Principles of Signals and Systems
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Lecture – 35
Conditions for Existence of Fourier Series – Dirchlet Conditions, Magnitude/ Phase Spectrum, Parseval's Theorem

Hello welcome to another module in this massive open online course. So, we are looking at the Fourier analysis and in particular the Fourier series of continuous time periodic function all right. So, in this module let us start by looking at the conditions for the existence of the Fourier series.

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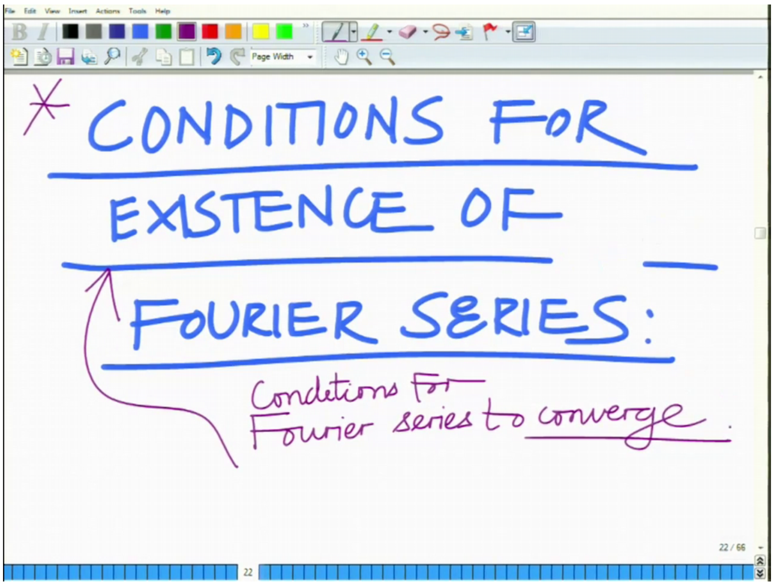

So, what we want to start with is the conditions for existence of Fourier series.

Under what conditions thus the Fourier series converge, no gas conditions for existence of the Fourier series. Basically known as a conditions for the convergence, for the Fourier series to converge that is conditions for the Fourier series to converge and these are also known as the these are also the Dirichlet, the Dirichlet conditions these are also known as the Dirichlet after the mathematician Dirichlet ok.

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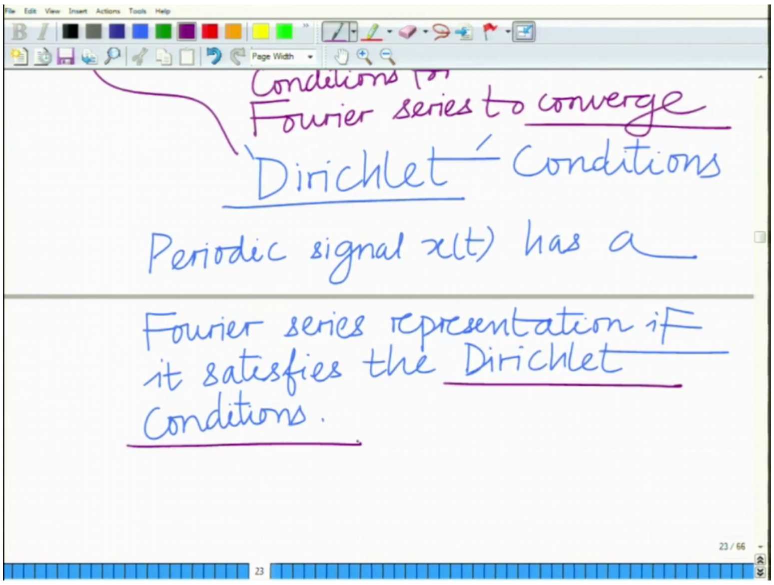

So, these are known as the Dirichlet conditions. So, next we has a Fourier series representation that is a periodic signal x t, has a Fourier series representation, if it satisfies the Dirichlet conditions. If it satisfies the Dirichlet conditions and these are as follows that is the Dirichlet conditions which are to be satisfied for the existence of the Fourier series are as follows that is x t is absolutely integrable over a period.

