Principles of Signals and Systems
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Lecture – 27
Z-Transform of LTI Systems – Causality, Stability, Systems Described by Difference Equation

Hello, welcome to another module in this massive open online course. So, we are looking at Z transfer, the Z transform and its various properties let us continue our discussion with the system function for a discrete time LTI system.

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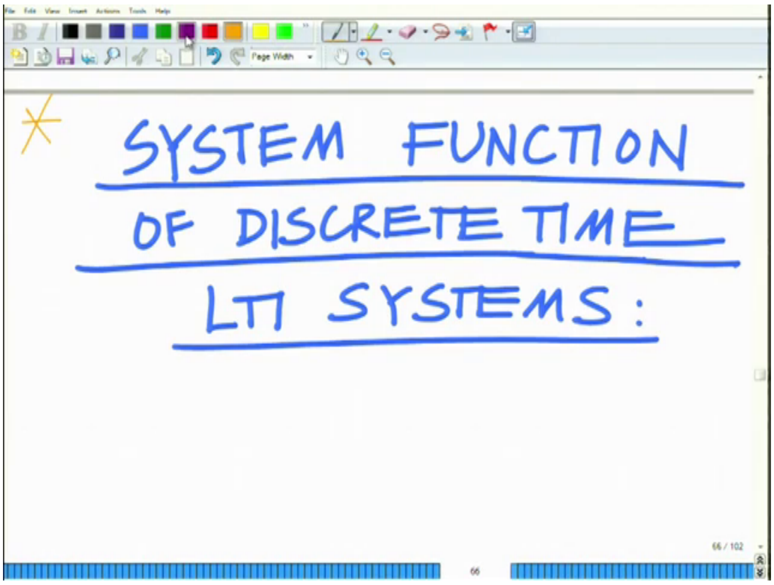

So, we want to continue our discussion on the Z transform by looking at the system of because remember the Z transform is used to analyze discrete time signals and systems.

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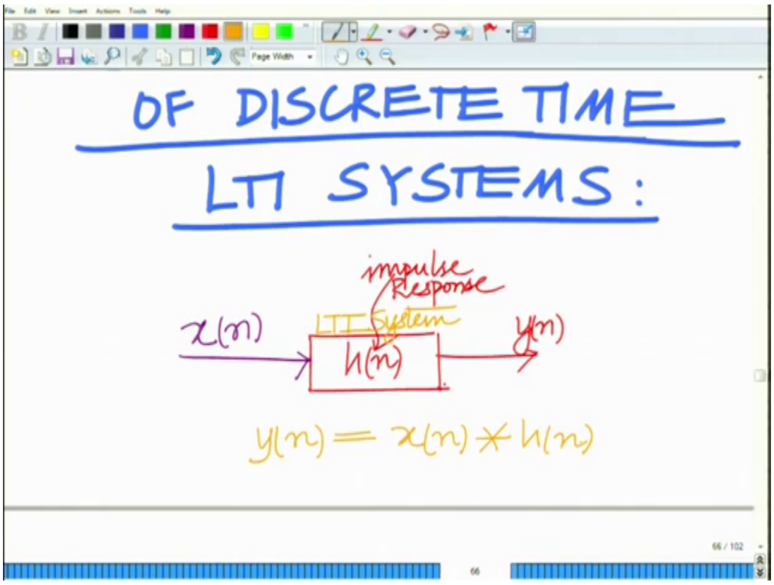

So, to look at the system transfer function of a discrete time or of discrete time LTI systems, and the similar once again through the discussion on the Laplace transform, but this time considering a discrete time LTI system and discrete signals. So, we have x n which is given as input to a discrete time LTI system with impulse response. So, this is your impulse response of the discrete time LTI system and the output is y n.

So, we have we are considering an LTI system with impulse response h n, and x n is the input to this LTI system, and y n is a corresponding output. Then we know that the output y n is given as the convolution of the input with the impulse response. So, what we will have is at y n equals x n convolved with h n. And from the previous properties of the Z transform, we know that convolution in time is basically multiplication in the transform domain.

