Principles of Signals and Systems

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Lecture – 32
General Inversion Method for Inverse z -Transform Computation – Method of Residuess

Hello welcome to another module in this massive open online course. So, we are looking at the z transform let us continue our discussion on the z transform look at another technique to compute the inverse transform, this is known as the general inversion method. So, this is the inverse z transform.

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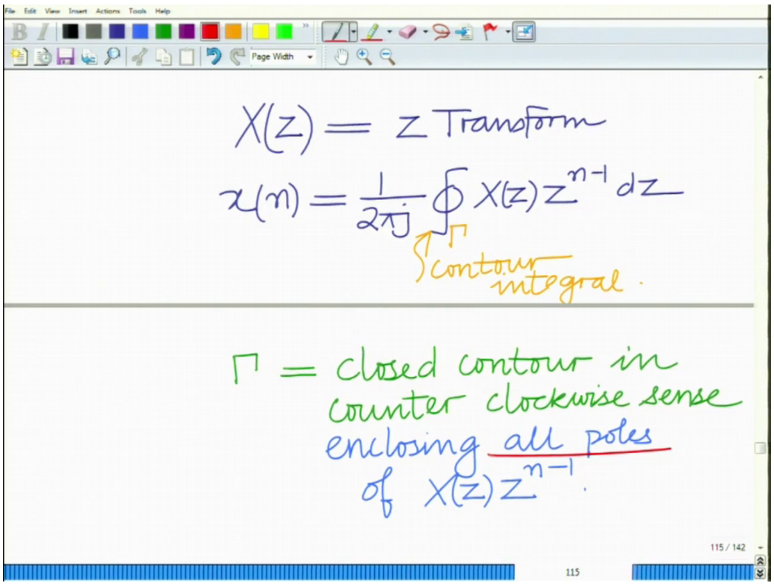

So, what we want to look at we want to look at a technique to compute the z transform through the what is known as the general inversion technique, the general inversion method to compute the inverse z transform is as follows what we would like to do is basically let us say X z is the z transform given z transform, then x n can be computed as the integral or let me write it over 2 pi j this is a contour integral X z, z to the raise to n minus d z and this integral that you are seeing over here this is what is known as a contour integral evaluated over a contour in the z plane.

And what is this and this is evaluated over this contour gamma, and this gamma is basically a closed contour in counterclockwise sense in closing all the poles, and this is important in closing all the poles correct. So, this gamma so this integral has to be evaluated over this contour closed contour gamma, which is a closed contour in the counterclockwise or basically the anti clockwise sense it encloses all the poles of X z into z raised to n minus and has to be in the region of convergence of this z transform.

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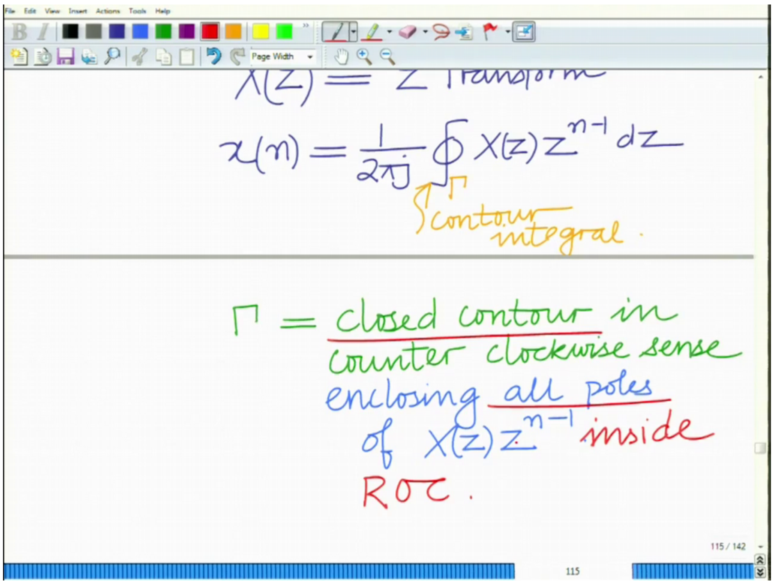

And this has to be inside the ROC inside the region of convergence. So, this is basically you can show that x n equals over 2 p i j the integral the contour integral over this contour gamma of X z times z n minus d z where gamma is a closed contour. So, closed contour in the anti-clockwise sense encloses all the poles of X z z n minus and the contour is inside the ROC. So, the contour lies inside the so, gamma lies inside the ROC.

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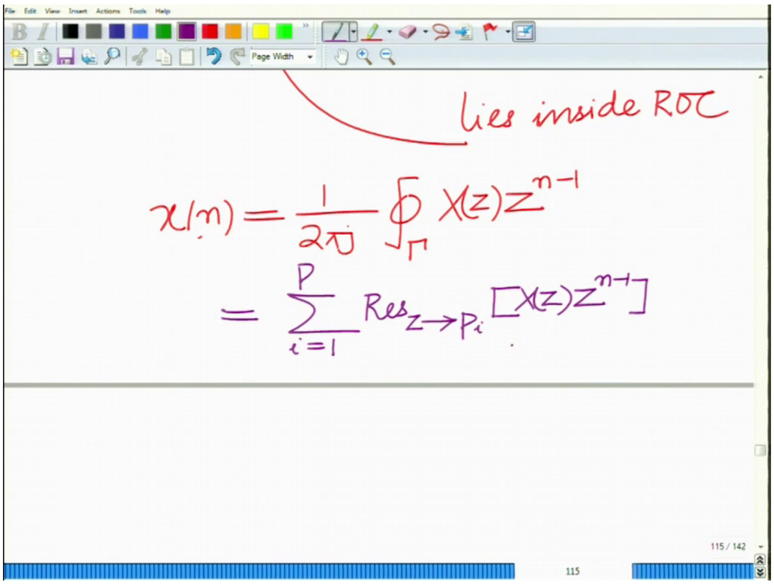

Let us make that contour lies inside the and it can be shown that this can be simplified as follows this is over 2 p i j contour integral over gamma X z z raise to n minus 1, which is equal to summation i equals to p summation over the residues z ready residues evaluated at each pole p i of the quantity X z raised to z raise to n minus 1.