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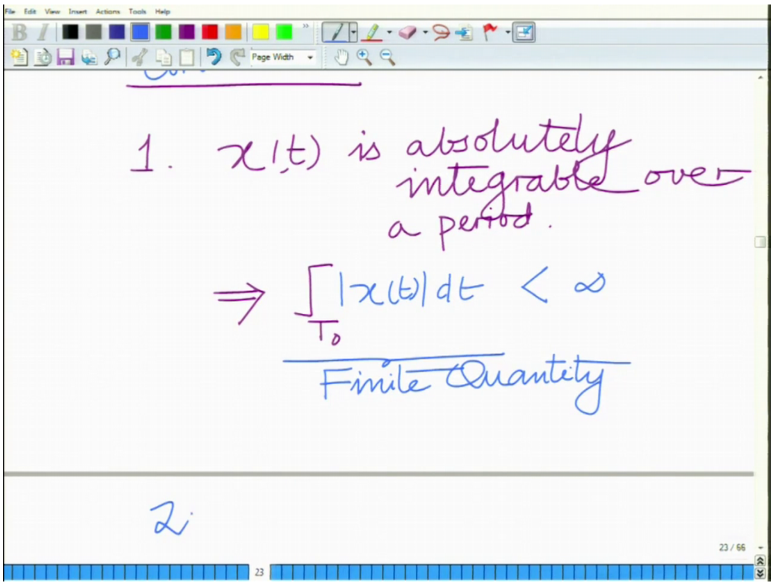

This implies that is if you look at this integral over a period of magnitude of x t d t this has to converge or this has to be a finite quantity. So, over a period if you integrate magnitude of x t that is integral only period t of duration T naught continuous period of duration T naught where T naught is a fundamental period correct integral over any fundamental period t naught magnitude x t d t has to be a finite quantity. So, this has to be a finite, this has to be a finite quantity and 2 x t has a finite number of maxima or minima.

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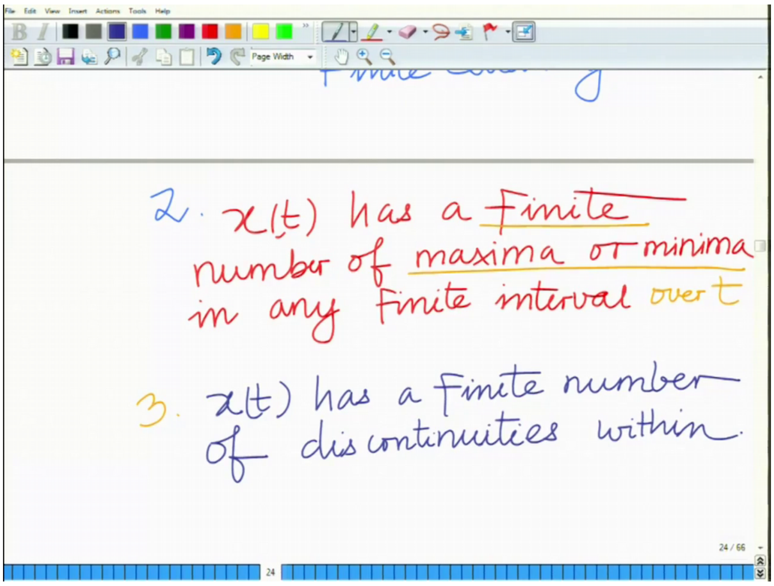

So, the other condition is x t has a finite number of maxima or minima, finite number of maxima in any finite interval in any in any finite number of maxima or minima, in any finite interval over time. In any finite interval over t and the third condition that is the number of that is we look at any finite any interval of a finite duration the number of maxima and minima there is a number of extremum, number of extrema of this signal x t has to be final ok.

And the third condition is that x t has a finite number of discontinuities, the signal x t has a finite the number of discontinuities, x t is a finite number of discontinuities within any finite interval of duration t.

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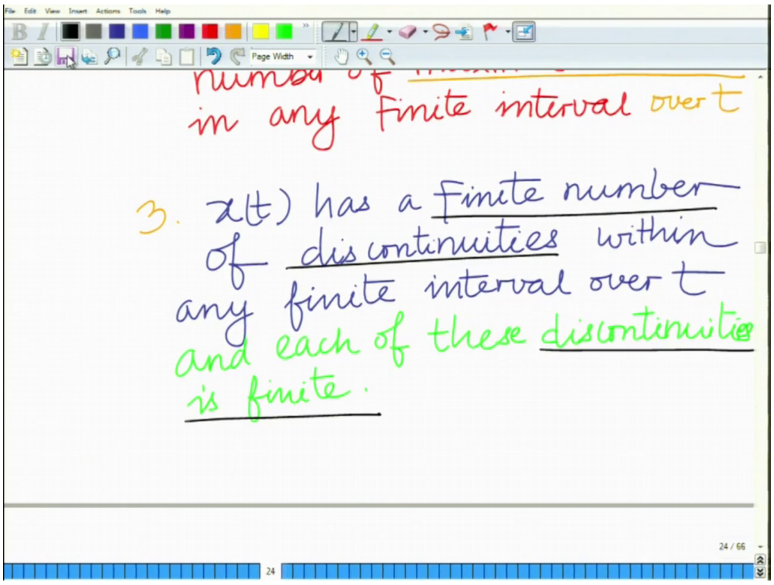

