Principles of Signals and Systems
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Lecture - 44
Fourier Analysis Examples - Complex Exponential Fourier Series and Trigonometric Fourier Series of Periodic Triangular Wave, Periodic Convolution

Hello, welcome to another module in this massive of online course. So, we are looking at example problems in Fourier analysis let us continue this discussion alright.

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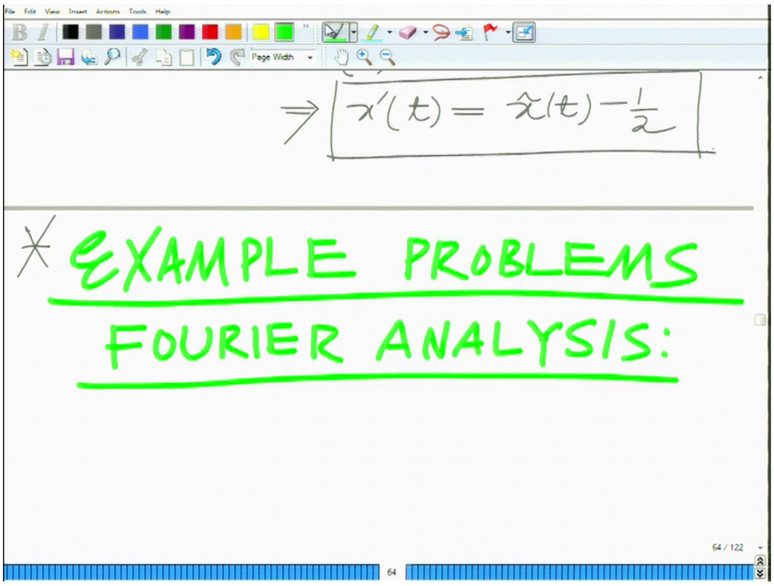

So, we are looking at example problems in Fourier analysis and we have seen previously, I mean we have currently looking at the Fourier series expansion of this.

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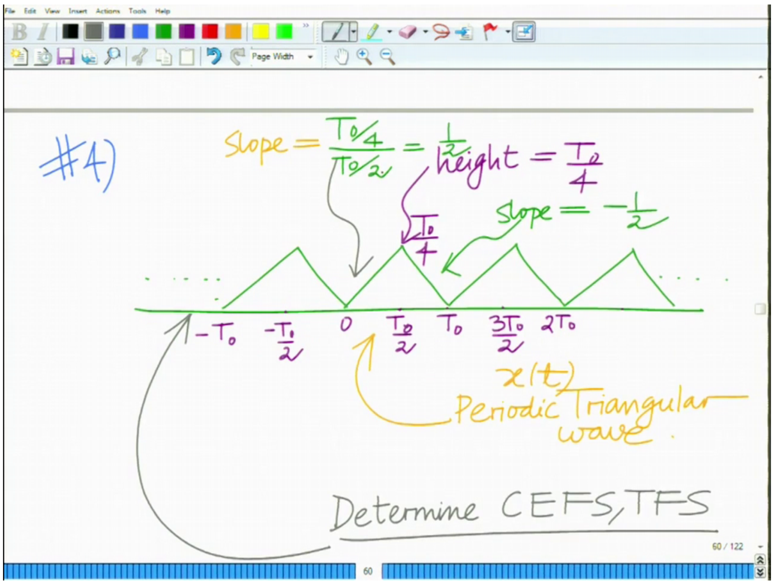

Periodic triangular wave and we have seen this triangular wave.

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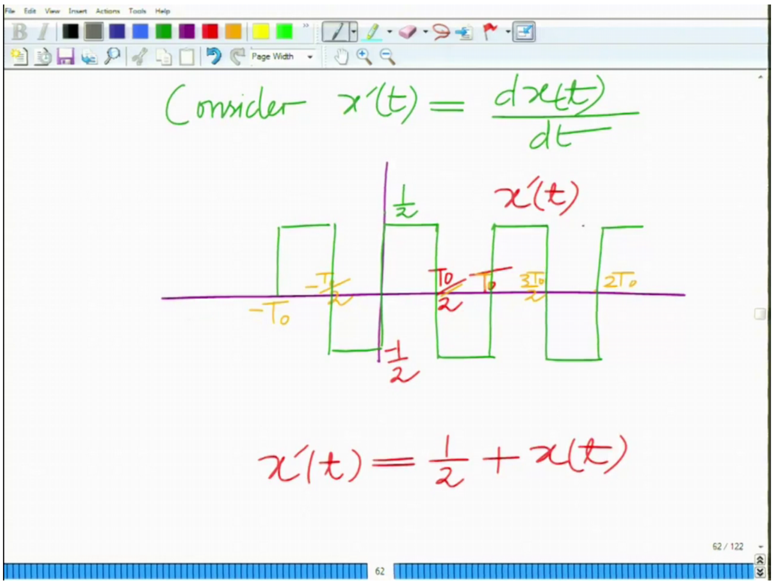

If we look at the derivative x prime t and consider x tilde t, which is half plus x prime t follow I am sorry x prime t, is we have x prime t x prime t and if we consider x tilde t which is half plus I am sorry I need to x tilde t which is half plus x prime t, that is given by this periodic square wave that we have seen earlier ok.