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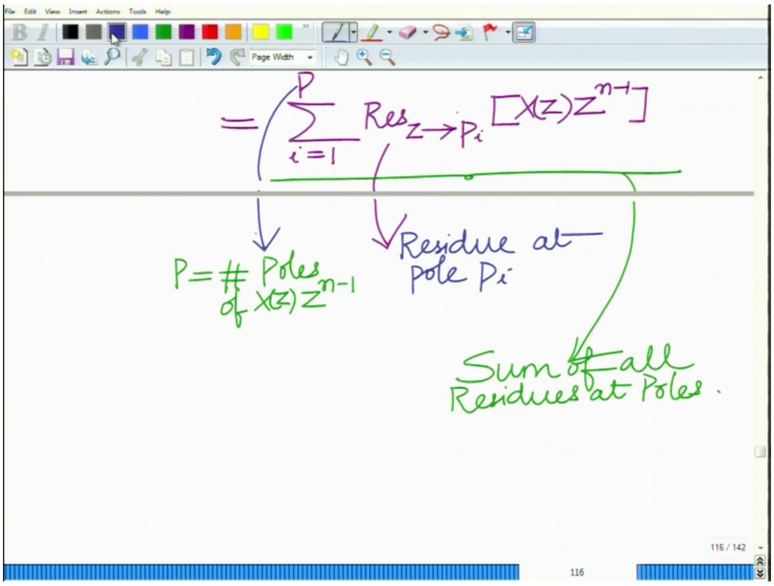

So, this denotes the residue what is known as the residue I will define this shortly this is the residue at pole p i and this quantity p equals the number of poles of X z into z raised to n minus 1.

And therefore, what this is basically the sum of all the residues evaluated at the poles sum of all the residues at poles summation over all residues at poles of X z z raised to n minus 1.

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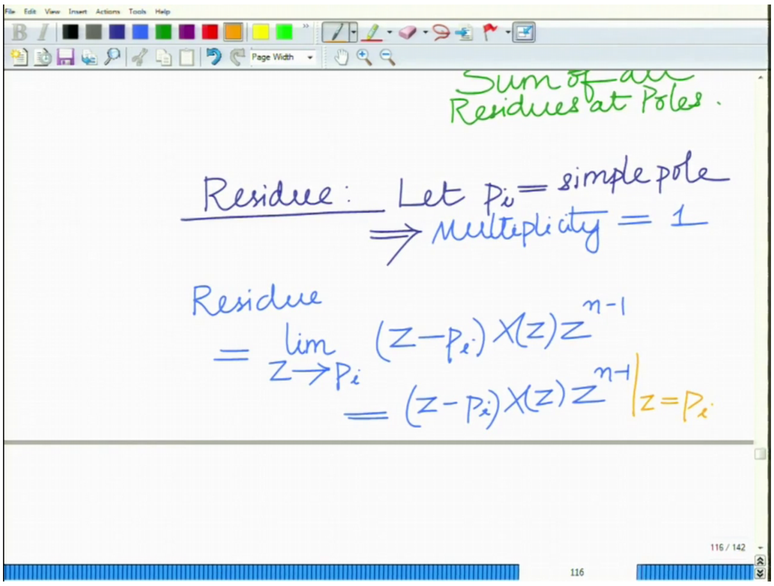

Now what is the residue, now if p i is a simple pole that is basically multiplicity equals implies that is it is a pole of multiplicity that is you only have a factor of z minus p I, implies the multiplicity equals. The residue equals of this simple pole is limit z tends to pi z minus p i X z of z raised to n minus or in most cases can be simply evaluated as z minus p i into X z into z raised to n minus that is evaluated at z equal to p i ok.

So, this is the residue at a simple pole p i that is a pole of multiplicity 1, this is simply given as x minus z si or no sorry z minus p i into X z into z raised to n minus evaluated at z is equal to p.

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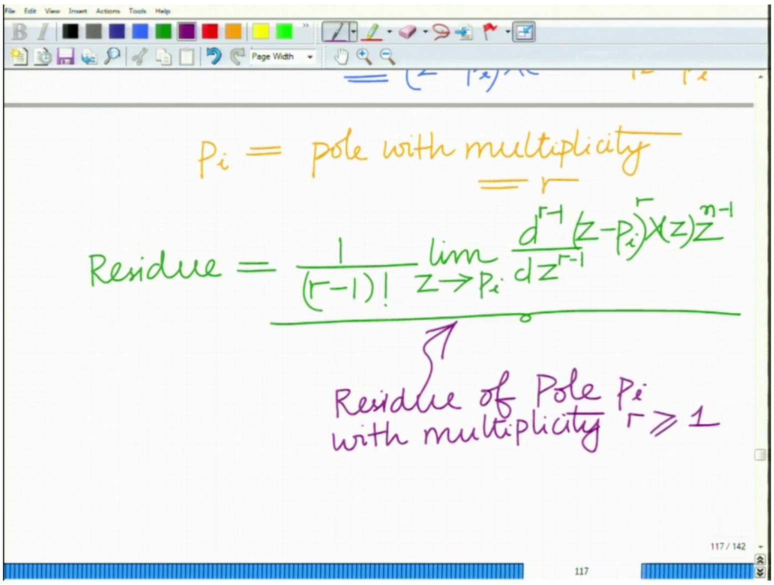

Now, for pole with multiplicity r where r is greater than or equal to now for any general multiplicity for pole will multiplicity, let p i equals pole with multiplicity r, then the residue of this equals you know this is over r minus factorial with multiplicity r greater than equal to 1 over r minus factorial the z limit z tends to pi d raise to r minus over d z raise to r minus z minus p r raise to r X z into z raised to n minus 1.

This is the residue of a pole with multiplicity r greater than or equal to that is with any general multiplicity r with greater than equal to 1 all right. So, we evaluate the residues all right of the transfer of X z into z raised to n minus 1 at the different poles, and then we sum all the residues at these poles that gives us the value of x n all right. So, to understand this better let us do a simple example, let us do a few examples all right?