Any finite interval over t and each of these discontinuities is finite and moreover that it has a finite number of discontinuities, it has a finite number of discontinuities and each of these discontinuities is finite that is a third Dirichlet condition. So, these are the 3 Dirichlet conditions, Dirichlet conditions that is your first is that x t is absolutely integrable.

That is integral over any fundamental period t naught of the magnitude of x t has to be finite all right and x t has a finite number of maxima or minima in any inter finite interval of finite duration over time and finally, x t has a finite number of discontinuities and each of at each of these points the x t the signal x t is finite in any interval of finite duration over time ok.

Now, an important thing to note is that, these are sufficient conditions right; these are not necessary.

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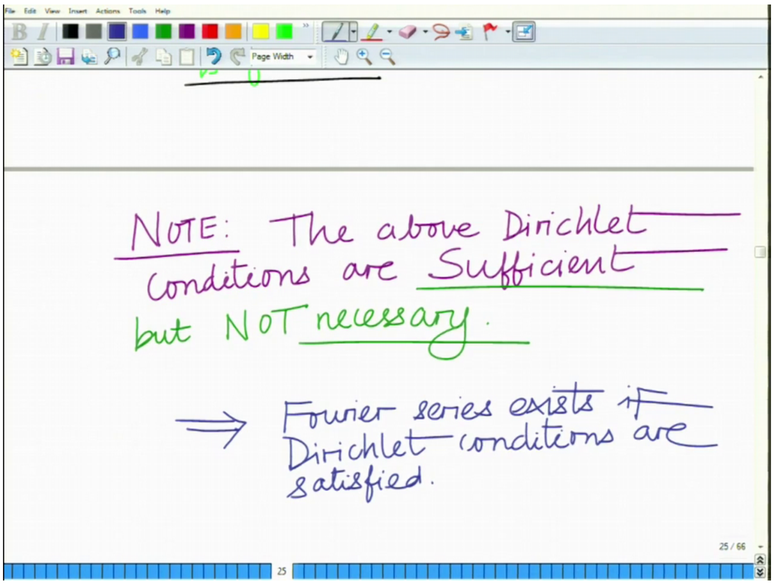

Note the above Dirichlet conditions, conditions are sufficient these are sufficient, but not necessary implies that the Fourier series exists.

If the Dirichlet conditions are satisfied, but not the other way round that is it is not that the Dirichlet conditions are satisfied if the Fourier series exists. So, these are sufficient conditions all right, which means that if these conditions are satisfied by this by the signal x t then the by a periodic signal x t by a continuous periodic signal x t then the Fourier series exists all right.

So, these are summary these are the conditions for the Fourier series to exist for a continuous periodic signal x t all right. Let us also look at the next topic which is the amplitude and phase spectrum.

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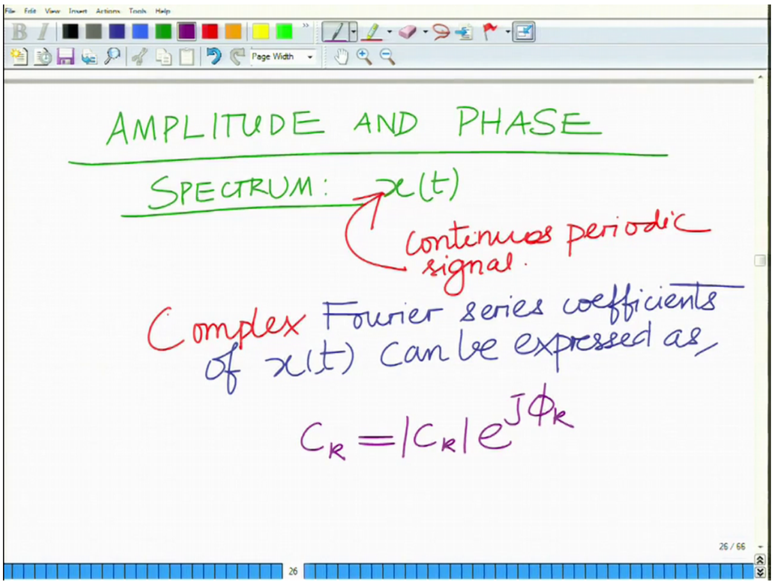