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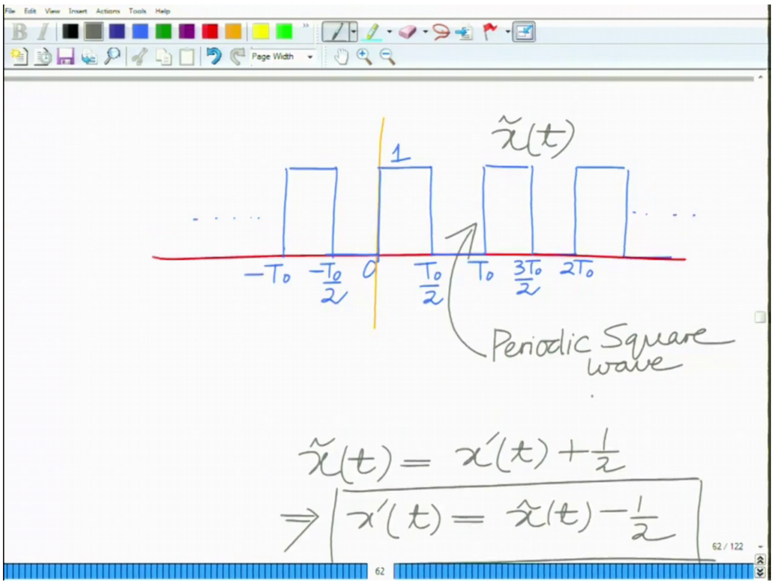

So, we have x tilde t is x prime t plus half and therefore, x prime t that is derivative of x t is x tilde t minus half.

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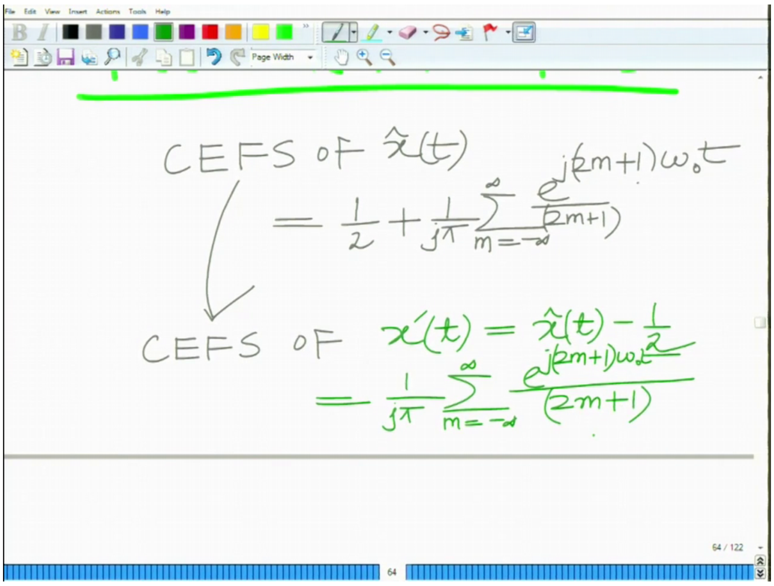

And we know the CEFS of the x tilde t we have already defined or we already derived the complex exponential Fourier series of x tilde t and the complex exponential Fourier series of x tilde t, this is given as half plus 1 over j pi summation m equals minus infinity to infinity 1 over twice m plus 1. In fact, times e raise to j 2 m plus 1 omega naught t.

This is the complex exponential Fourier series of x tilde t therefore, this implies the CEFS the complex exponential Fourier series of x prime t which is equal to x tilde t minus half, that is equal to now subtracting half you can readily seen that is 1 over j pi simply m equals minus infinity m equals m equals minus infinity to infinity e raise to j 2 m plus 1 omega naught t by 2 m plus 1.

And now what we have is now. So, we have derived the c. So, we have been able to derived the CEFS the complex exponential Fourier series of x prime t. Now also we realize that x prime t is nothing, but the derivative of x t ok.

And we have already seen this that is if we have the complex exponential Fourier series of x t or let us look at this.

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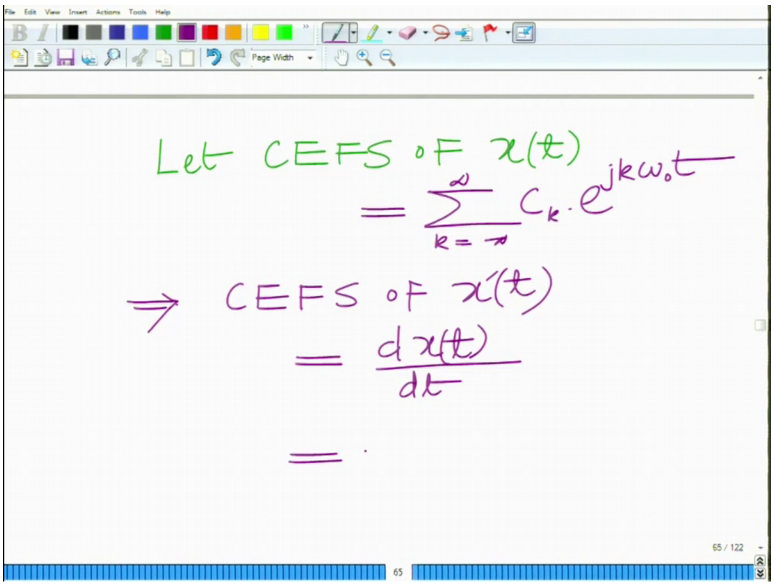

Let us say the complex exponential Fourier series of x t which is the triangular wave Let the CEFS of x t which is the original triangular wave x t equal to let us denote this by summation k equals minus infinity C k e raise to j k omega naught t.

Now, this implies the complex exponential Fourier series of x prime t which is equal to d that is the derivative of x t.

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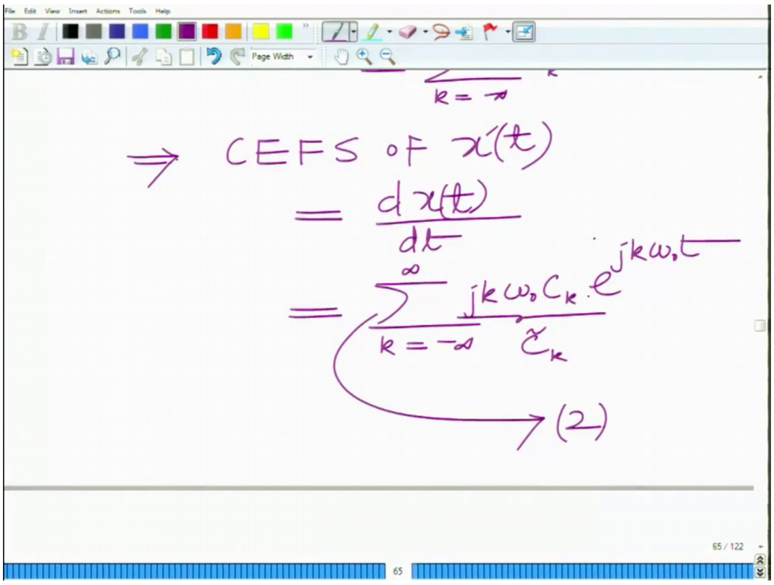

That will be given as now differentiate this, you have summation k equals minus infinity to infinity, if we differentiate this you have j k omega naught correct you have j k omega naught C k e raise to j k omega naught t.

