Principles of Signals and Systems
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Lecture - 37
Fourier Transform of Exponential, Unit Step Function, Properties of Fourier Transform – Linearity, Time Shifting, Frequency Shifting, Time-Reversal

Hello, welcome to another module in this massive open online course. So, we are looking at the Fourier transform and the properties of the Fourier transform of continuous signals all right. So, let us continue our discussion.

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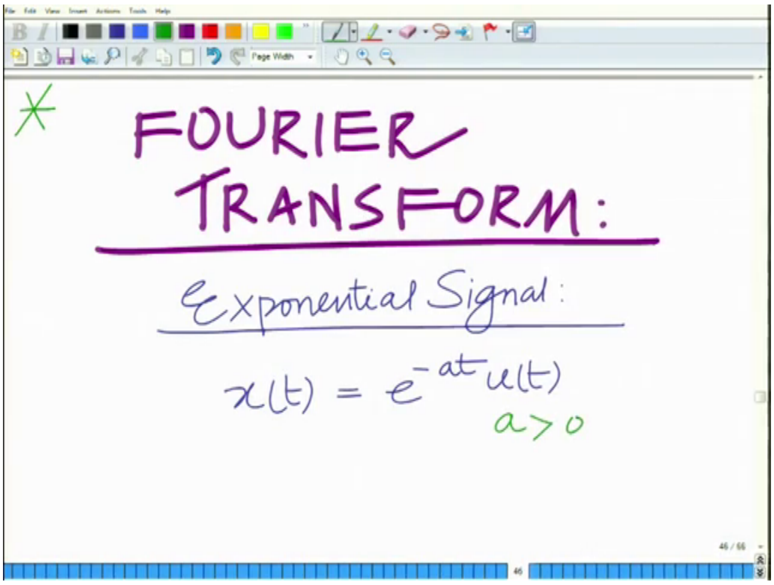

So, we are looking at the Fourier transform and the properties of the Fourier transform. We have looked at the Fourier transform of the impulse signal ok.

Let us now look at the Fourier transform of the exponential of an exponential signal; that is a signal of the form xt equals e raised to minus t ut assuming that a is a positive quantity a is greater than 0. Now, notice that this signal is absolutely integrable ok.

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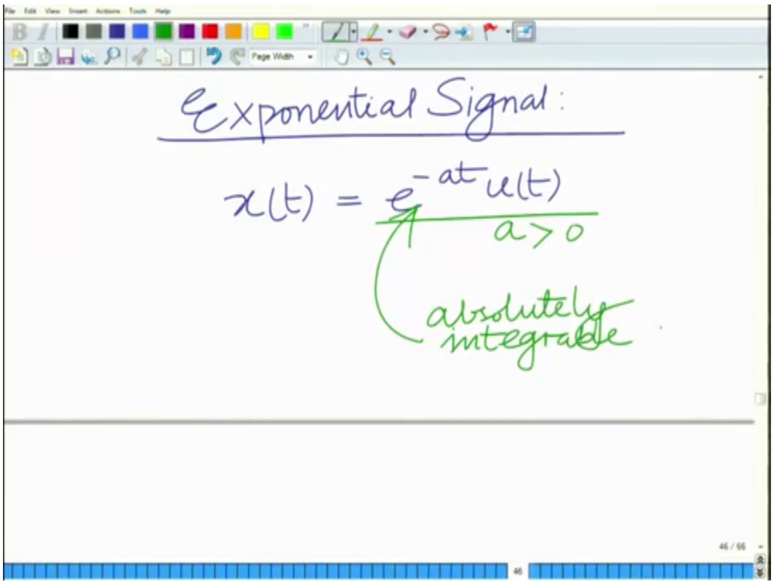

Notice that this is an absolutely if you look at this signal this signal is an absolutely integrable signal this is an absolutely integrable signal and in fact, if you look at the Laplace transform of this.

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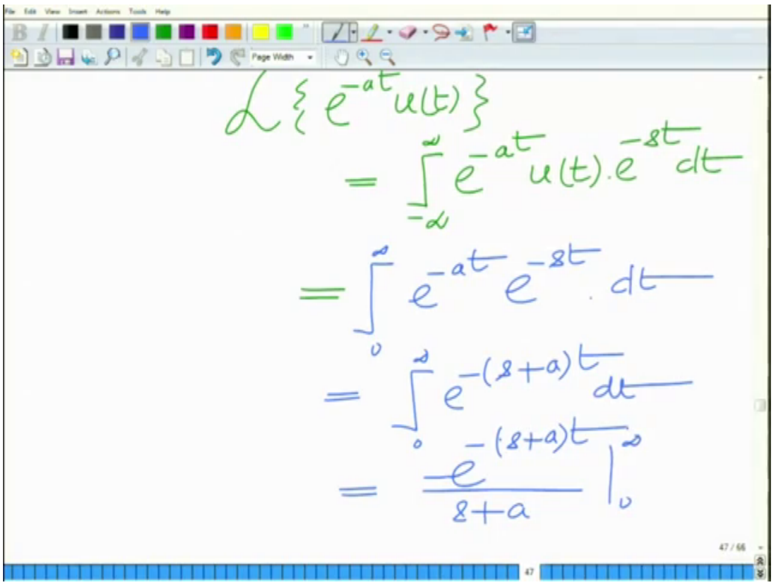

You can check the Laplace transform of this the Laplace transform of this is Laplace transform of e raised to minus t u t this is integral minus infinity to infinity or integral. In fact, let me just write it integral minus infinity to infinity e raised to minus at u t e raised to minus s t dt

Now, ut is nonzero only for t greater than or equal to 0. So, this reduces to the integral 0 to infinity e raise to minus at and for t greater than equal to 0 this is 1. So, so e raise to minus at e raised to minus stdt in that is integral 0 to infinity e raised to minus s plus a dt which is 1 over s plus a times e raised to minus s plus a. In fact, minus 1 over s plus a e raised to minus s plus a 2 e raised to minus s plus a t evaluated between the limit 0 to infinity.