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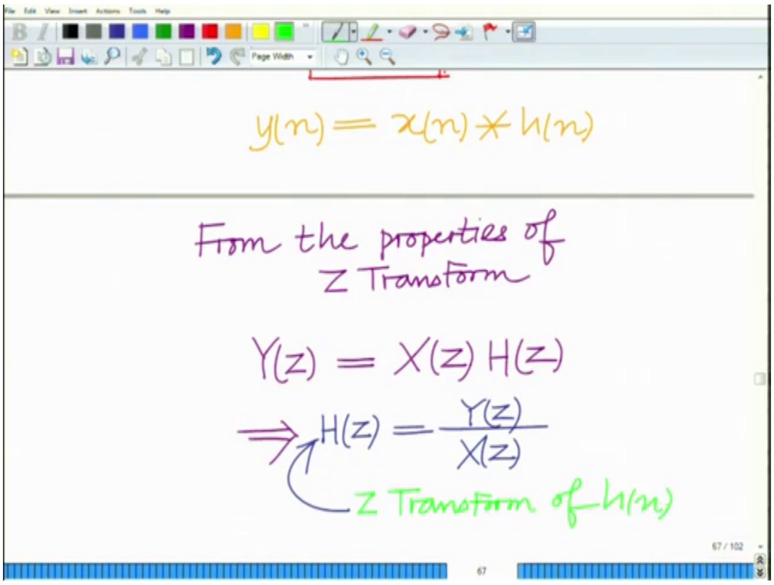

Therefore, from the properties of the Z transform from the properties of the Z transform what follows is at y of z equals the multiplication of the Z transforms X of Z times H of Z implies the transfer function H of Z equals Y of Z divided by X of Z. Where H of Z remember H of Z this is the Z transform of the impulse response h of n that is the impulse response of the LTI system.

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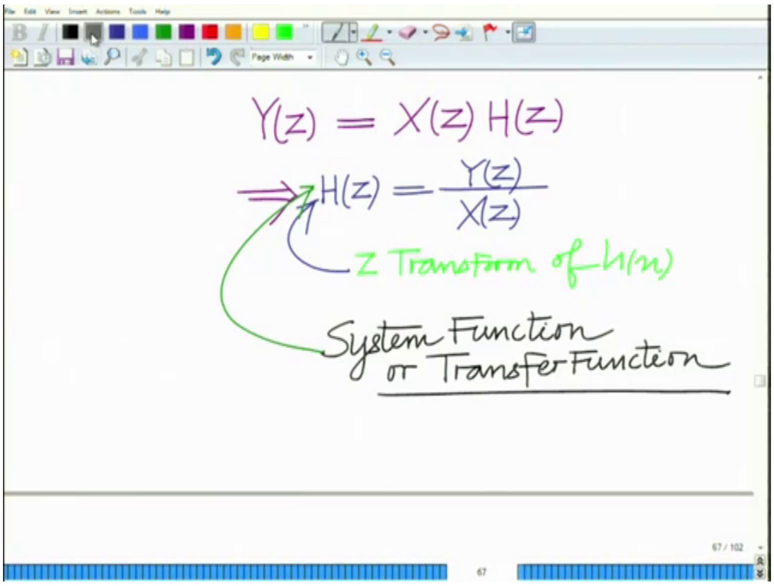

At this H of Z is known as this H of Z is known as the system function or the transfer function this is known as the system function or the transfer function of the discrete time

LTI system. Now, let us look at the properties of the discrete time LTI system based on its system function or the transformation. So, what you want to explore now are the properties of the LTI system.