And this is basically your C tilde k you can call this coefficients of the complex Fourier series have C tilde k. Now, let us call this as expression number 2 and we already have a expression for this complex exponential Fourier series of x prime t from above let us call this as expression 1.

Now, from expression 2 and expression 1 comparing the coefficients now.

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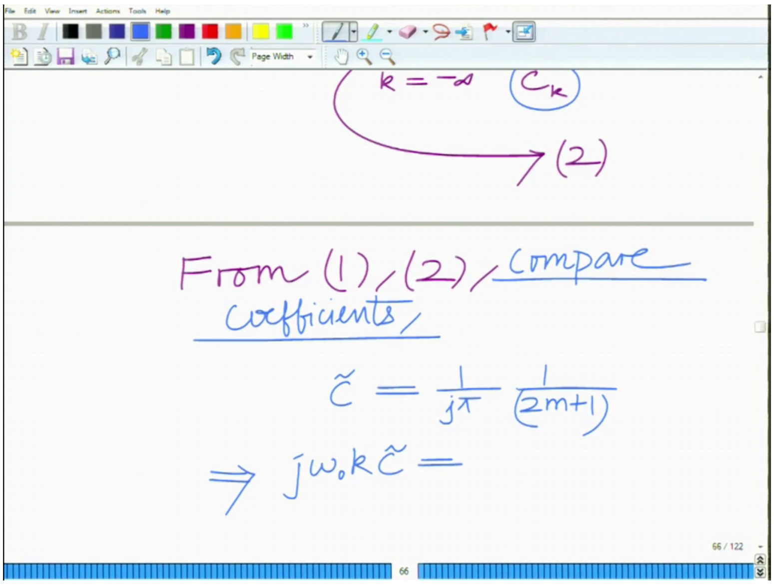

Because both of them are complex exponential Fourier series of x prime t or the derivative of x t, from 1 and 2 what we can do is? We can compare the coefficients, because both are the complex correct what we are doing is we are comparing the Fourier [coe/coefficients] coefficients of the complex [exp/exponential] exponential Fourier series.

Because the CEFS because the complex [exp/exponential] exponential Fourier series representation, because both give us the CEFS of x prime t. So, what we can conclude is that the coefficients must be equal in particular the coefficients of the terms e raise to j k e raise to j k omega naught t. So, what we have is comparing the coefficients we have that this coefficients C tilde k must be equal to this coefficient of e raise to j 2 pi e raise to j 2 m plus 1 omega naught t.

Or in other words we must have C tilde k equals coefficient of e raise to j k omega naught t which is basically if we look at that coefficient that is nothing, but 1 over j pi times 1 over 2 m plus 1, but remember as you can see from all C tilde recall C tilde is nothing, but J omega naught k C tilde ok. Now, therefore, now off course this remember this C tilde this is C tilde k.

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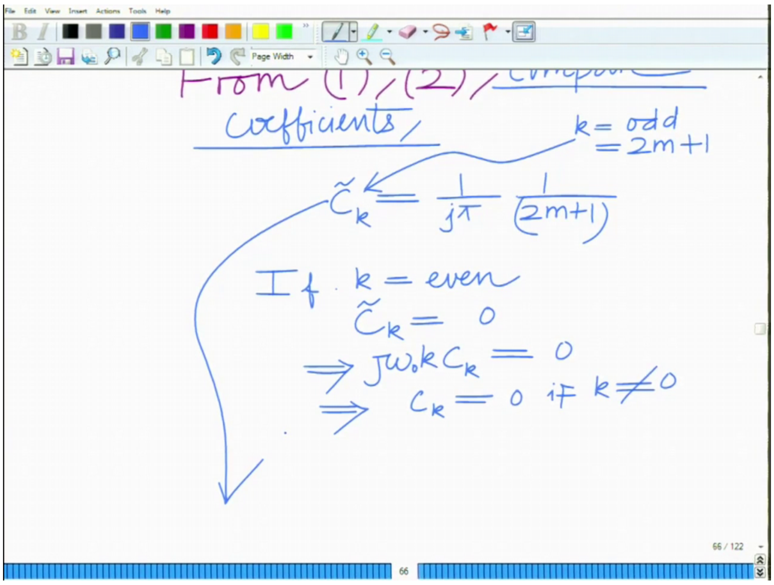

This is only for k k is even or k equal to 2 m plus 1 this only for k equals odd sorry this is only for k equal to odd that is 2 m plus 1.

If k is even you can see C tilde k because this exists only defined that is CEFS you can see of x prime t contains only the coefficients corresponding to odd k that is the k of the form 2 m plus 1. So, we will have C tilde k equals if k is even we have C tilde k equals 0 implies J omega naught k C k equals 0 implies C k equals 0 if k naught equal to 0 ok, this is the story for k naught equal to 0 k is naught equal to 0 then this implies that it has to be the case that C k is equal to 0.

