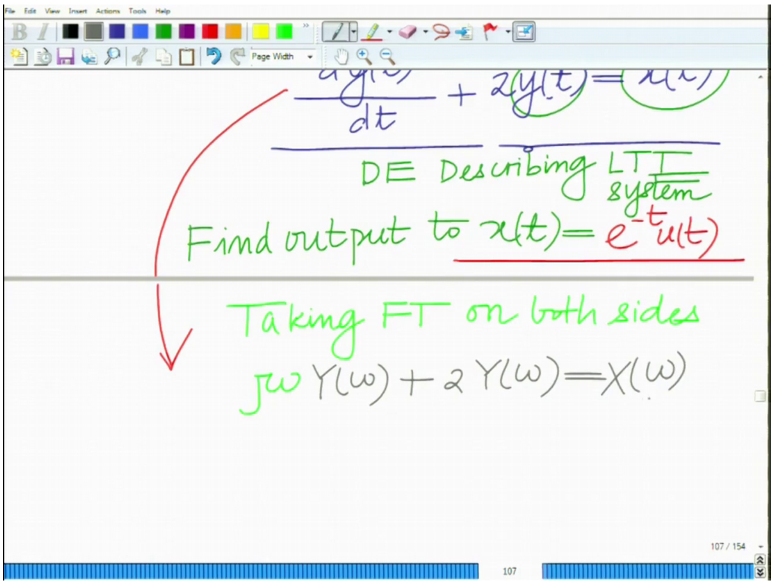

So, taking the Fourier transform on both sides what we have taking the Fourier transform both sides what we have is j omega, y omega because the Fourier transform of the derivative dy t or dt is j omega times y omega j omega y omega plus twice y omega where y omega is Fourier transform y t equals X omega is the Fourier transform the input xt.

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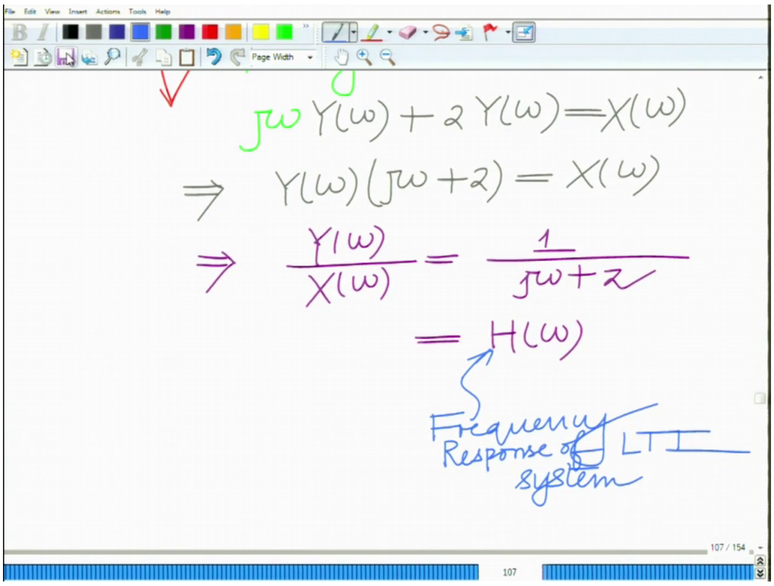

This implies that y omega into j omega plus 2 equals X omega which implies y omega by X omega which we also call the frequency response of the system is 1 over j omega by 2 which is also equal to remember h omega; the Fourier transform of the impulse response and this also known as the frequency response of the system ok.

There is also known as the also known as the frequency response of the LTI system.

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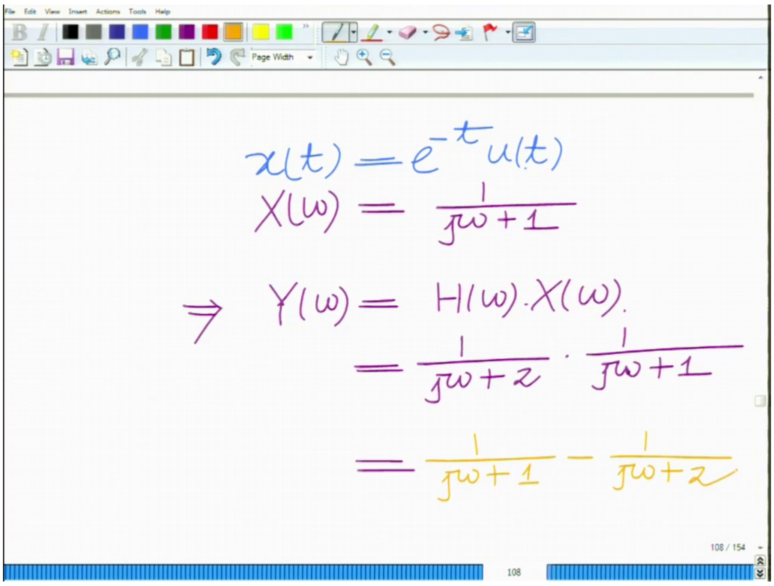

Now, we have the given signal xt is e raised to minus t u t we have this signal and we have X of omega equals 1 over j omega plus 1 that is the Fourier transform of xt which implies that the output y omega to this input is h omega times X omega which is basically h omega we already know that is 1 over j omega.