So, what we want to look at now is the amplitude and phase spectrum of this continuous periodic signal, amplitude and phase spectrum of x t we remember we are still looking at we are still looking at a continuous periodic signal, they are looking at a continuous periodic signal.

Now, the complex Fourier series coefficients of x t remember these are complex, the complex of x t can be expressed as magnitude this can be expressed as ck equals, magnitude ck e raise to j phi k that is this is the magnitude of ck and this is the phase of ck, this is the phase.

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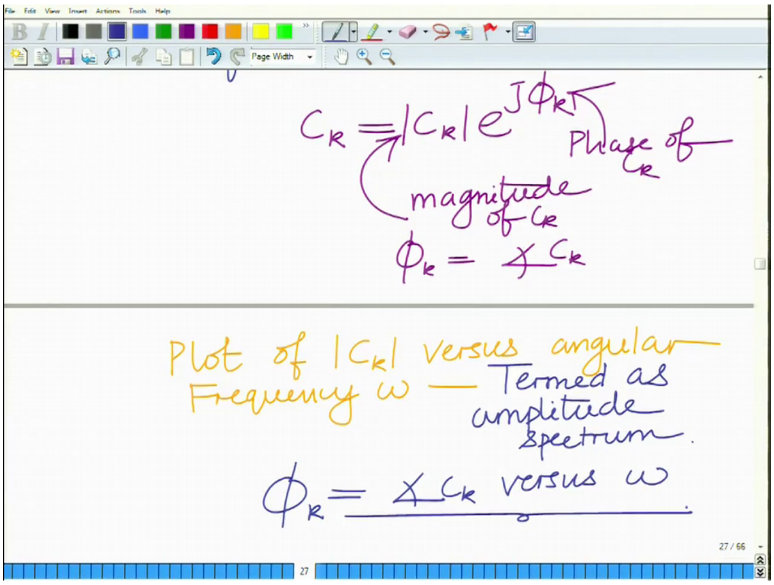

In fact, you can also write phi k equals the angle of this coefficient ck ok.

Now, plot, if you plot this magnitude ck versus the frequencies the discrete frequencies that is k omega naught remember the coefficients e k are defined for the fundamental frequency omega naught and multiples of the fundamental frequency that is at harmonics corresponding to omega naught. So, if you plot this magnitudes of the Fourier coefficients ck versus k omega naught all right verses or verses omega this is termed as the magnitude spectrum of the signal x t. So, plot of, plot of magnitude ck versus angular frequency omega correct angular frequency omega this is termed as the amplitude spectrum.

This is termed as a amplitude spectrum and similarly my plot of phi k that is which is the angle of ck versus omega this is term as your phase spectrum the plot of phi k versus omega that is the phase, versus omega this is termed as the phase spectrum

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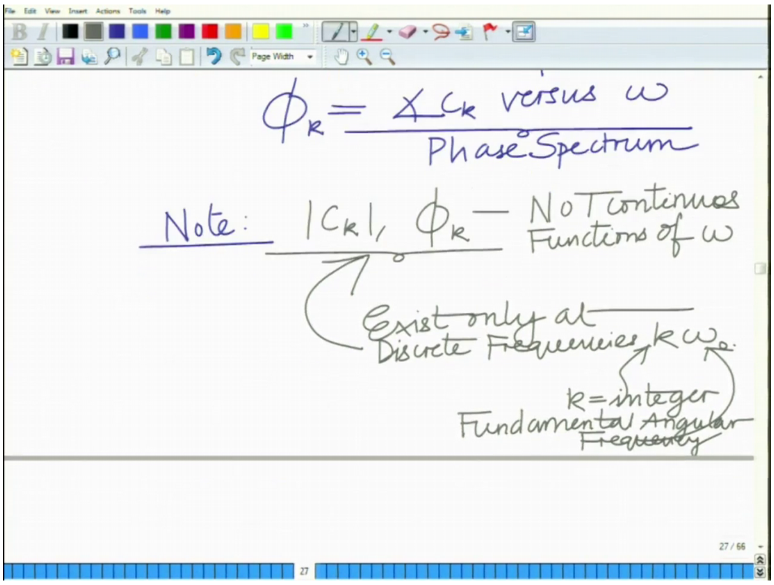

Now, observe that these only exist that is this magnitude ck phi k only exists at discrete set of points that is at k times, omega naught where omega naught is the fundamental angular frequency therefore, this is also known as a discrete frequency spectrum or also known as a line spectrum ok.