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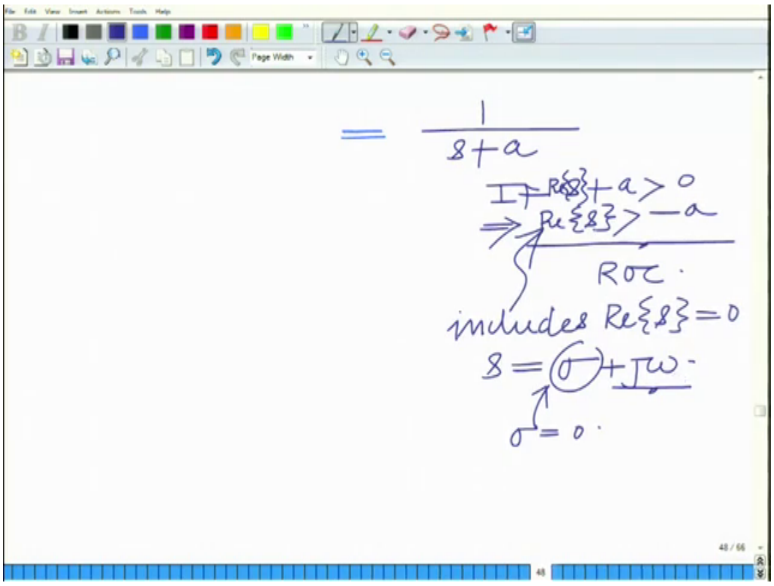

And this is equal to if s plus a is greater than equal to 0, this is equal to 1 over s plus a; if s plus a greater than 0 implies that is remember real part of s plus a greater than 0; this implies that the roc therefore, from the Laplace transform you will remember that real part of s is greater than minus a

Now, since a is positive since a is positive this includes real part of s equal to 0 includes ok. So, what we are observing is that; if you look at the roc of the Laplace transform that is real part of s is greater than minus a, but since we have said a is greater than 0; that is a is a positive quantity therefore, real part of s greater than some negative quantity which implies that real part of s is equal to 0 is in the ROC.

And, now if you look at this the complex frequency s equals sigma plus J omega real part of s equal to 0 implies sigma equal to 0. So, implies J omega is in the roc.

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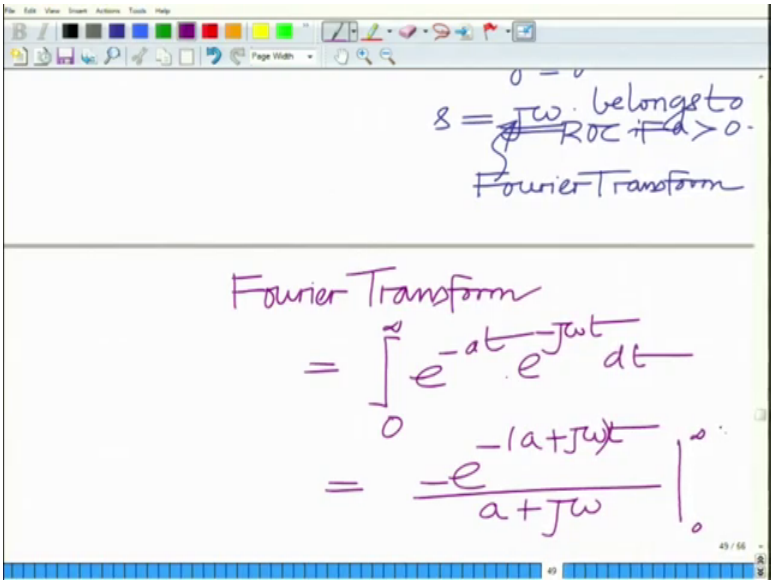

So, sig[ma] s equal to J omega belongs to the roc if a is greater than 0 and this is nothing, but s equal to J omega this is nothing, but a s equal to J omega we are nothing, but the Fourier transform and. In fact, if you evaluate the Fourier transform you will observe that the Fourier transform equals integral minus infinity to infinity e raised or directly write integral 0 to infinity e raised to minus at e raised to minus J omega t dt, which is e raised to minus a plus J omega t divided by a plus J omega evaluated between the limits 0 to infinity.

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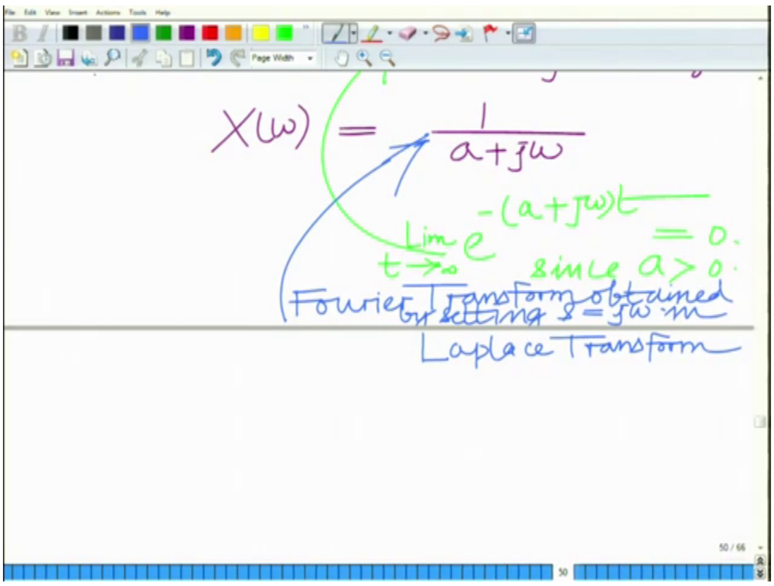

And this would be 1 over a plus J omega and this is your x of omega is 1 over a plus J omega.