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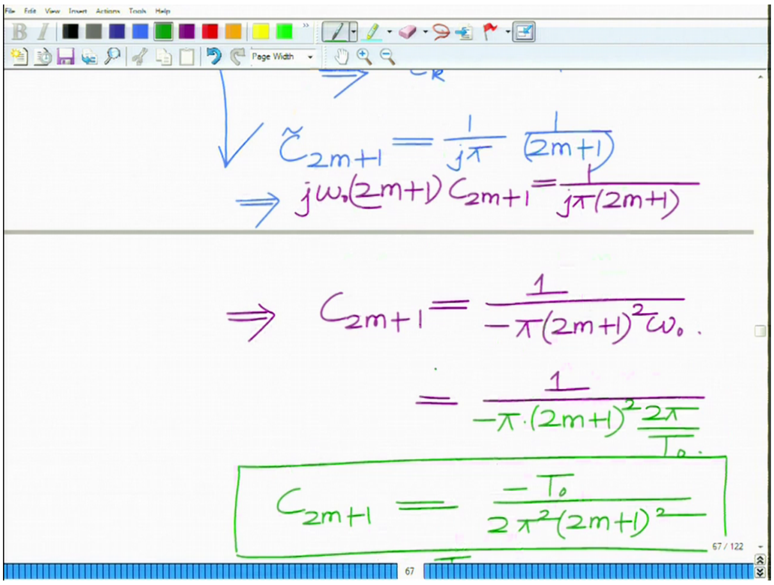

On the other hand for C tilde k for k is even then we have C tilde of 2 m plus 1 equals 1 over j pi 1 over 2 m plus 1, which implies that you have j omega naught k which is 2 m plus 1 C tilde or C of 2 m plus 1 equals 1 over j pi of 2 m plus 1.

Which implies C of 2 m plus 1 where k is odd k equals to 1 is 1 over this is 1 over minus pi into 2 m plus 1 square, 2 m plus 1 square into omega naught and this can further be simplified as 1 over minus pi 2 m plus 1 square omega naught remember is 2 pi by T 0, which is which implies is that C of 2 m plus 1 equals minus T 0 over twice pi square 2 m plus 1 square. So, this is the result that we have relating to the regarding the coefficients C 2 m plus 1 and C k is 0 by the way if k is odd.

So we have the coefficients C we have derived the expression for the coefficient C k. So, the coefficient C k in the complex exponential Fourier series of the original periodic triangular wave and C k is 0 if k is odd and if k is even that is in case of the form 2 m plus 1 then we have derived this expression for C of 2 m plus 1.

The only coefficient that is remain in our s C naught because remember we cannot derive C naught using this technique, because C naught when we differentiate x t remember the d C component C naught is constant that vanishes. So, that you can derive that using the from the derivative that is exponent.

So, what we do for that is we simply a evaluate the d C coefficient remember the d C coefficients C 0 Equals 1 over T 0 integral 0 to T 0 x t d t.

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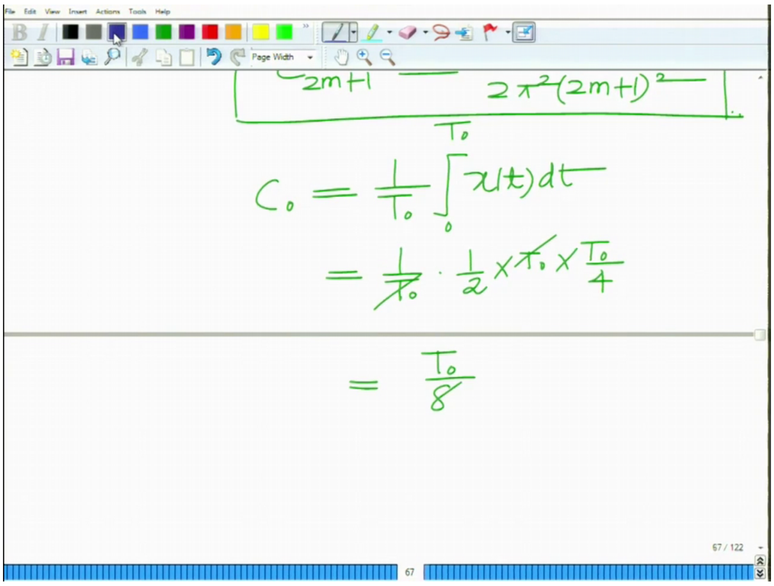

Which is 1 over T 0 the area under x T, which you if can see this is the triangular wave in a interval of at duration T 0 the triangular wave the area is half base into height that is half into T 0 into T by 4. So, this is simply going to be 1 over T 0 into the area half into T 0 into T 0 by 4 which you can basically see is T 0 over 8 ok. So, this C 0 equals.

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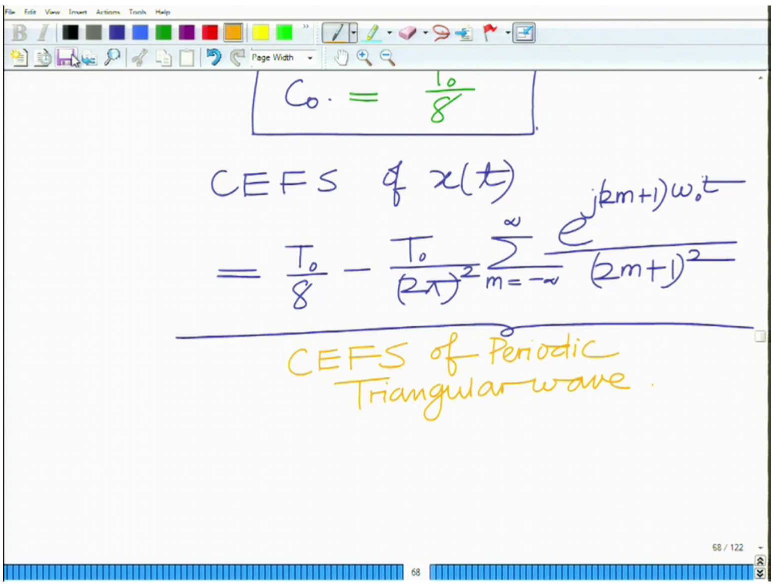

T 0 over 8 and therefore, the CEFS of x t the CEFS of original triangular wave that is x t is T 0 by 8 minus T 0 by 2 pi whole square summation m equals minus infinity to infinity e raise to j 2 m plus 1 omega naught t, by 2 m plus 1 square. So, this is the CEFS of the original triangle wave periodic triangular wave, this is the complex exponential Fourier series of the periodic triangular wave all right.