So, note so note that ck. So, note that magnitude ck comma phi k these are not continuous functions of omega of the angular frequency omega, these exist only at discrete set of frequencies, k omega naught where k is an integer and omega naught equals fundamental angular frequency ok.

These only exist at a discrete set of frequencies and therefore, hence termed as the discrete frequency spectra or line spectra.

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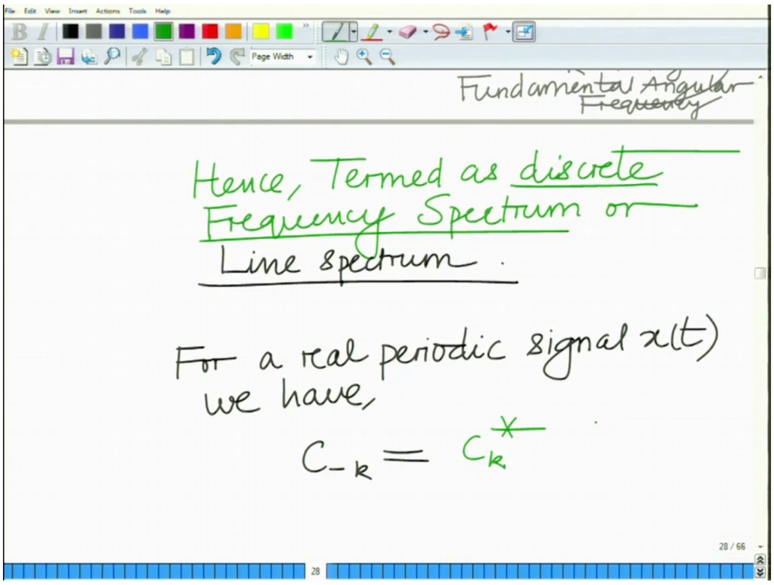

Hence termed as a discreet frequency spectrum or what is also known as a line spectrum, discrete frequency spectrum or a line spectrum. Now, further if you look at a real periodic signal, when the signal is real remember we have said that the Fourier coefficients right the Fourier coefficients that is C of minus k is the conjugate of the Fourier coefficient C of k corresponding to k of corresponding to the frequency k omega naught. So, for a real signal in fact, for a real periodic signal for a real periodic signal, x t we have the coefficient C of minus k equals C k conjugate.

Now, what this means is this implies.

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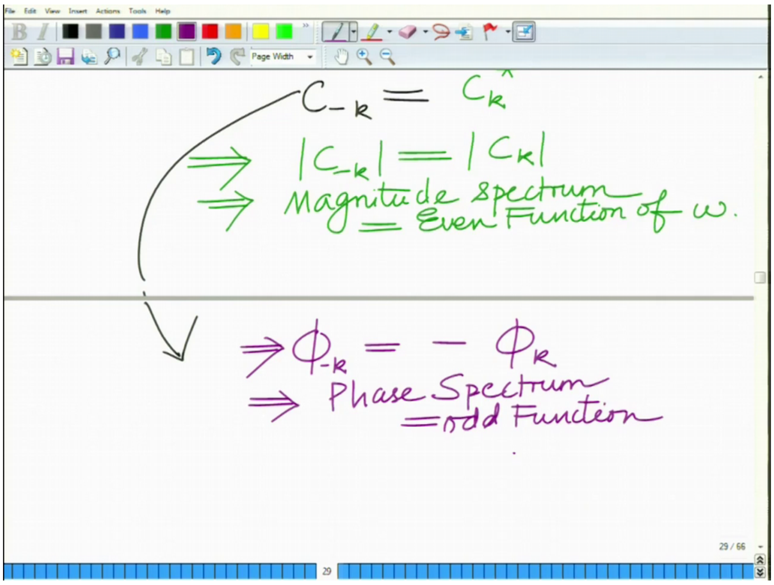

Firstly this implies that magnitude C of minus k equals magnitude because they are complex conjugate of each other magnitude of C of minus k equals magnitude equals magnitude C of minus C of k, implies the magnitude spectrum is an even function of frequency magnitude spectrum is an even function of omega. Further we have since they are complex conjugate of each other this also implies what this also implies is that the phase or angle of C of minus k phase of minus k is minus of phase of k.