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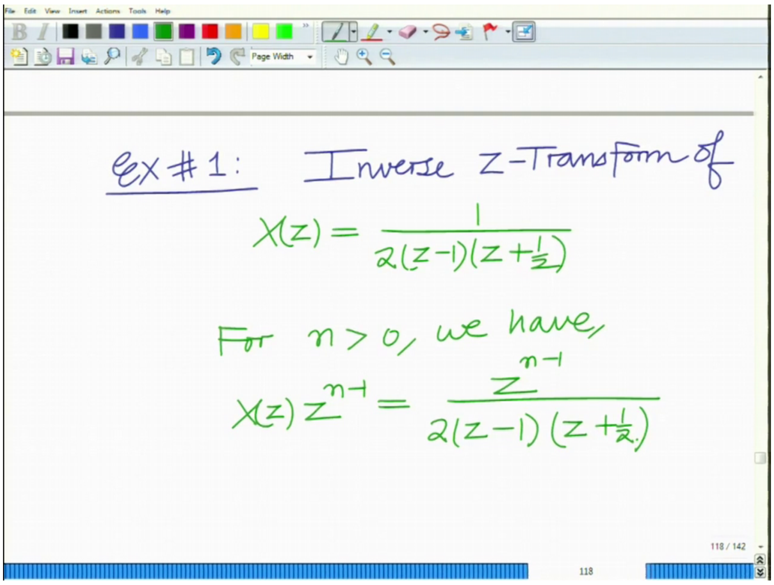

So, using this general inversion technique for instance; let us do this simple example let us evaluate the inverse z transform of X z equals over twice z minus into z plus half. Now for n greater than 0, we have x into z raised to n minus 1 equals e raised to n minus 1 divided by twice z minus 1 into z plus half.

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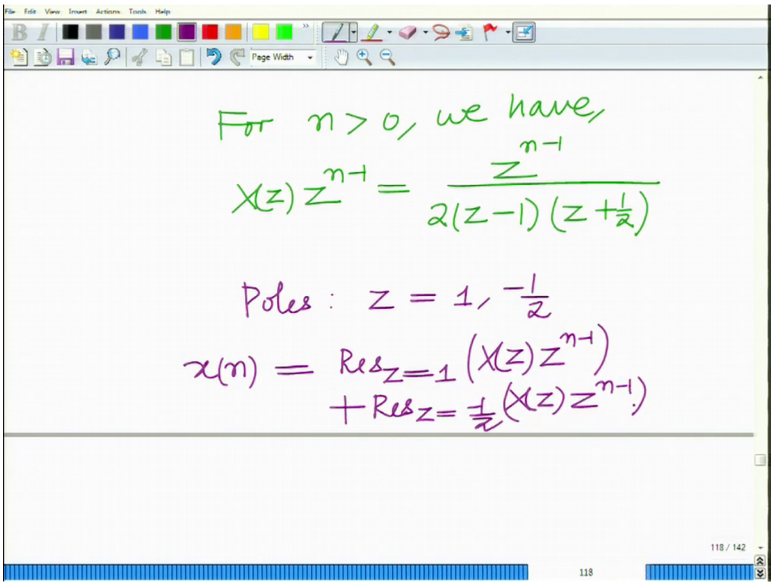

Now, the poles are it z equal to z equal to minus half is equal to comma minus half therefore, x n is the sum of the residues evaluated at these poles that is basically you will have well the residue evaluated at z equal to 1 of we have 2 poles correct. So, we have the sum of the residues residue evaluated at z equal to a 1 of X z into z raised to n minus plus the residue evaluated z equal to minus half of X z into z raised to n minus 1. Now, the pole evaluated at no residue at z equal to 1 or z raise to n minus 1 equals.

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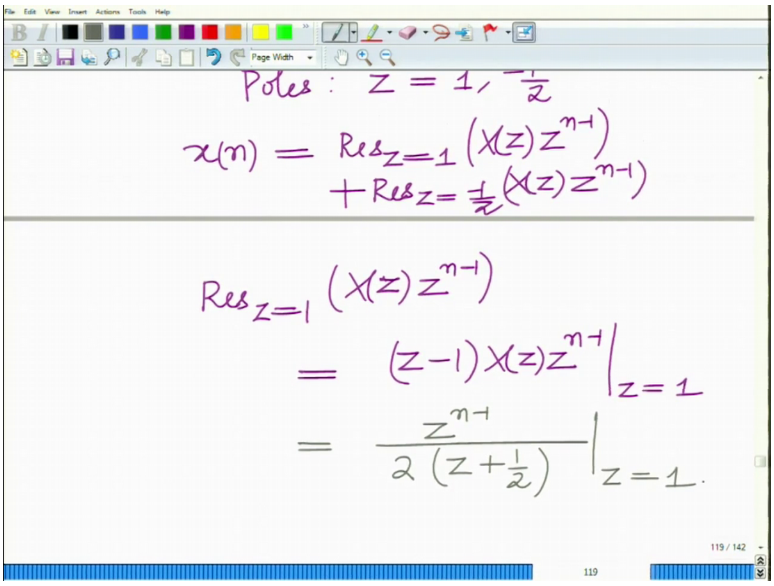

Basically well z minus 1 into X z into z raised to n minus 1 evaluated at z equal 1 to this is equal to well z raised to n minus 1 over twice z plus half evaluated at z equal to 1, and this is equal to well z raised to n minus 1 raise to n minus 1.

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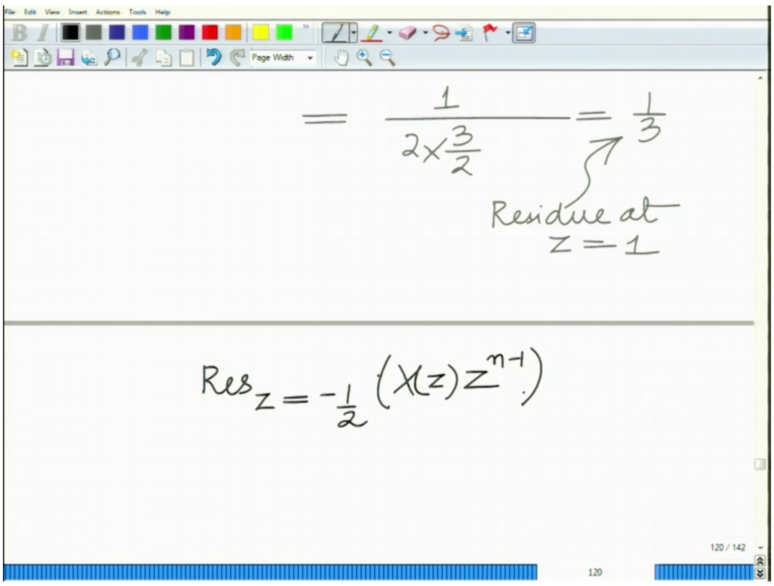

That is simply divided by 2 into 1 plus half that is 3 over 2 which is equal to basically 1 over 3 ok. So, that is basically you are residue at z equal to 1. Now let us look at the residue at z now the residue at z equals, now that evaluates the residue it z equal now the residue it is equals minus half of X z into z raised to n minus 1, the residue at z equal to minus half that is basically z minus p i or z plus half into X z into z raised to n minus 1 evaluated it z equal to minus 1 half ok.