And now again similarly 1 can readily evaluate the TFS that is the trigonometric Fourier series and the TFS is given as.

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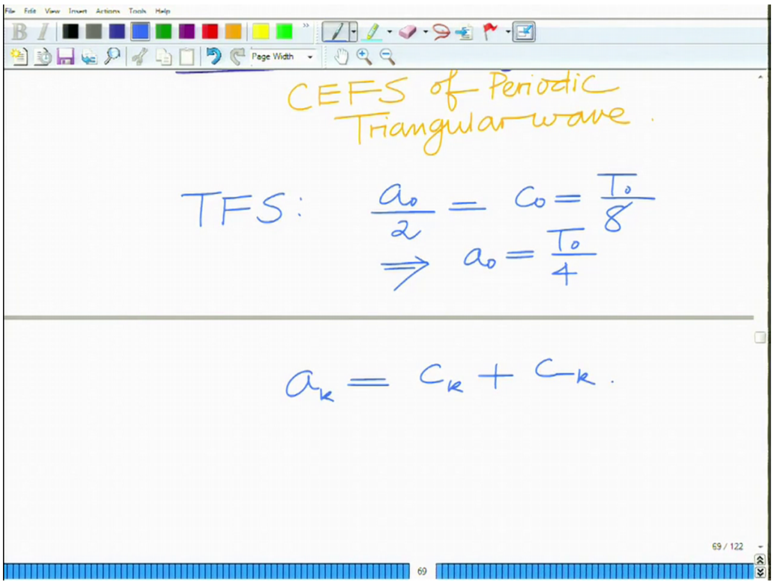

Now, coming to the TFS remember we have a naught by 2 equals C 0 equals T 0 by 8, which implies a naught equals T 0 by 4. Now a k for k naught equal to 0 equals C k plus C of minus k naught off course if C k is odd.

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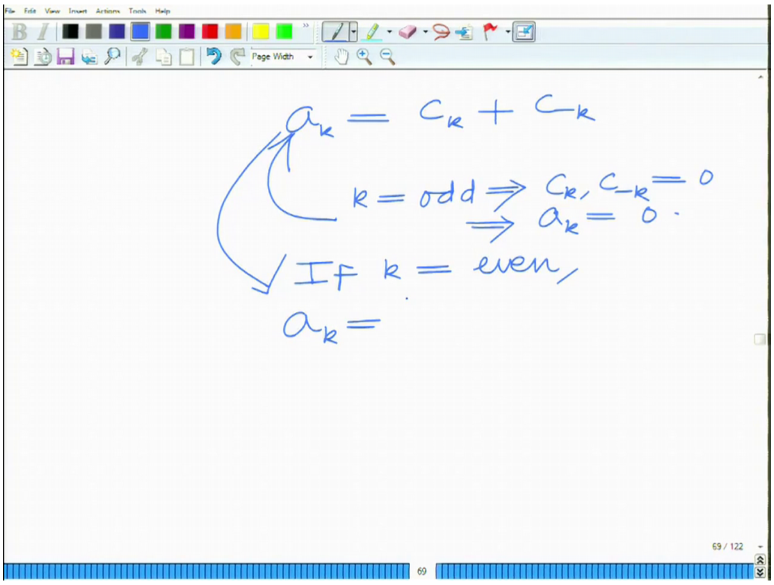

C k C minus k equals to 0 this implies a k equals 0. Now if k is even now if k is even what will happen is we have a k equals well we have a k equals or I am sorry if k is even is k is even.

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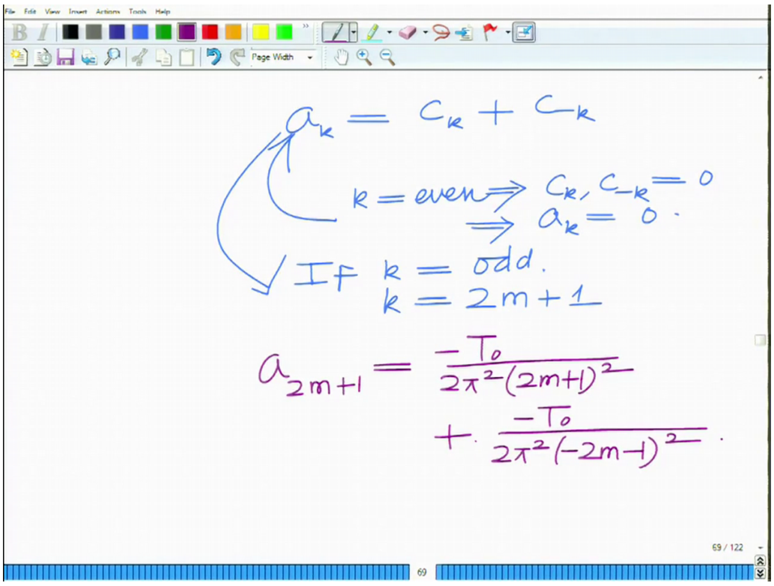

C k C minus k is 0 if k is odd we have a k equals that is k equal to is of the form 2 m plus 1, then a k or a of 2 m plus 1 equals well this is minus T 0 by 2 pi square into 2 m plus 1 whole square plus C of minus k, which is minus T 0 over 2 pi square minus 2 m minus 1 whole square which you can see is basically.