This implies that the phase spectrum is an odd function. So, we have magnitude spectrum which is even function the phase spectrum which is an odd function this is for a real periodic signal x t all right. Let us now look at the power of this periodic signal and the property of the power.

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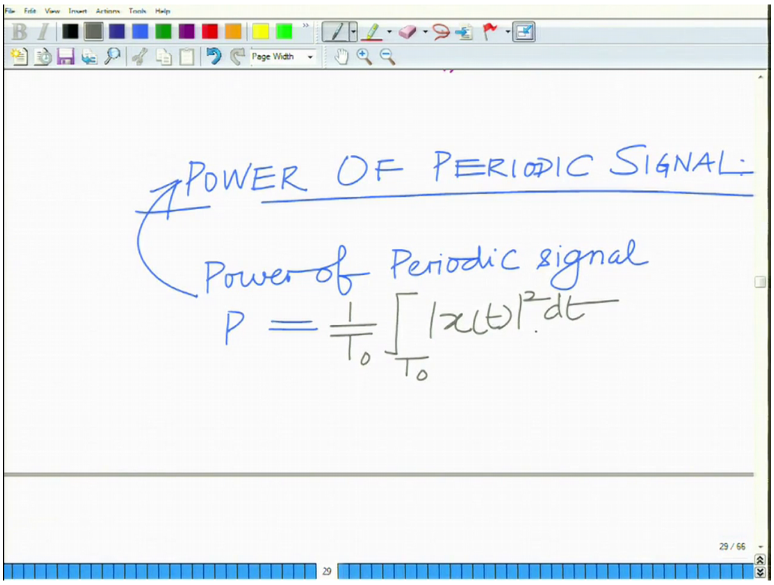

So, let us now look at another aspect that is the power of the.

Let us now look at this other aspect, which is the power of the periodic signal and we have power of the periodic signal, this is equal to this is given as P equals on1e over T naught, this is defined as integral 1 over T naught or any fundamental period t naught magnitude x t square d t this is the power of the periodic signal x t.

Now, what we are going to do is we are going to prove a property using the Fourier series. So, consider the Fourier series the Fourier series of x t.

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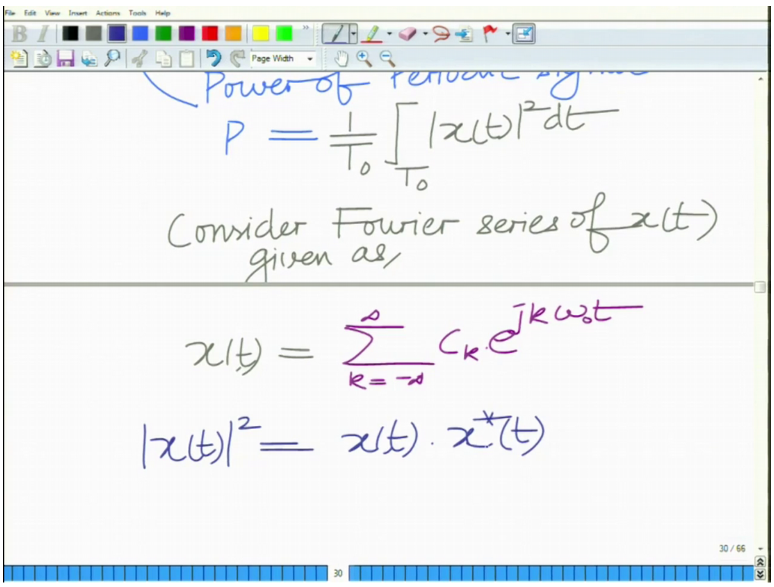

Now, consider the Fourier series of x t given as, x t equals summation k equal to minus infinity to infinity C k e raised to j k omega naught t. Let us say this is the Fourier series, then magnitude of x t square is remember magnitude of a quantity, magnitude of quantity square is simply x t times x t conjugate which is equal to, now first I am going to write x t.