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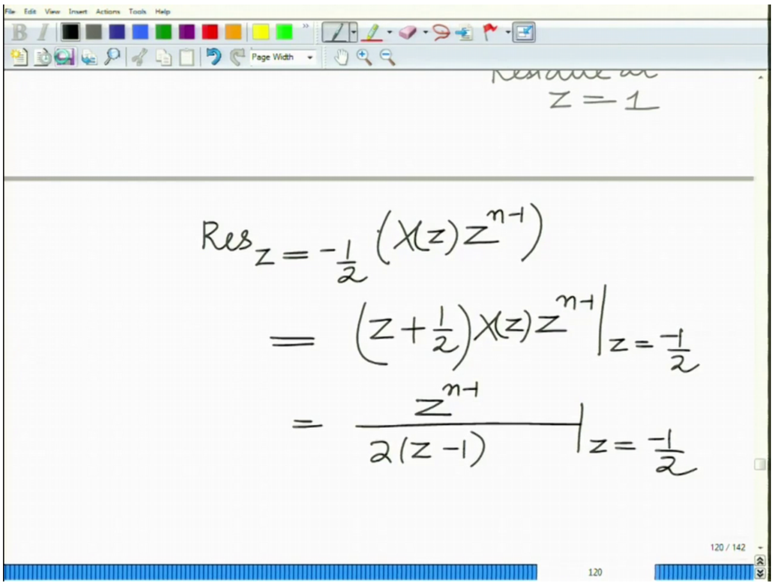

So, that is the residue at z equal to minus half, that is basically you are that will be z raise to n minus 1 over twice z minus 1 evaluated at z equal to minus half.

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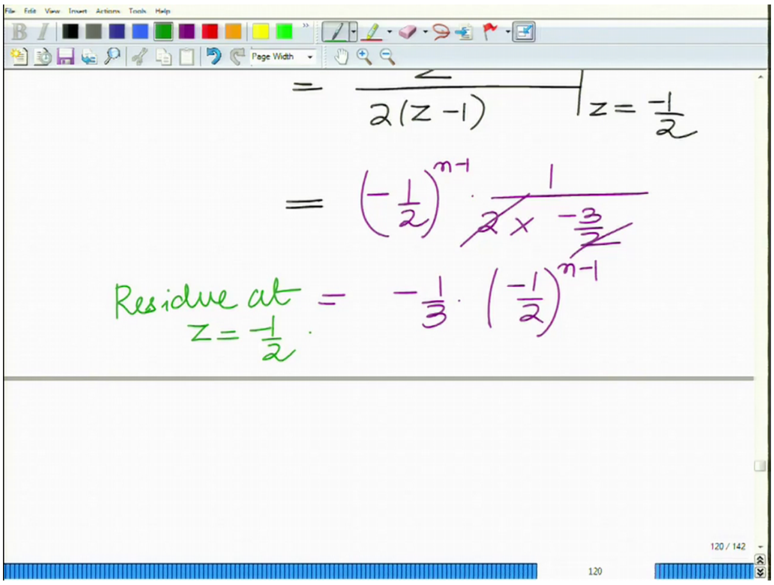

So, this will be your z raise to n minus half n minus 1 over, twice z minus half minus 1 that is minus 3 over 2. So, this is basically minus by 3 into minus half raised to n minus 1, and this is the residue at z equal to minus half.

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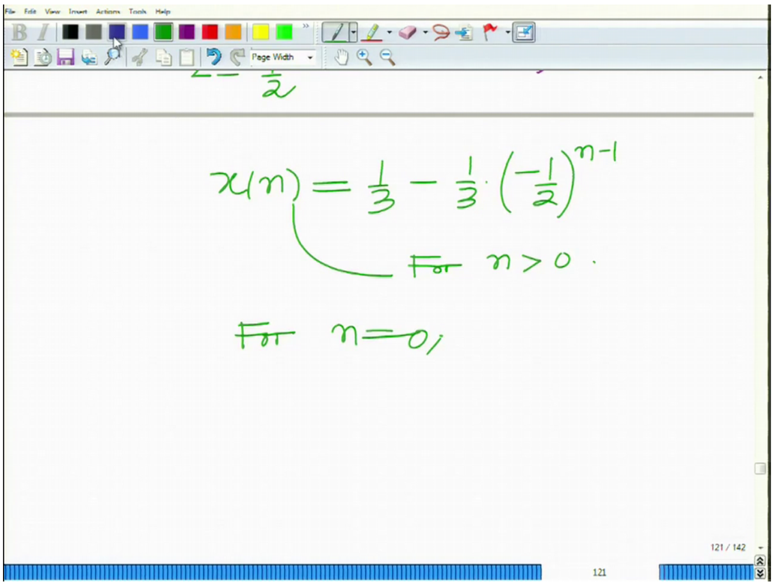

Therefore x n is simply the sum of these 2 residues that is a residue is equal to 1, z equal to 1 plus residue it is equal to minus half that is 1 over 3 minus 1 over 3 minus half raised to n minus 1. So, remember this is for n greater than 0, now for for n equal to 0 what we obtain is the following.