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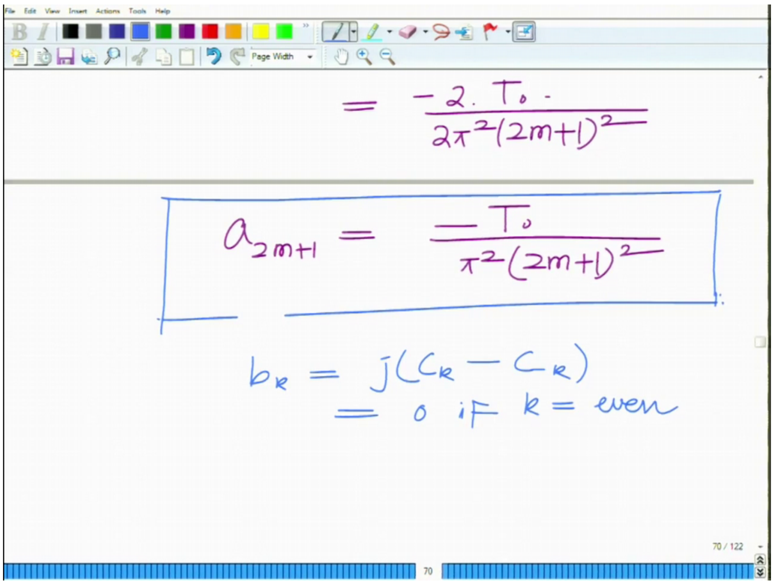

Minus twice T 0 by 2 pi square into T m plus 1 whole square which is equal to minus T 0 by 2 pi 2 pi square I am sorry minus T 0 by pi square by 2 m plus 1 whole square this is your a 2 m plus 1 ok.

So, this is your a 2 m plus 1. And similarly now coming to b k b k equals J C k minus C minus k equals 0 if k is even because k is even C k and C minus k are both zeroes so, b k is 0. If k is odd then b k is.

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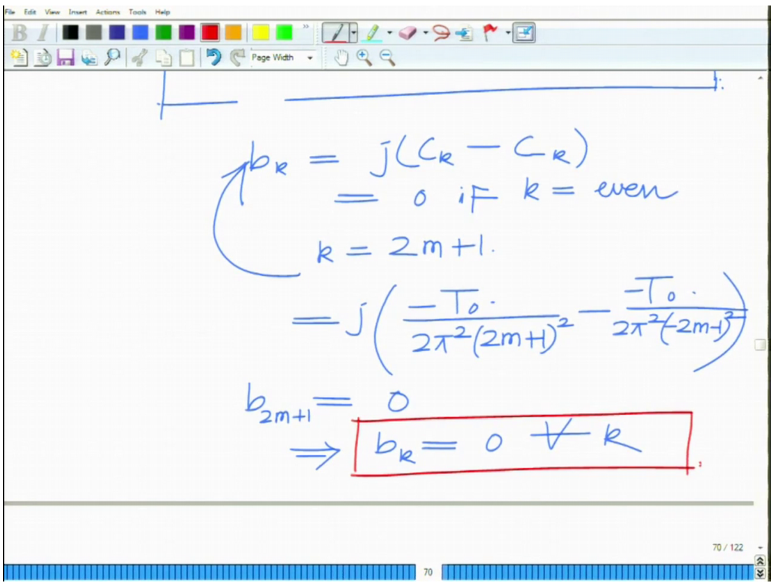

Then b k is basically j times well minus T 0 by 2 pi square 2 m plus 1 whole square minus T 0 or minus see minus k minus minus T 0 by 2 pi square minus 2 m minus 1 whole square which you can clearly is 0.

So, b of 2 m plus 1 0 which implies b k is 0 for k even or which means b k is 0 for or k.

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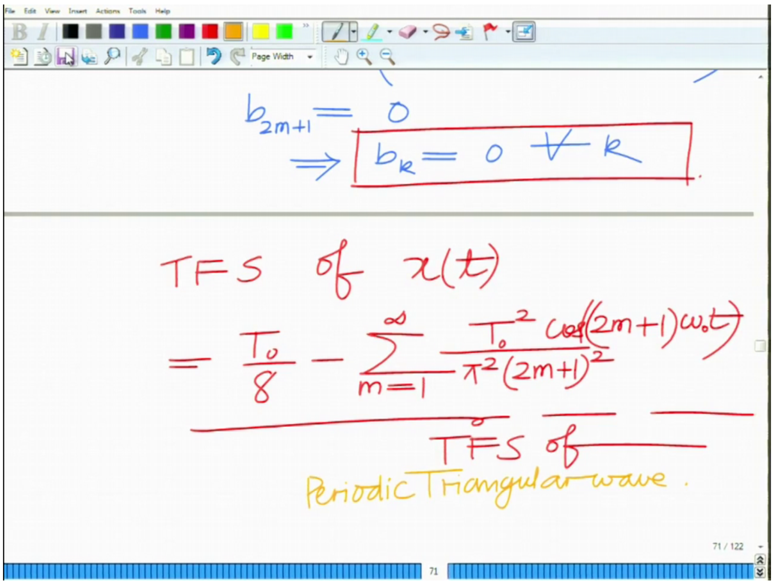

Therefore, the TFS of the periodic triangular wave x t T 0 by 8 minus summation m equal to 1 to infinity T 0 square by pi square into 2 m plus 1 square into cosine 2 m plus 1 omega naught t, that is your TFS this is a TFS of the periodic triangular wave this is your p a TFS of the periodic triangular wave ok.

So, that is basically completes is slightly elaborate the procedure to derive it is likely elaborate it is more simplistic and more intuitive than trying to rather the direct approach of deriving the CEFS the TFS that trying to employ, the triangular wave and trying to integrate the triangular wave by multiplying by the various complex exponentials or the various harmonics and the integrating to derive the various coefficients of the CEFS ok.

Slightly it is an interesting and very intuitive that is to consider the derivative of the triangular wave which is square wave and then approach the CEFS from that triangle.

All right let us proceed to the next problem in the next problem is fairly simple, but yet it again another intuitive problem that is problem number 5.