And, here we are assuming that we are setting e raised to minus a plus J omega t limit t tending to infinity equals 0 since is greater than 0 that is for a greater than 0. Remember this is a decreasing exponential it is absolutely integrable therefore, it is Fourier transform exist and. In fact, the Fourier transform is obtained by setting s equal to J omega in the Laplace transform remember the Laplace transform is 1 over s plus a set s equal to J omega you obtain 1 over J omega plus a which is. In fact, the Fourier transform ok.

So, the Fourier transforms. So, it is absolutely integrable and observe that the Fourier transform Fourier transform is observed is obtained by setting by setting s equal to J omega in the Laplace transform is obtained by setting s equal to J omega in the Laplace transform and also note that this follows since x of t the signal x of t is absolutely neighbor.

Now, let us see a scenario where that is not to let us consider a signal which is not absolutely integral.

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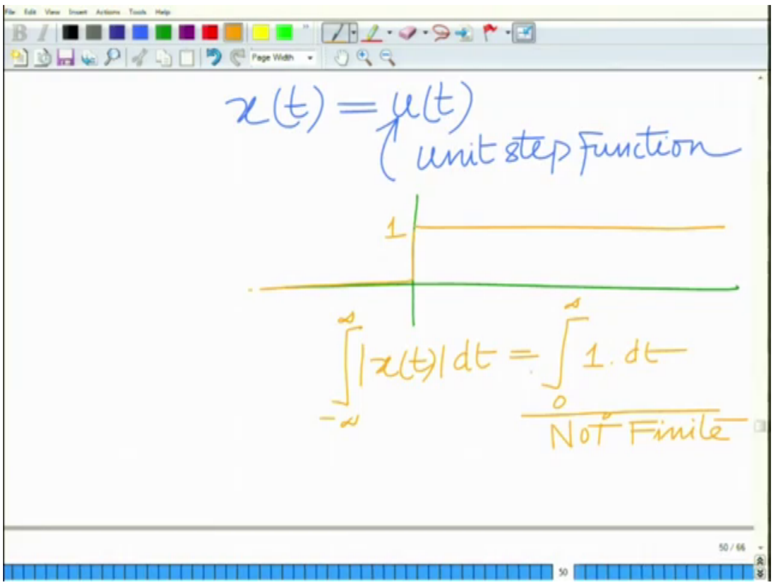

For instance consider xt equals the unit step function xt equals the unit step function and that looks like this if you look at 0 for t less than equal to 0 for t greater than equal to 0 0 for t less than 0 this is 1 for t greater than or equal to 0.

And, now if you look at integral minus infinity to infinity magnitude xt dt that is integral 0 to infinity times 1 dt and this is not finite you can clearly see this is not finite.

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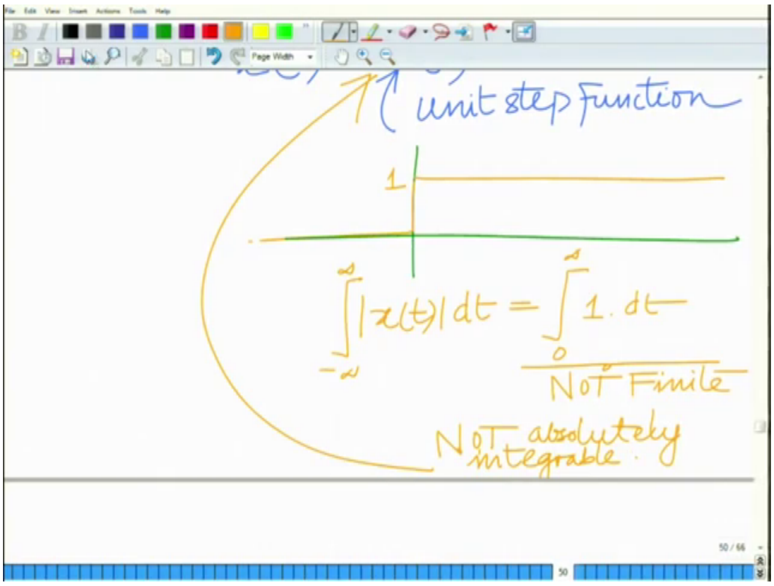

And therefore, this signal is not absolutely integrable not absolutely this is not absolutely integrable.