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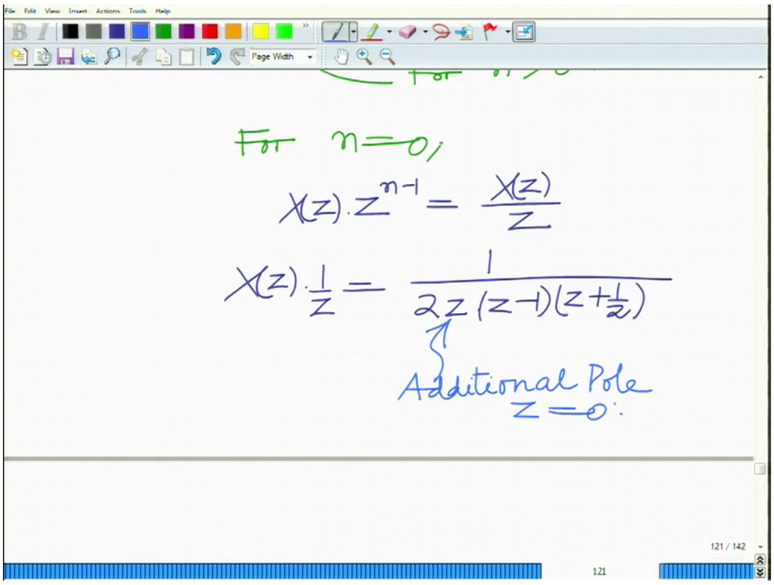

So, consider X z into z raised to n minus 1 n equal to 0. So, n minus 1 equals minus 1. So, this is simply X z over z, now X z into 1 by z or X z over z equals over twice z into z minus 1, into z now note that because of this at n equal to 0, because of this division by z or X z by z we have this additional pole at z equal to 0 alright. So, because of this z raised to n minus 1 n equal to 0 so, that will be z raised to minus 1. So, because of that factor we have an additional pole at as equal to 0 which we have to take into consideration while evaluating the sum of the residues ok.

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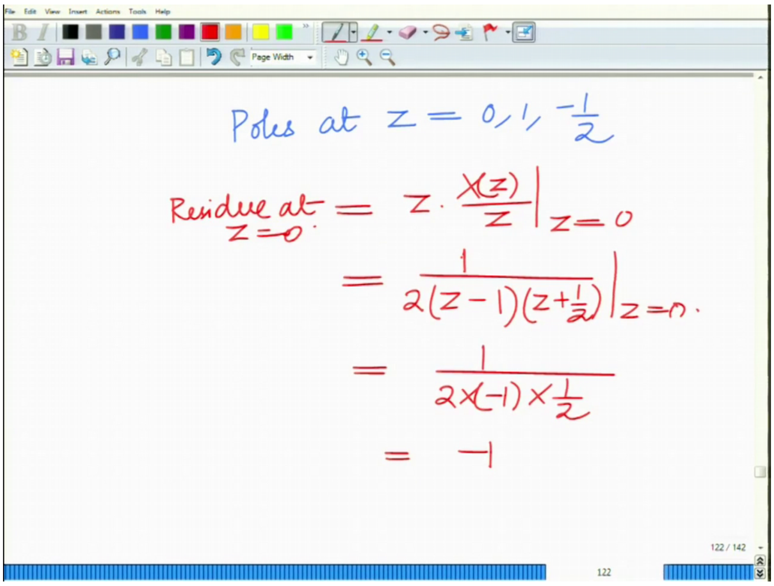

Now, therefore we have poles at z equal to 0 comma 1 comma 1 minus half, now x 0 equals z into X z over z evaluated at that is X z into n minus 1, which is X z over z evaluated at z equal to 0 which is basically you are 1 over no this is not X z this is a residue at 0, 1 over this is basically your 1 over twice z minus 1 z minus 1 into z plus half evaluated at z equal to 0. So, this is 1 over twice minus 1 into half equals minus 1. So, this is basically your minus 1.

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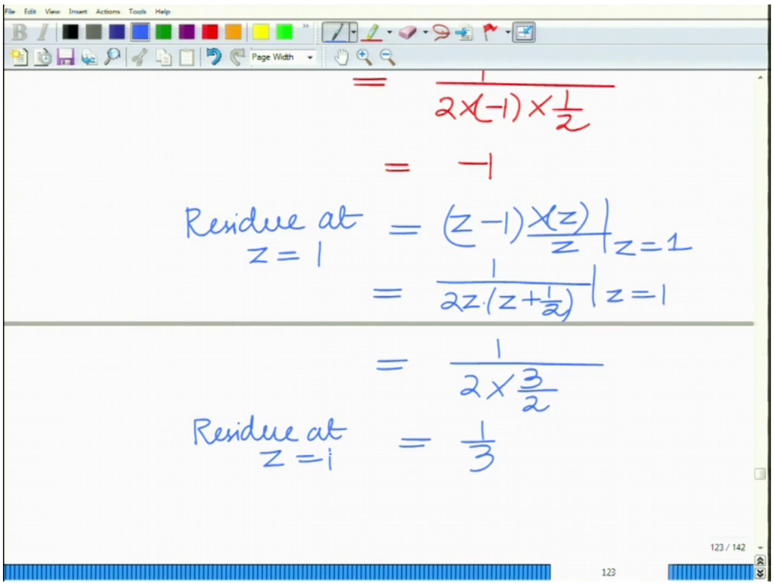

Now, residue at z equal 1 to this will be z minus 1 into X z over z evaluated at z equal to 1. So, this will be basically over twice z into z plus half evaluated at z equal to 1.

So, this is over twice z is 1 so, twice z plus half 3 by 2 that is 1 over 3 this is your residue at z equal to 1, and the residue finally residue at z equal to minus half is z plus half into X z over z evaluated at z equal to minus half.

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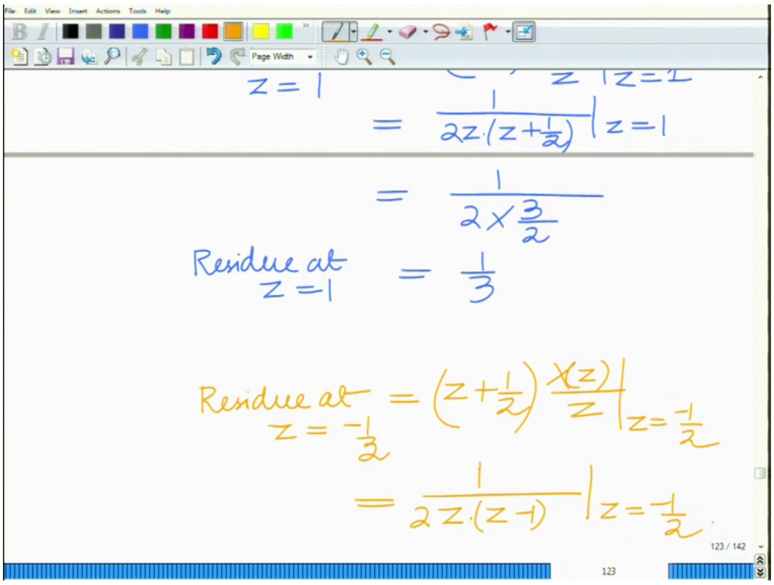

So, this will be 1 over twice z into z minus 1 evaluated at z equal to minus half.