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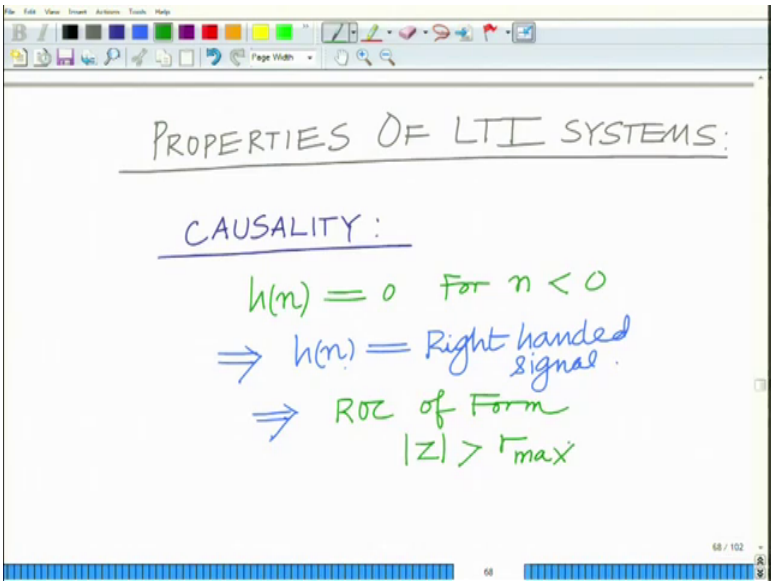

So, we want to look at the properties of the LTI systems the first property is causality we want to consider a causal LTI system. So, first properties causality, this implies that h n equal to 0 correct. So, an LTI system is a causal system, if the impulse response is 0 for n less than 0 that is the discrete time impulse response h of n is 0, for n less than 0; it is nonzero only for n greater than or equal to 0. Now, this means that h n you can see is a left-handed signal is a right-handed signal. Therefore, the ROC - the region of convergence of the Z transform H of Z must be of the form magnitude Z greater than R max, where r max remember as for the properties of the ROC is the maximum of the poles of H of Z. So, this implies h of n equals the right-sided or right-handed signal implies ROC is must be of the form magnitude of Z greater than r max, where r max remember from what we saw yesterday from the properties of the ROC if you remember.

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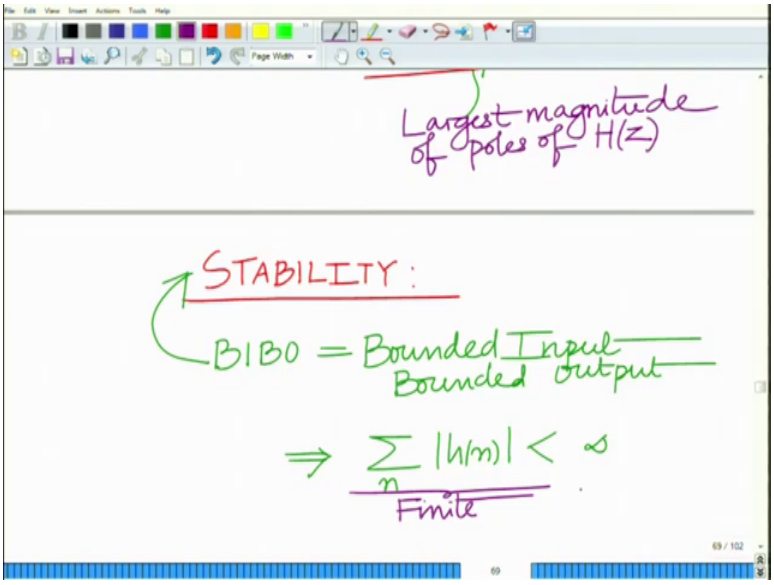