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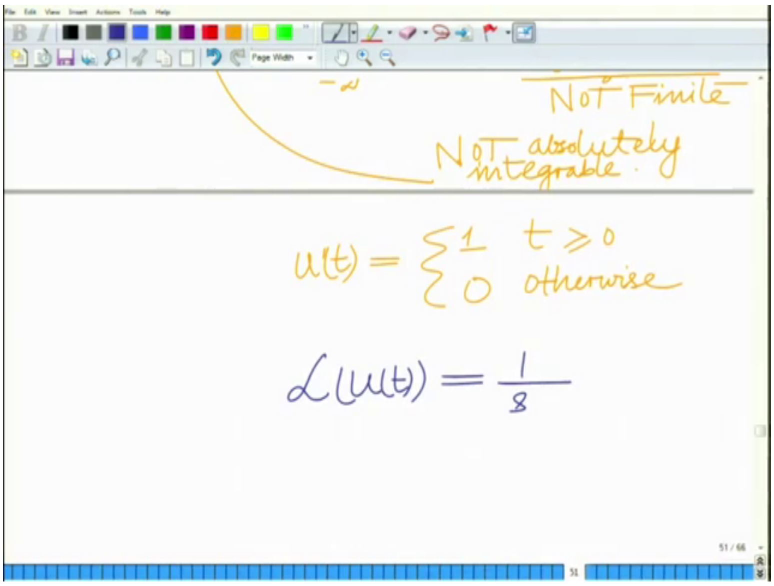

And. In fact, if you look at the Laplace transform. So, u t remember ut let me just write down the expression for ut ut equals 1 for t greater than equal to 0 this is 0 otherwise and. In fact, you will remember that the Laplace transform of u t equals 1 over s ok. So, the Laplace transform of u t is 1 over s and the region of convergence roc of the Laplace transform remember the roc of the Laplace transform is a real part of s greater than 0, which means the sigma that is a real part of this complex frequency has to be greater than 0 all right.

So, it does not equal sigma. So, it does not equal include sigma equal to sigma equals 0, in the region of convergence therefore, the J omega axis is not present in the region of convergence of this signal ok, which means that you cannot employ the straight Fourier transform in a straightforward fashion ok, region of convergence ROC.

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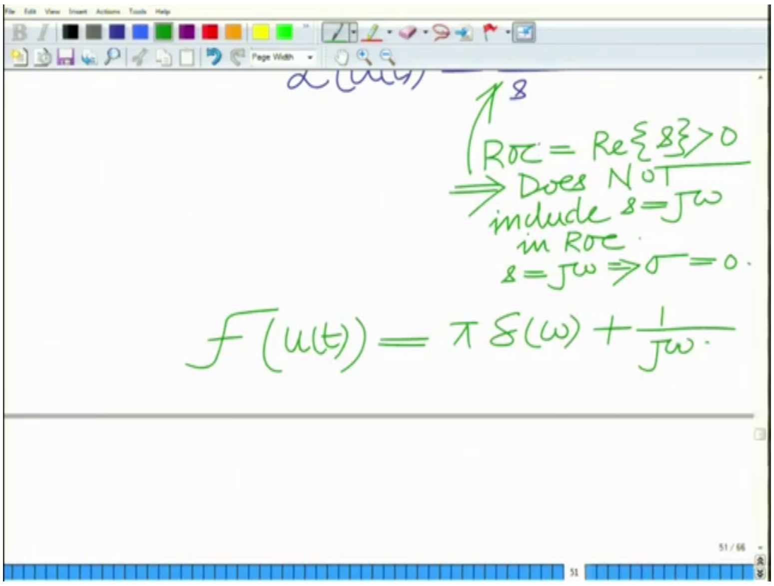

A real part of s greater than 0 implies does not include s equal to J omega in roc s equal to J omega implies sigma equals 0; that is a real part of s that is equal to 0.

And in fact, if you look at the Fourier transform you cannot evaluate this in a straightforward fashion, but the Fourier transform of this exists and the Fourier transform it can be shown to be given as pi of delta omega plus 1 over J omega. In fact, if you have if you set s equal to J omega in the Laplace transform all you will obtain is 1 over J omega. So, you will miss this factor which is pi times delta omega.

In fact, this is therefore, you can see that the Fourier transform is not obtained by setting s equal to J omega in the Laplace transform; because well s equal to 0 does not that is real part of s equal to 0 is not in the region of convergence and you can also see that this function is not absolutely integrable. So, naughty this is not obtained from Laplace transform not obtained from the Laplace transform by setting.

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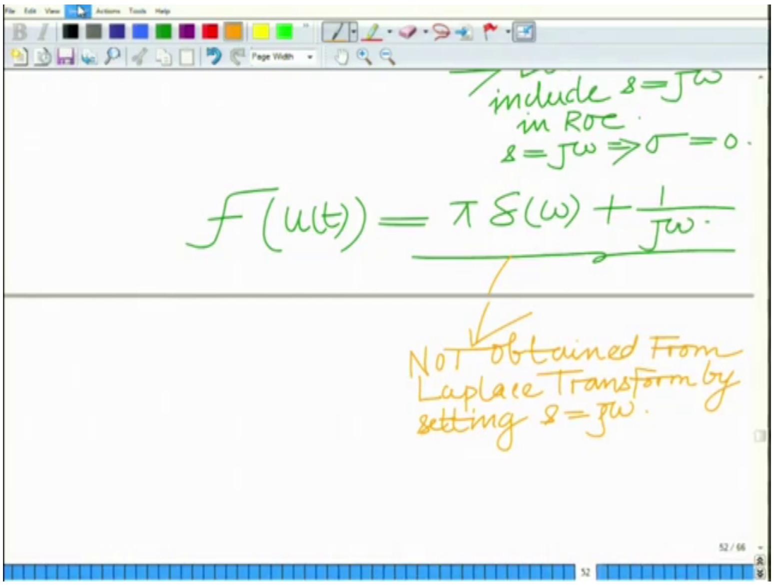

S equal to J omega there is a Laplace transform is not obtained by replacing s equal to J omega in the. So, the Fourier transform is not obtained from by replacing s equal to J omega in the Laplace transform; because it is not absolutely integrable ok.