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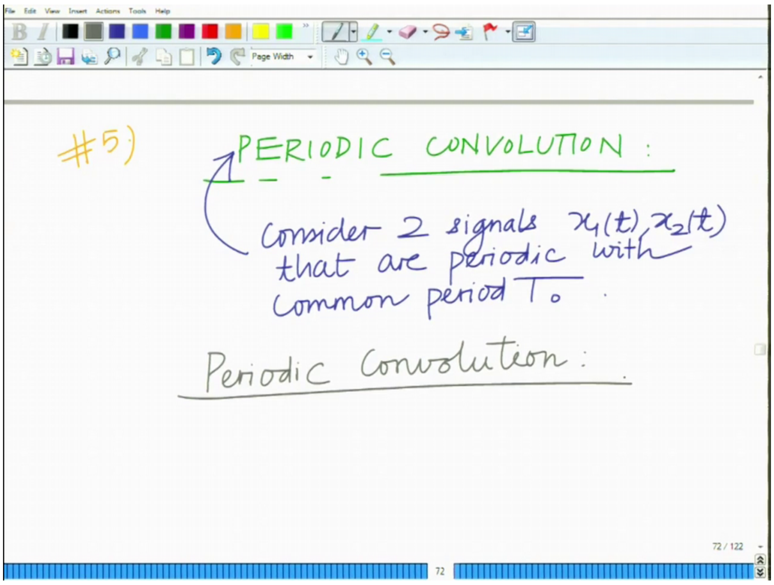

And this problem is regarding periodic convolution. Now consider is periodic convolution, now consider 2 signals x 1 t and x 2 t which are periodic with common period T 0, with the common period T 0 now the periodic convolution is defined as now the periodic convolution of these 2 signals with common period T 0 is x tilde t equals x 1 t convolved with x 2 t and this is symbol for this periodic convolution.

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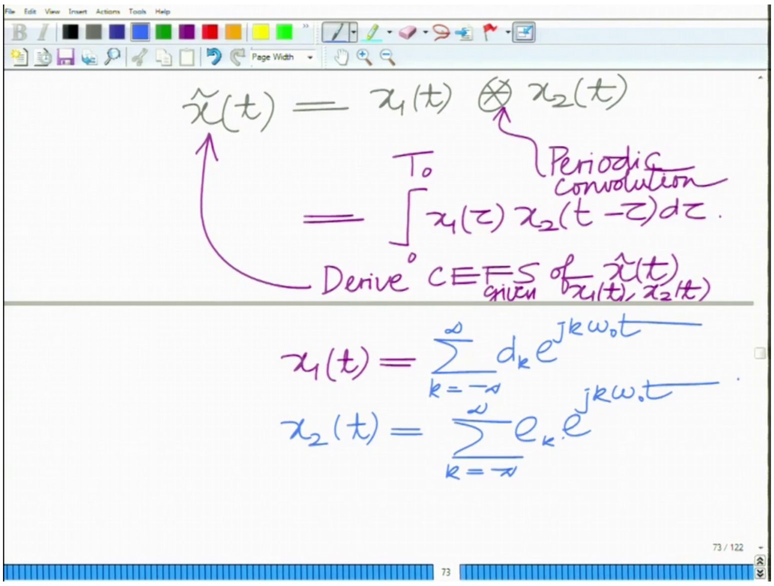

This is your periodic convolution and this is done over this is convolution basically over a period over any period that is x 1 tau x 2 minus tau.

So, while in the conventional convolution we are integrating from a tau from minus infinity to infinity remember this is for 2 periodic signals with the common period T 0 and integration is over only 1 period the integration is over only 1 period and now let us derive a property of this periodic convolution let now what we want to do is derive the CEFS derive CEFS of x tilde t given the CEFS of x 1 t and x 2 t that is, let us say x 1 t equals summation k equals minus infinity to infinity d k e raise to j k omega naught t, x 2 t equals summation k equals minus infinity to infinity e k e raise to j k omega naught t.

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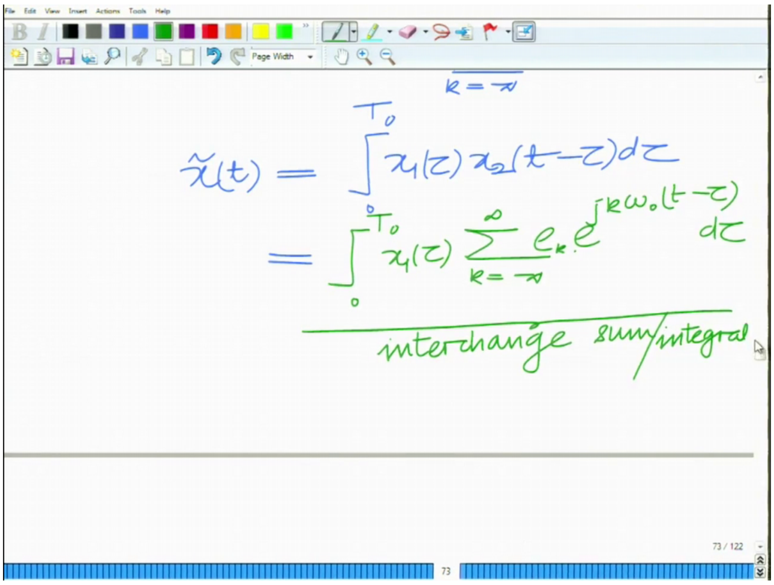

Then now remember x tilde t is simply the periodic convolution which is integrals over 1 period 0 to T 0 x 1 t or x 1 tau x 2 t minus tau t tau, which is integration over 1 period x 1 tau now it is now x 2 tau you substitute the CEFS.

So, that gives the summation k equals minus infinity to infinity C k e raise to j k omega naught t minus tau d tau. And, now you interchange the summation and the integration and that gives us and if you interchange, that is if you interchange sum and the Integral that gives the summation k equals minus infinity to infinity e k integral 0 to T 0.