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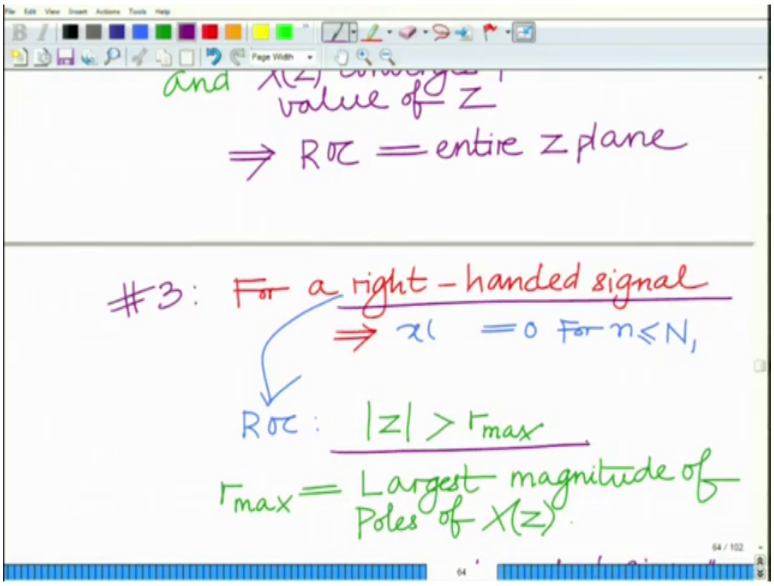

From the properties of the ROC for the right handed signal magnitude of Z must be greater than r max. Where r max is the largest magnitude of the poles of X of Z that is the largest magnitude of poles of X of Z r max is the poles of H of Z where H of Z is the transfer function that is what we have just seen. So, the ROC is of the form magnitude h, magnitude z is greater than r max.

Now, let us look at stability that is the other important thing stability now for the system to be BIBO stable. Remember stability is one of the most popular definitions is BIBO stability bounded input bounded output stability, bounded input bounded BIBO stability. For BIBO stability, we have we require summation over all n that is n ranging from minus infinity to infinity magnitude summation of magnitude of h of n must be less than if it that is it must be at this quantity must be a finite quantity cannot be an infinite quantity, it has to be a finite quantity. Now, we can show that this implies that for a BIBO stable system the transfer function that is H of Z must include the unit circle in its ROC that in its region of convergence.

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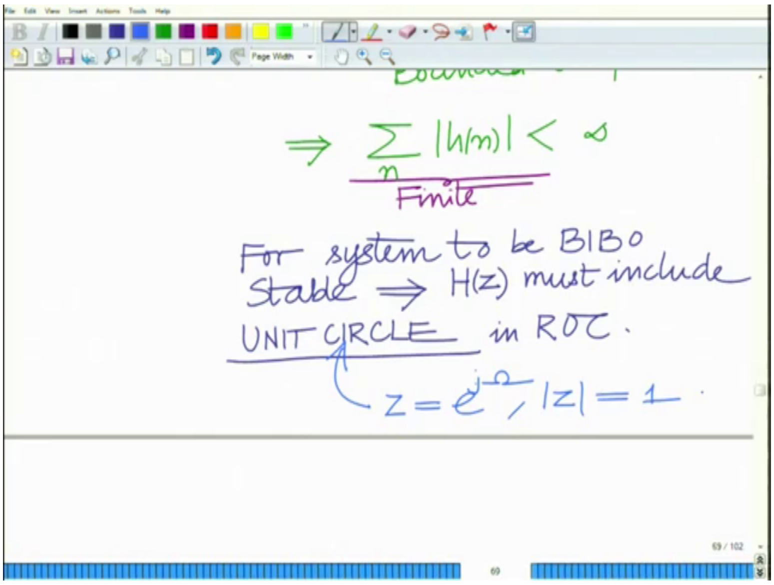

So, this implies and this can be easily shown which we wrote this as shortly that is for system to be BIBO stable that implies H of Z must include the unit circle. Remember unit circle is all the values of Z which are of the form Z equals e raise to j omega that is magnitude of Z is one that is unit magnitude any arbitrary phase. So, there is unit circle comprises of all Z all values of Z such that Z equals e power j omega that is basically satisfying the property magnitude of Z equals 1 that is r equals remember every Z can be represented in polar coordinates as r e raise to j omega. If r equals 1, which is the magnitude of Z then that corresponds to Z on the unit circle.