Let us, now continue our discussion let us look at the properties of the Fourier transform or let me just start this on a different page we want to look at the properties of the Fourier transform.

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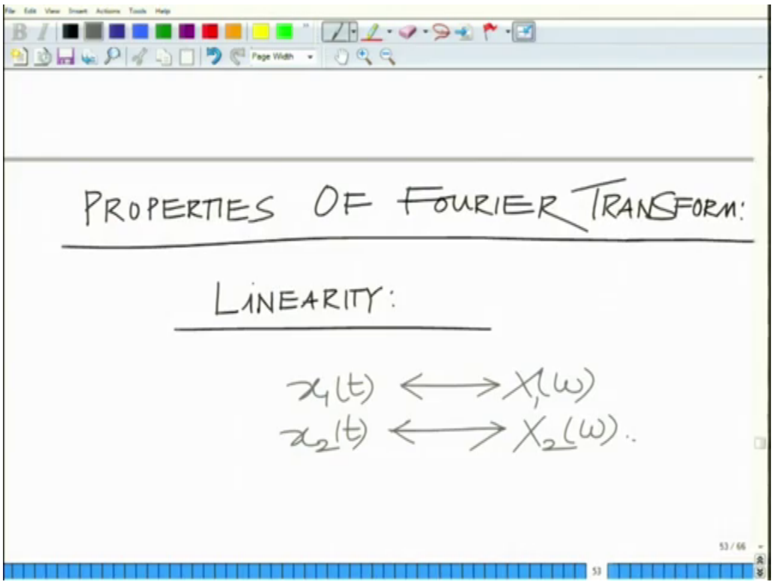

And the first property that we want to look at is what is termed as the linearity property of the Fourier transform; it is very simple if you have 2 signals x 1 t with Fourier transform X 1 omega omega is angular frequency x 2 t which has the Fourier transform X 2 omega.

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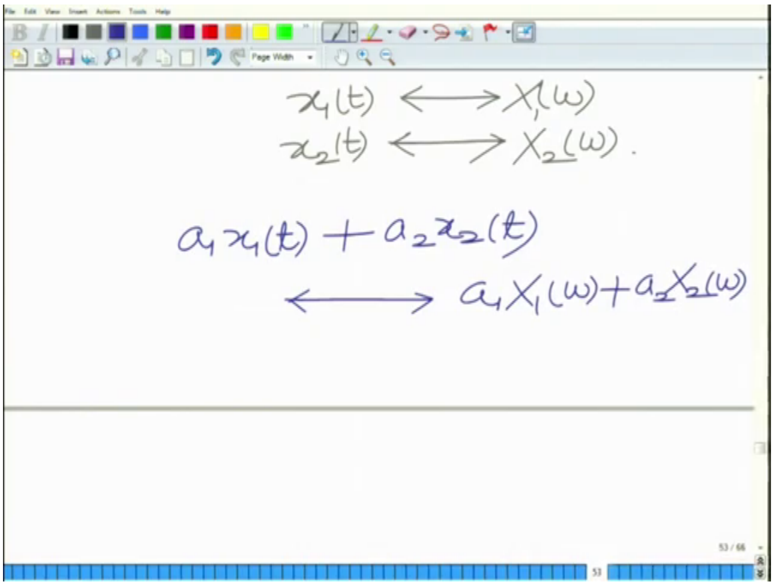

Then x 1 t plus x 2 t or a 1 x 1 t plus a 2 x 2 t has the Fourier transform a 1 x 1 omega plus a 2 x 2 omega ok; this is the Fourier transform.

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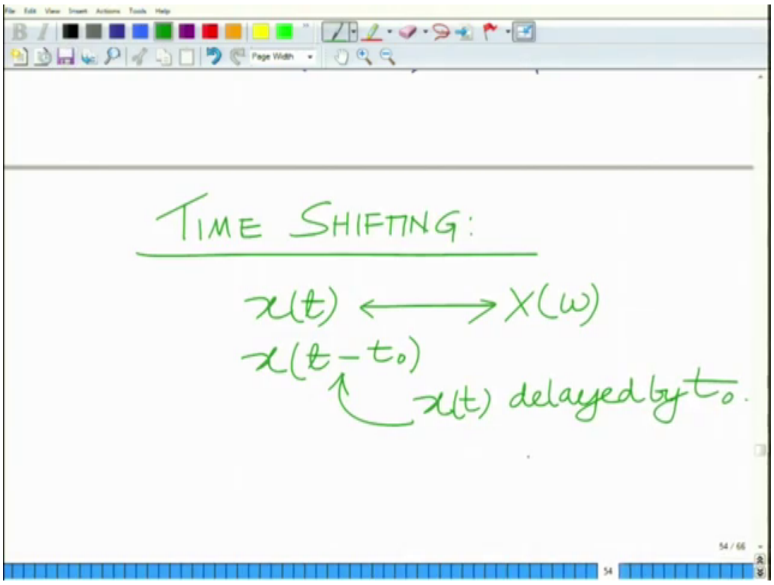

The time shifting property let us look at another property which is the time shifting property which is xt has the Fourier transform X of omega consider a time shift x of t minus t naught that is this is x of t x of t delayed by t naught that is the signal x of t minus t naught. So, we are considering x of t which is the Fourier transform X of omega capital X of omega what is the Fourier transform of x of t minus t naught and this can be obtained as follows; now let us call this Fourier transform as x tilde of omega