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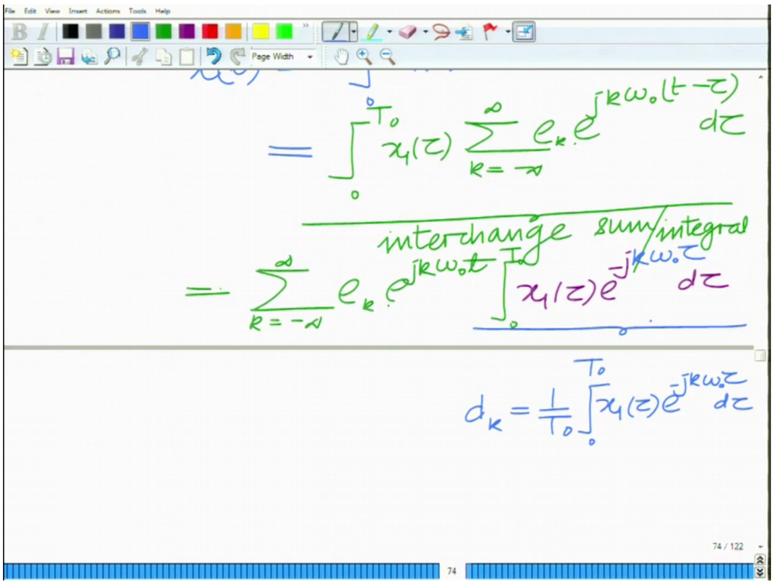

Ah well let us write it this way we can split the e raise to j k omega naught t. So, e raise to j k omega naught t it does not depend on tau comes out of the integral what is remaining is 0 to T 0 x 1 tau integral x 1 tau e raise to minus j omega naught tau d tau.

And we can now see this quantity which is nothing, but integral x 1 tau multiplied by e raise to minus j k omega naught tau d tau over 1 period tau not is nothing, but the complex exponential Fourier series of the co complex exponential that is the CEFS coefficient of x 1 tau that is the k th coefficient of the complex exponential Fourier series of x 1 tau which is nothing, but your d k.

So, at this. In fact, not d k, but this is. In fact, T 0 times d k because you can see d k equals 1 over T 0 tau naught T 0 integral 0 to T 0 x 1 tau e raise to minus j k omega naught tau d tau.

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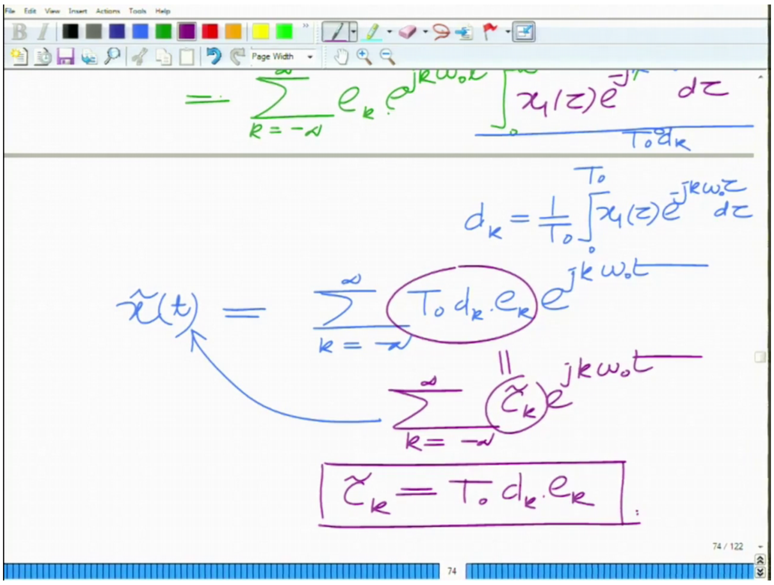

So, this is in fact, your. So, this is. In fact, T 0 d k. So, finally, you get this is summation k equals minus infinity to infinity T 0 d k e k e raise to j k omega naught t. This is the complex exponential Fourier series which implies that if we have the complex exponential Fourier series coefficients, if we have the complex exponential Fourier series of this as k equals minus infinity to infinity C tilde k e raise to j k omega naught t we would have T 0 d k e k equal to C tilde k, which implies C tilde k complex exponential Fourier series coefficient of x tilde k which is the periodic convolution of x 1 T and x 2 t is T 0 C k d k this is the CEFS coefficient k th CEFS coefficient of periodic.

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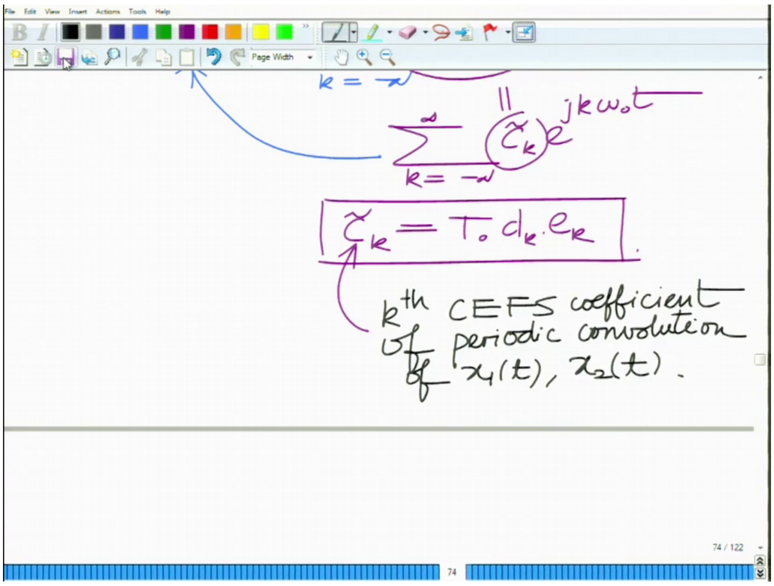

This is the k th CEFS coefficient of the periodic convolution of x 1 t and x 2 t. Let us say that completes basically that completes our discussion of examples for the discrete Fourier series that is the complex exponential Fourier series and the trigonometric Fourier series of continuous time periodic signals in subsequent module, that is starting with the next module we look at examples for the Fourier transform, that is the Fourier expansion for or the Fourier analysis for continuous time a periodic signal all right. So, let us stop here and continue in the later modules.

Thank you very much.