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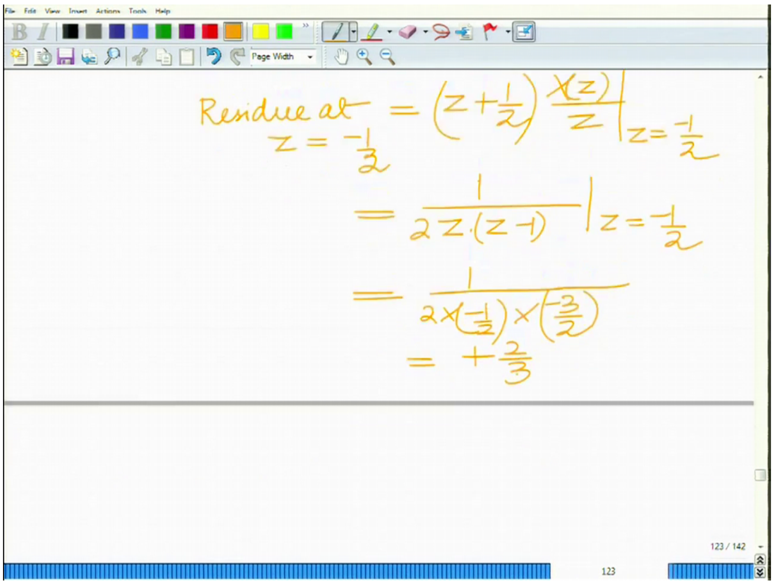

So, this will be 1 over twice into minus half, into z minus 1 equals minus 3 over 2. So, this will be plus 2 over 3 actually. So, x of 0 is basically the sum of residues.

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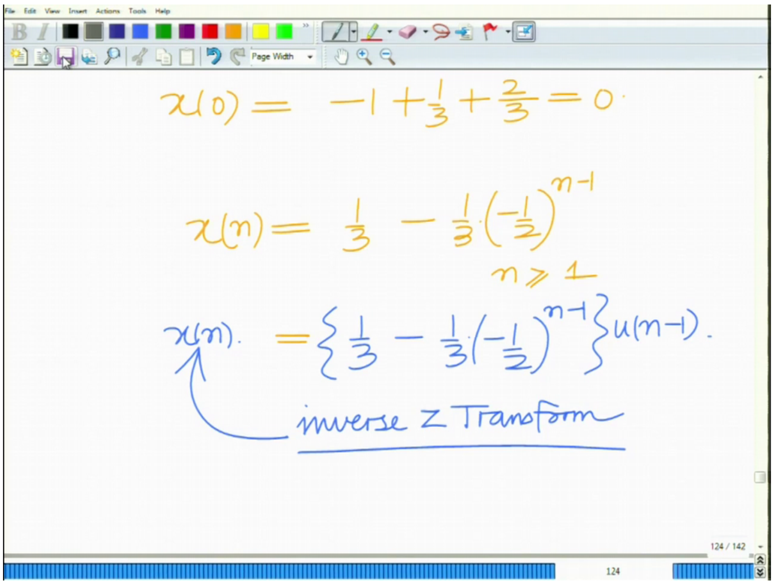

So, now you get for n equal to 0 x of 0 equal sum of residues at z equal to 0, x of 0 is the sum of residues at the poles remember there is an additional pole at z equal to zero.

So, the sum of their residues at z equal to 0 is equal to 1 at z equal to minus r, which is basically minus 1 plus 1 by 3 plus 2 by 3 this is 0. So, x of 0 equals 0 so, x of 0 equals 0 and therefore, what we obtain is basically x of n equals 1 by 3 minus 1 by 3 into minus half to the power of n minus 1 for n greater than equal to 1 which can be written as 1 by 3 minus 1 by 3 into minus half to the power of n minus 1 into u n minus 1 all right. So, this is basically your inverse z transform, evaluating using the general inversion technique, this is your inverse z transform ok.

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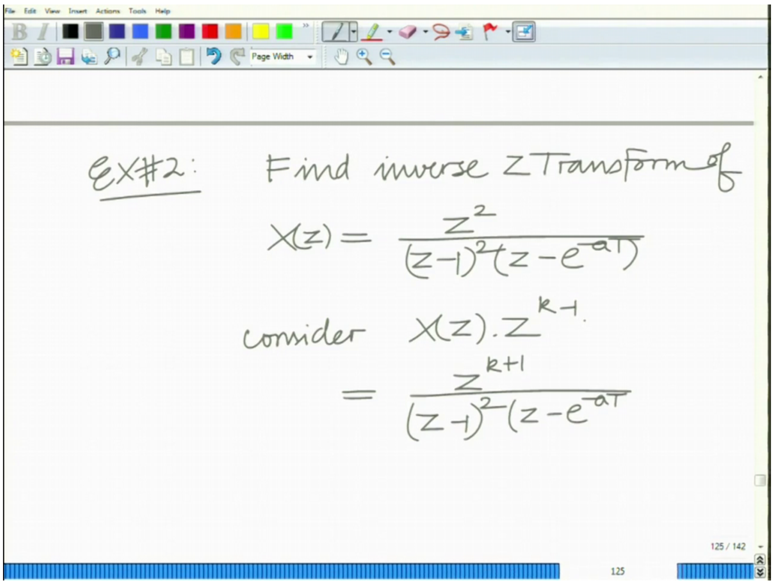

Let us now look at another simple example again, example number 2 find the inverse z transform of X of z equals z square over z minus 1 square z minus e raised to minus a T ok.

Now, consider X of z into z raised to n minus 1 z raised to k minus 1 to evaluate x k this is equal to well z square into z raised to k minus 1.