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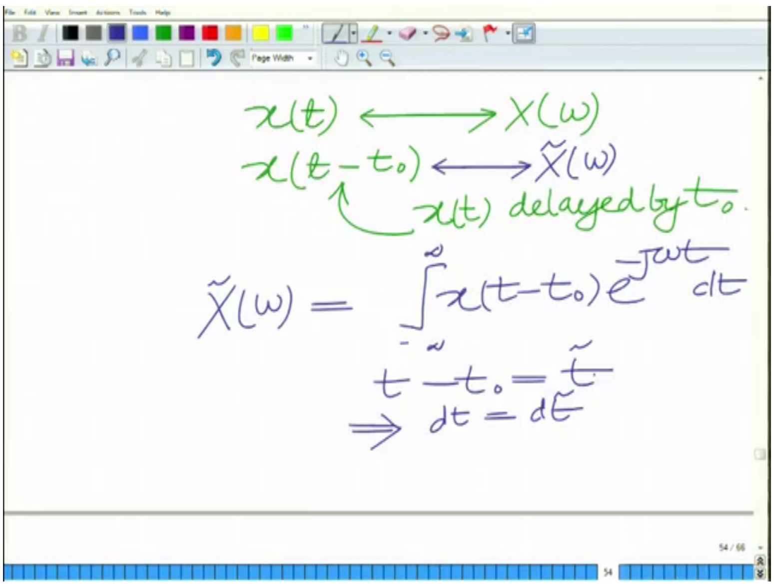

Now X tilde of omega is minus infinity to infinity x of t minus t naught e raised to minus J omega t dt. Now set t minus t naught equals t tilde that implies dt equals d t tilde and therefore, now if you simplify this.

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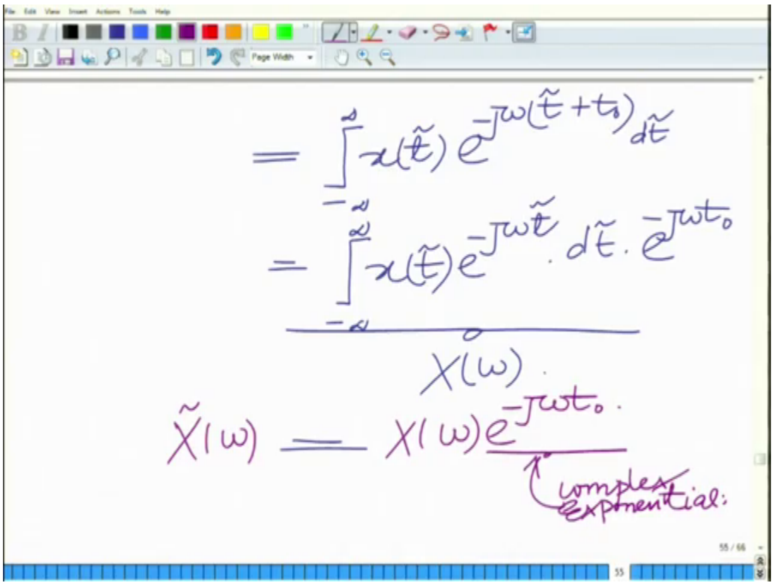

I can write this as minus infinity to infinity x of t minus t naught is x of t tilde minus J omega naught t is basically t minus t naught is t tilde. So, this is basically t tilde t is t tilde plus t naught dt is d t tilde which is basically now minus infinity to infinity x of t tilde e raised to minus J omega t tilde t tilde e raised to minus J omega t naught and if you look at this this is nothing, but X of omega. So, we have finally, that X tilde omega equals X of omega e raised to minus J omega t naught and this is basically you can see that this is equal to a complex exponential there are complex exponential ok

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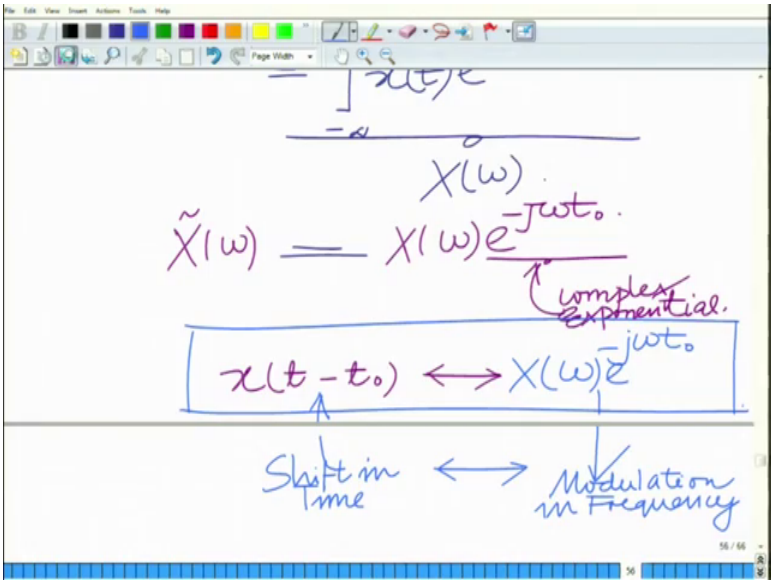

So, basically what we are seeing is that x of t minus t naught has the Fourier transform X of omega e raised to minus J omega t naught. So, this is the Fourier transform; that is basically shift in time which implies multiplication by or modulation this complex exponential is nothing, but modulation. So, this is modulation in shift in time basically results in a modulation in the frequency all right. So, if a shift in time xt minus t naught the signal it is delayed by denote the corresponding Fourier transform is multiplied by this factor e raised to minus J omega complex exponential which is e raised to minus J omega t naught in the frequency ok.