We already seen what is h omega h omega is 1 over j omega plus 2 times 1 over X omega that is 1 over j omega plus 1 which now I can express this now I define the inverse Fourier transform of y omega to derive the corresponding signal yt and now I can use partial fraction expansion to conveniently derive yt. So, I can write this as 1 over j omega plus 1 minus 1 over j omega plus 2 and this is basically the partial fraction expansion this is the PF expansion ok.

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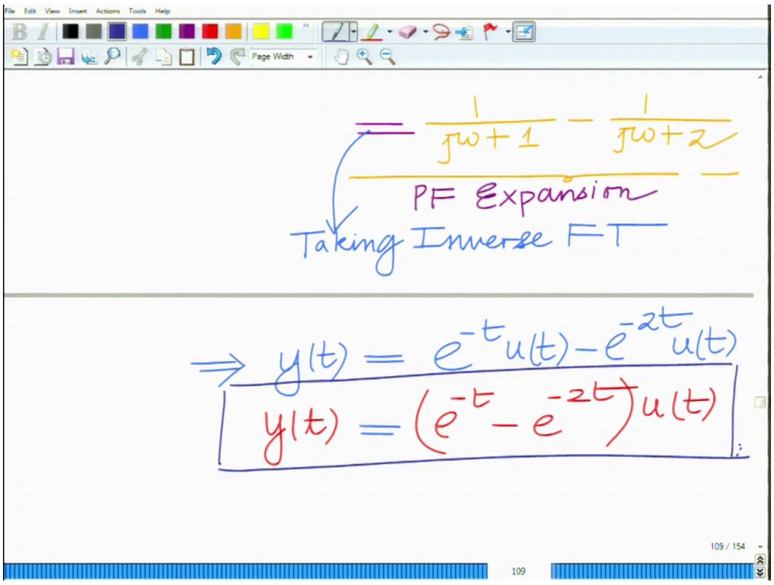

Partial fraction expansion and taking now the inverse Fourier transform, taking now the inverse Fourier transform what we have yt equals inverse Fourier transform 1 over j omega plus 1 that is e power minus t ut minus the inverse Fourier transform of one over j omega plus 2 e raised to minus t u t which can basically be represented as e raised to minus t minus e raised to minus e raised to minus 2 t u that is basically the that is basically output signal to the given input ok.

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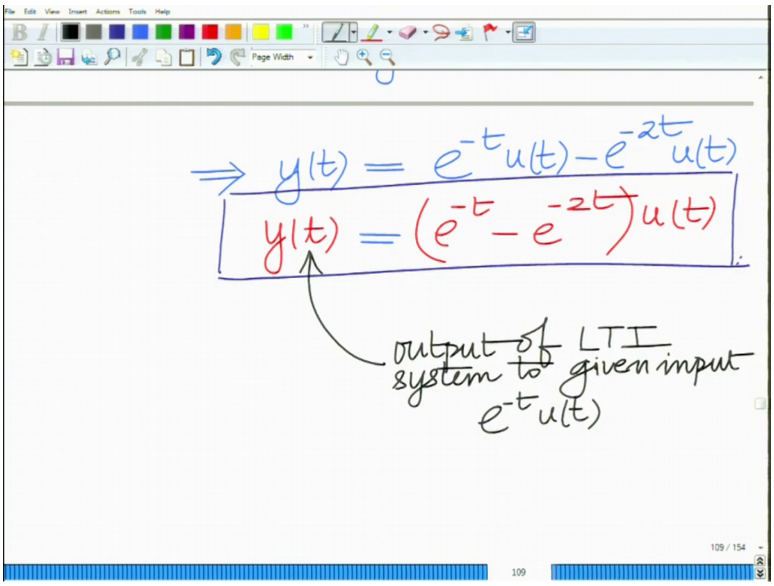

So, this is your output for the given input xt equals e raised to minus 2 t e raised to minus t u t that is e raised to minus t ut ok. So, that is output to the given input e raised to minus t u t all right. So, in this module we have done seen a couple of other examples of the Fourier transform first; we have looked at the Fourier transformation pulse. And next we have also looked at the Fourier transform or how to use the Fourier transform the frequency response of LTI systems to derive or to deduce the output signal for a given input signal alright.

We will stop here and the subsequent modules we look at other important applications of the Fourier transform starting with what is known as the Bode plot to basically get an idea or basically visually pictorially represent the properties or the frequency response of an LTI system alright. So, we will stop here.

Thank you very much.