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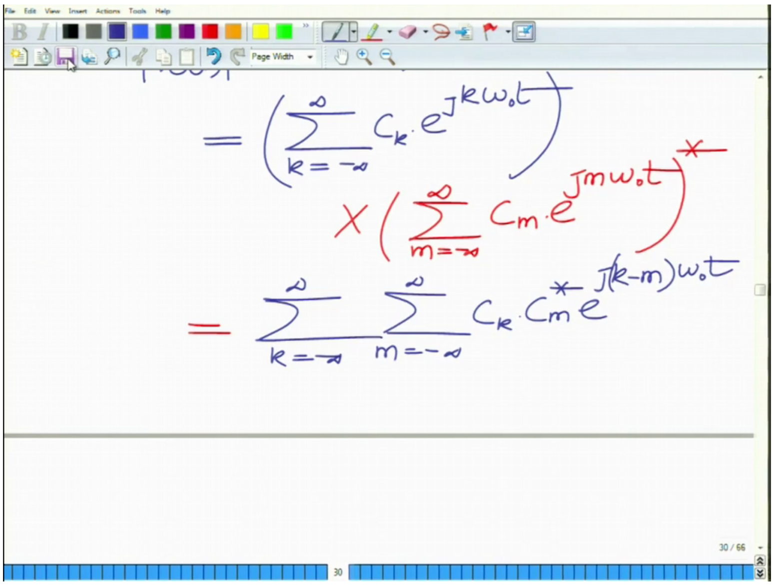

Yes which is we already have the expression in terms of the Fourier series that is summation over k, ck e raised to j, e raised to j, k omega naught t multiplied by x t conjugate. I am just going to change the index that is summation m equal to I can change the index summation m equal to minus infinity to infinity C of m, e raised to j m omega naught t whole conjugate and.

Now, if I open the brackets and expand this what I am going to have is I am going to have summation k equal to minus infinity to infinity, summation m equal to minus infinity to infinity, C k cm conjugate times e raised to j k minus m omega naught t. So, this is what I have. So, what I am doing is basically I am representing x t in terms of its Fourier series considering x t into x conjugate of t and that gives me magnitude x t square ok.

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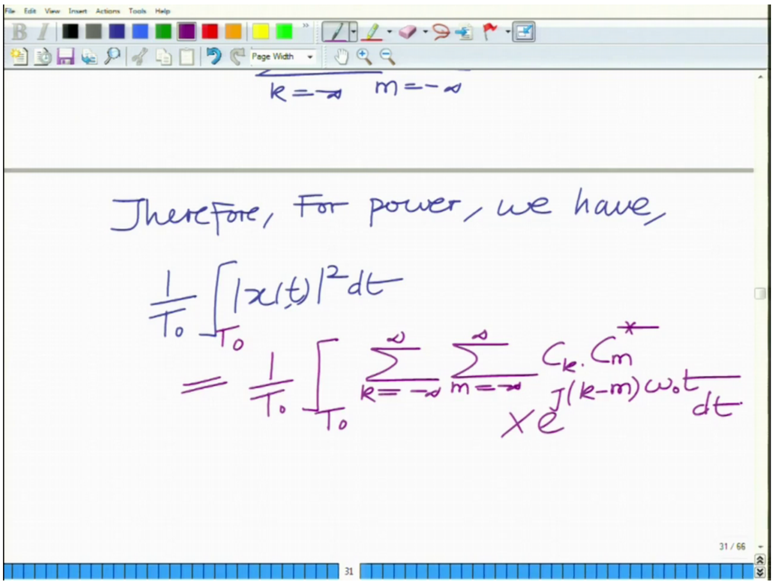

And now, therefore, for the power therefore, for the power we have 1 over t naught magnitude x t squared dt, this is 1 over t naught substitute the expression above for magnitude x t square. This is 1 over T naught that is integral, remember this integral is over any duration t naught any period or any time interval contiguous time interval of duration t naught summation m equals minus infinity to infinity, ck cm conjugate times e raised to j k minus m omega naught t dt and.