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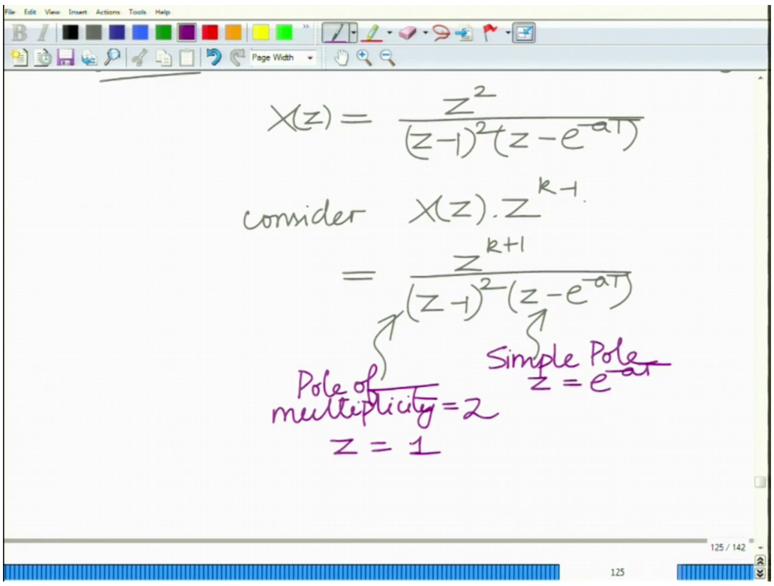

So, z raise to k plus 1 divided by z minus 1 square into z minus e raised to minus a Ts now, note that this has a pole of discuss thus there is a simple pole at z equal to e raised to minus a T, and there is a pole of multiplicity 2 at is equal to 1 all right.

So, there are 2 poles at z equal to 1 you have a pole of multiplicity 2 at z equal to 1 alright, there at 2 poles at z equal you have a pole of multiplicity 2 because, your factor is z minus square, and at e raised to minus a T you have a pole of multiplicity 1.

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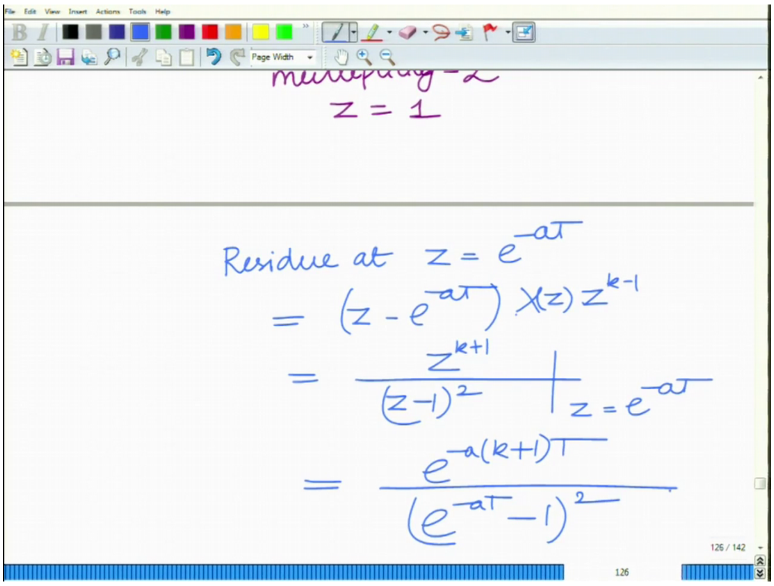

Now, first let us find the simple residue at z raised to z equal to e raised to minus a T that is basically simply z minus e raise to minus a T into X z into z raised to k minus 1, that is basically your z raised to k plus 1 divided by z minus 1 square evaluated at z equal to e raise to minus a T, that is e raised to minus a k plus T over e minus a T, e raised to minus a T minus 1 whole square.

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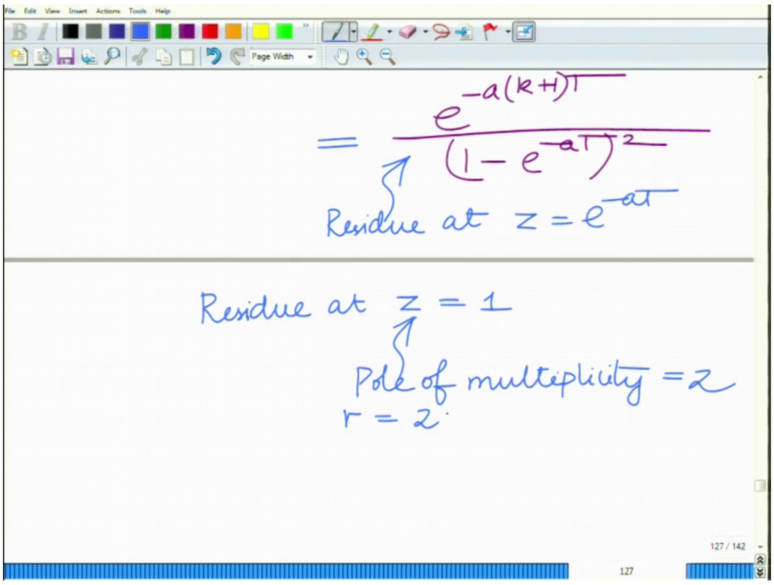

Which is also your e raised to minus a k plus T over 1 minus e raise to minus a T whole square, and this is basically your residue at z equal to e raise to minus a T.

Now, we still have to find the residue at z equal to 1, remember at z equal to 1 we have a pole of multiplicity 2. So, we have to use the general formula for the computing the residue for a pole at a pole with multiplicity r greater than or equal to 1 all right. So, here you have pole of multiplicity equal to 2 that is basically the formula you have to set r equal to 2, and that gives you what does that give us well.

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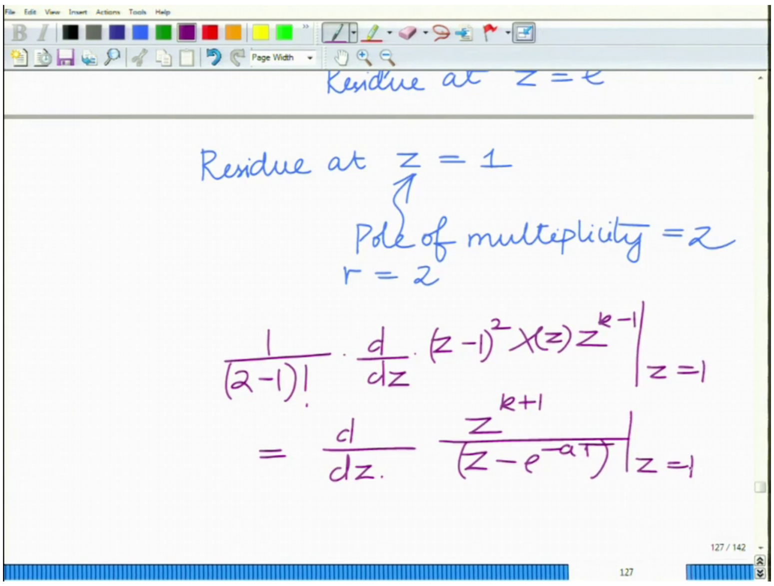

Residue is 1 over r minus factorial d over d to the r minus 1 that is d over d z of z minus 1 whole square X z into z raised to k minus 1 equals well d over d z. And this is evaluated at z equal 1 to and this is d over d z which reduces to do d z of z minus e minus a T evaluated at z equal to 1.