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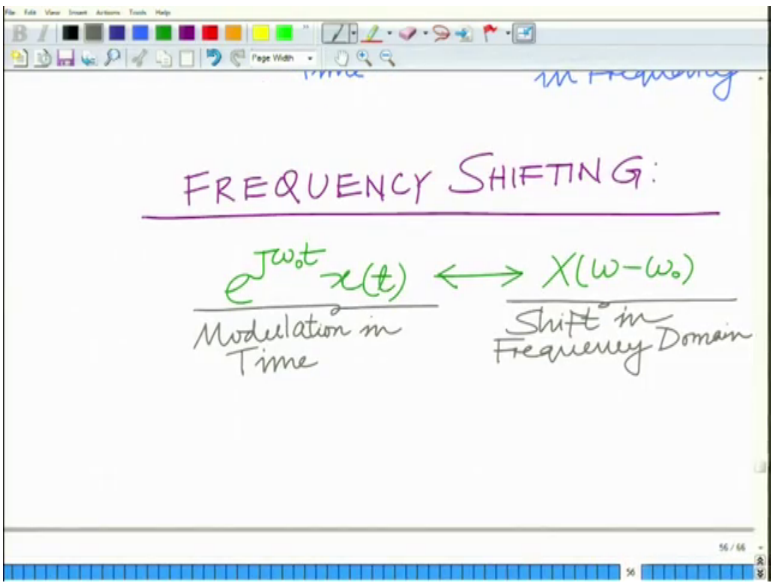

Now, let us look at another property which is the frequency shifting property of the Fourier transform. So, let us look at the frequency shifting property you know the frequency shifting property this states that again it is a dual of the time shift that is if you have the modulation if you have a modulation in time e raised to J omega naught t x of t in frequency you have x.

So, this basically shows and you can easily demonstrate this similar to what we have done above. So, this is basically modulation in time that is multiplying by this complex exponential e raised to J omega naught t this is equal to shift in frequency domain. This is basically a shift in the frequency this results in a shift in the frequency domain and then another interesting property is basically the time scaling property.

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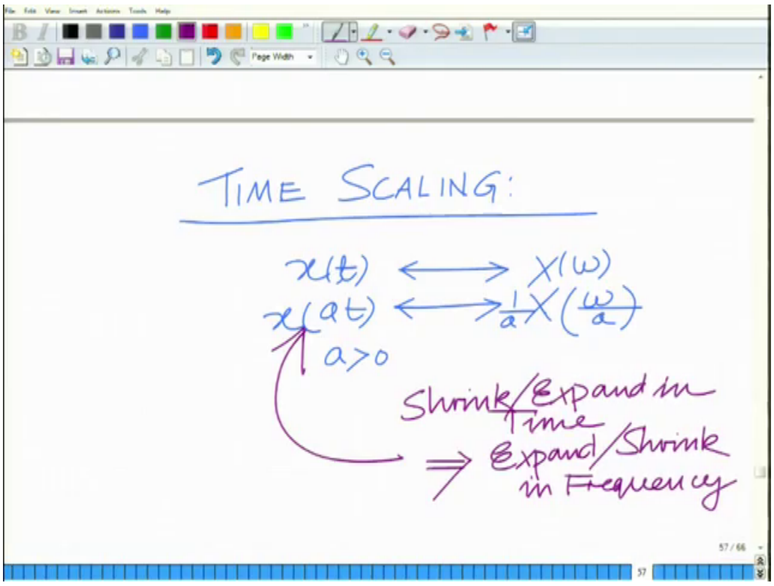

And this is as follows consider a greater than 0, that is a positive constant then now let us consider x of t ok. Let us start with that x of t as the Fourier transform X of omega then x of a t where a is a positive scaling factor that is assume that a is greater than 0 as the Fourier transform 1 over a and X of omega over a.

Now this is interesting because now look at this if a is greater than 0 all right you can see that x of at is a shrunk version of x of t that is x is a is for instance let us say 5 x of 5 times t is basically x of t the shrinks by a factor of 5 by a factor of 5.

So, in the frequency domain, what happens is you have x of omega over 5 which is means it expands by a factor of 5. Similarly if a is less than 1 then x of at is an expanded version of a and correspondingly freaks in frequency domain X of omega over a is a shrunk version of X of omega. So, basically expansion in the time leads to a shrinking in the frequency and similarly a shrinking in a time all right scaling down in time leads to an expansion or scaling up in frequency ok.

So, basically what we have is it is a very interesting property. So, shrink slash expand in time leads to the opposite effect expand slash shrink in frequency. So, what happens is if it is well localized in time that is it becomes more and more localized in time it becomes less and less localized in frequency all right and you can clearly see the example of this that is when in time it becomes extremely narrow. It becomes an impulse in time, then frequency it is spread over the entire frequency axis which is basically it is one the you can see that the Fourier transform for impulse in time is one that is spreads over entire frequency axis minus infinity to infinity.

As it starts shrinking in time as it starts as it starts expanding in time then starts shrinking in frequency all right and this know this is a well known property of the Fourier transform the time domain representation and it is Fourier transform that is time frequency localization all right. So, it is well localized in time then it is not well localized in frequency it is well localized in frequency or it is poorly localized a time that is it is wider in time then it shrinks in the in the then it is as it starts to become wider in the time domain it starts to shrink in the frequency domain ok.

So, this is an important property and you can see this as follows the proof of this is straightforward for a equal to greater than 0.