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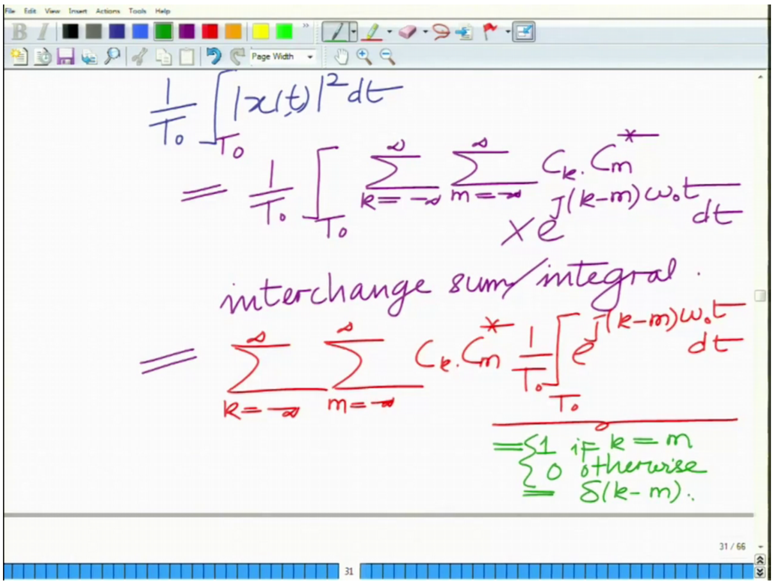

Now, if I integrate now if I interchange the sum slash integral we bring the summations outside. So, that will give me k equals minus infinity to infinity, summation m equal to minus infinity to infinity ck cm conjugate 1 over T naught integral over duration t naught e raised to j k minus m omega naught t t.

And we have seen this integral, that is integral of any harmonic of omega naught right any complex exponential with its frequency that is an integer multiple of omega naught you integrate it over the period t naught it is 0, unless all right unless k is equal to m in which case this is e raised to 0 which is 1 the integral over duration of t naught is t naught divided by t naught which is 1. So, this is equal to 1.

If k equals m and 0 otherwise implies this is basically the discrete impulse that is delta k minus m equals 1 if k equals m k equals to m and 0 otherwise.

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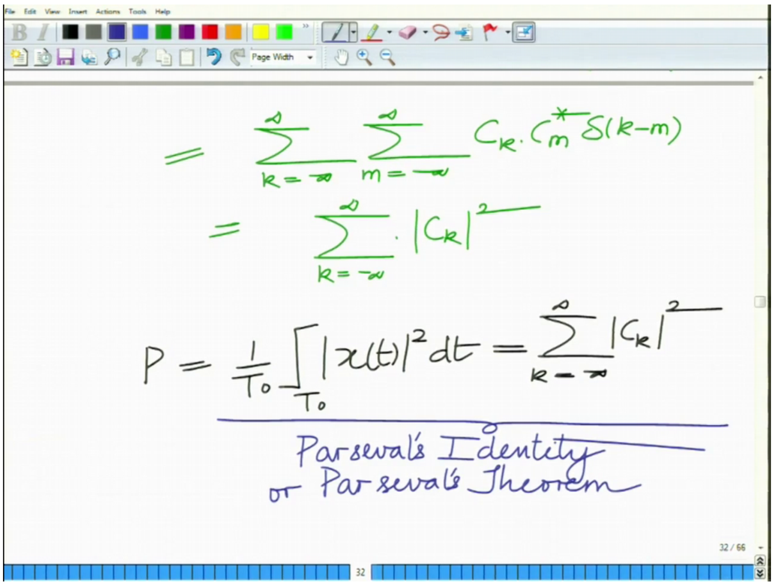

So, this is summation k equals minus infinity to infinity, summation m equals minus infinity to infinity ck, C will conjugate delta k minus m which is of course, now exists only that is now only the terms corresponding to k equal to m survive all the rest of the terms are 0.

So, this is summation k equal to minus infinity, ck into ck conjugate which is magnitude ck square and therefore, we have showed a very important property a very important and very interesting property which is basically that if I compute the power in the time domain of this periodic signal that is magnitude x t square d t that is equal to summation k equal to minus infinity to infinity magnitude C k square and this is known as the Parsevals identity or the Parsevals theorem.

So, this important property is known as the Parsevals identity or this is also known as Parsevals also known as Parsevals theorem all right. So, that is an interesting property. So, what we have seen in this module is we have looked at the conditions for that that is the Dirichlet conditions. Dirichlet conditions for the existence of which are the sufficient conditions for the existence of the Fourier series representation or the convergence of the Fourier series representation of a periodic signal, continuous time periodic signal x t and we have also seen the Parsevals relation which relates the power of continuous time periodic signal x t to its Fourier series representation all right. So, we will stop here and look at other aspects in the subsequent modules.

Thank you very much.