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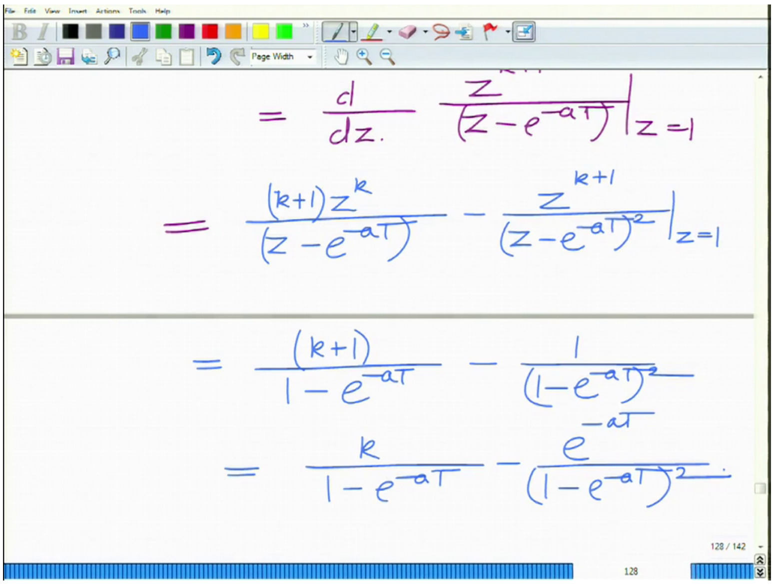

And this derivative is basically can be computed as follows that is k plus 1 into z raised to k by z minus e minus a T minus z raise to k plus 1 over z minus e raise to minus a T whole square evaluated at z equal to 1 substitute z equal to 1.

So, that gives us k plus 1 over minus e raise to minus a T minus 1 over minus e raise to minus a T whole square, which you can further simplify as 1 over k 1 over minus e raise to minus a T, minus e raise to minus a T by 1 minus e raise to minus a T whole square ok. So, this is basically the residue at z equal to 1.

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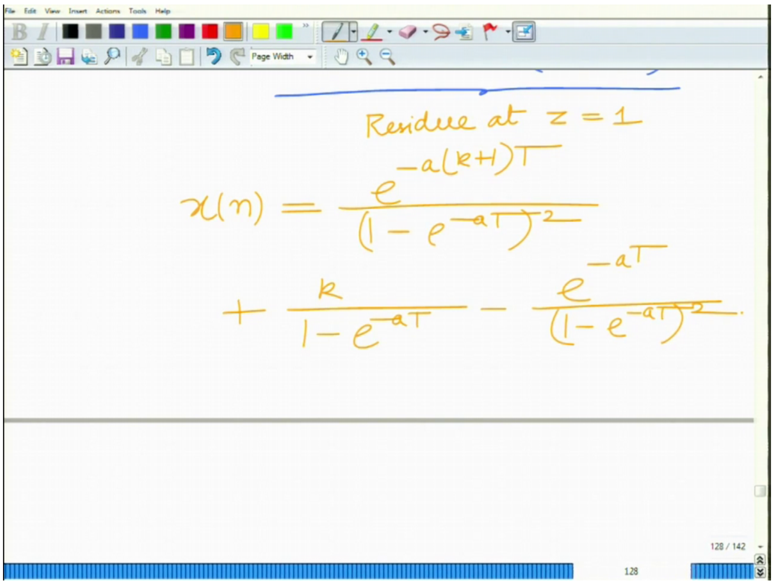

And therefore, finally, x n will be the sum of these 2 residues, that is e raised to minus the residue at e raised to is equal to e raise to minus a T, e raise to minus a into k plus 1 T over minus e raise to minus n a T whole square plus k over 1 minus e raise to minus a T minus e raise to minus a T divided by 1 minus e raise to minus a T whole square.

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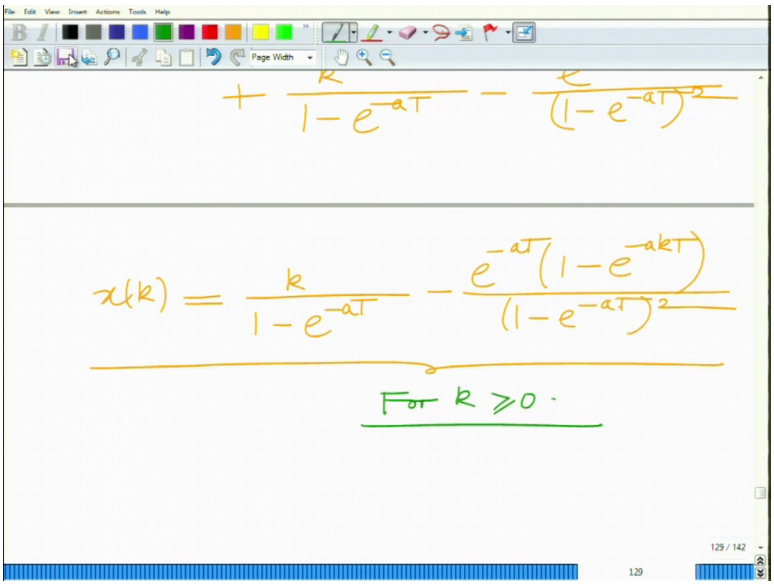

Which is now k over 1 minus e raise to minus a T, and you can simplify bringing the first and last terms together as e raised to minus a T into 1 minus e raised to minus a k T divided by 1 minus e raise to minus a T whole square. This is your x k for k greater than equal to 0 that is for, and this is valid for k greater than equal to ok.

So, basically that completes the problem all right. So, what we have seen here is basically we have seen a different technique to evaluate the inverse z transform that is basically using the general inverse technique, and we have seen a couple of that is using what is also known as the method of residues that is evaluating x n as the sum of the residues of the poles of X z into z raised to n minus 1, and we have seen a couple of examples to understand this technique better alright. So, we will stop here and continue in the subsequent modules.

Thank you very much.