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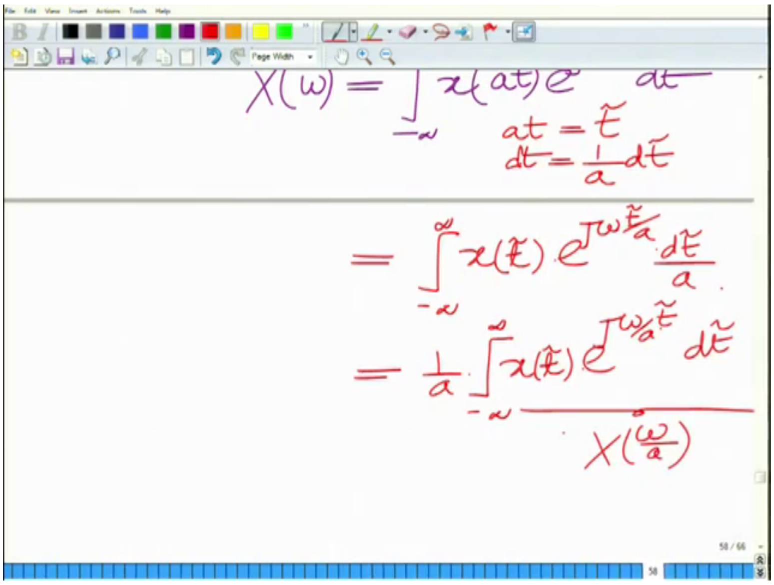

You have X tilde of omega the Fourier transform of x of a t is minus infinity to infinity x of a t e raised to J omega t dt set at equals t tilde which means dt equals 1 over a dt tilde which is minus infinity to infinity this is x of t tilde e raised to J omega t is t tilde by a d t tilde by a which is nothing, but 1 over a minus infinity to infinity x of t tilde e raised to J omega by a t tilde into d tilde this is nothing, but this is X of omega over a remember we are assuming a greater than 0 therefore, we have.

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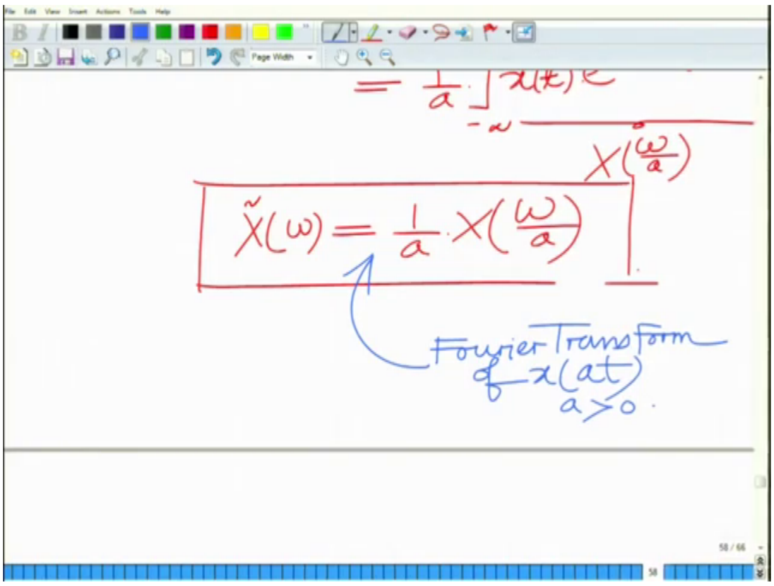

X tilde of omega equals 1 over a X of omega a.

So, this is the Fourier transform of x of a t where a is greater than 0 a is greater than 0 and similarly you can show, what is known as the time reversal property the time reversal property or the time reversal property the time reversal property of the Fourier transform states.

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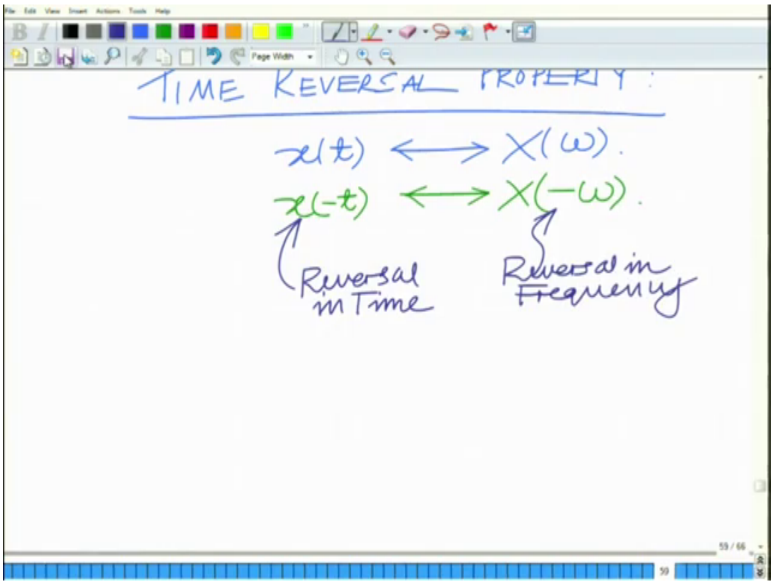

That if xt has the Fourier transform X of omega then you can show that x of minus t has the Fourier transform X of minus omega that is reversal in time leads to a reversal in frequency reversal in time leads to a reversal in frequency also this can also be shown very simply ah. So, we have reversal in time leads to a reversal in frequency this is the time reversal property of the Fourier transform all right.

So, what we have seen in this module is we have looked at some interesting properties. So, you consider continued our analysis of the Fourier transform we have looked at the Fourier transform the exponential and the unit step function and we also started looking at the at some of the interesting properties of the Fourier transform all right. So, we will stop here and continue in their subsequent modules.

Thank you very much.