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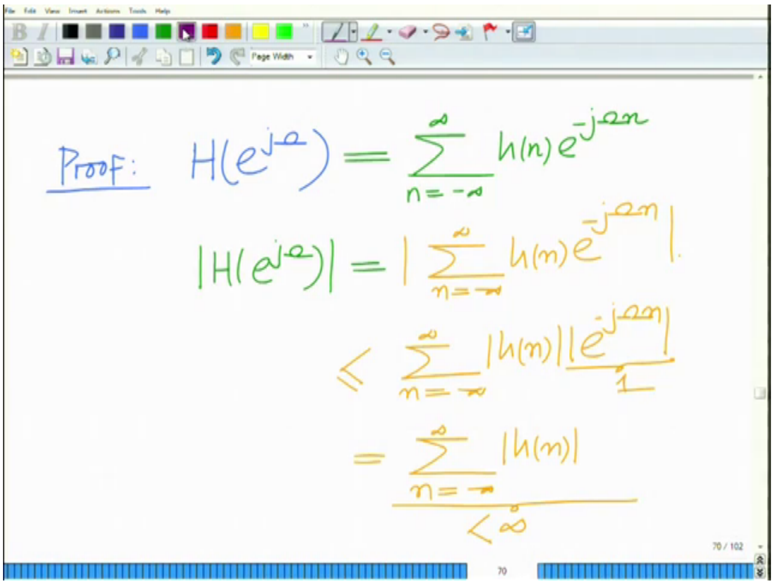

Now, let us look at a chose for a stable system. Let us look at for a BIBO stable system, let us prove this and the proof is very simple. So, magnitude H of e raise to j omega for any Z on the unit circle equals well n equals minus infinity to infinity h n Z raise to minus n and that is e raise to minus j omega n. So, magnitude H of e raise to j omega this will be magnitude either n equals minus infinity to infinity h of n e raise to minus j omega n.

Now, this magnitude of a sum is less than or equal to the sum of its magnitudes. So, this is less than or equal to summation n equals minus infinity to infinity magnitude h n minus e raise to n. Now, this quantity is 1 magnitude e raise to minus j omega 1. So, this is n equals minus infinity-to-infinity magnitude h of n which we know since the system is BIBO stable, we know this quantity is less than infinity that is this is a finite quantity, this quantity is this is a finite quantity

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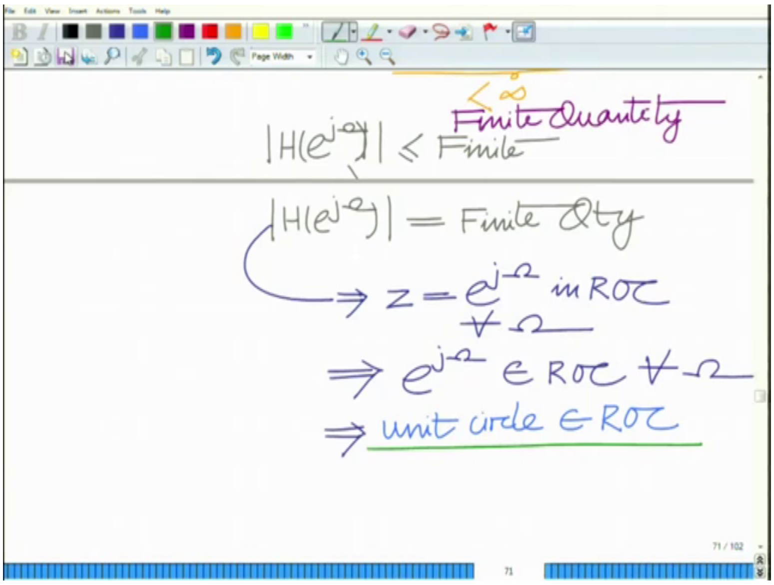

And therefore, this implies magnitude e raise to j omega is less than or equal to a finite quantity which implies magnitude of h raise because this is greater than or equal to 0. This is positive quantity because is a magnitude so this is less than which is which means that this is equal to a finite quantity which means that this has to converge for every Z is equal to e raise to minus j omega and it has to be a finite quantity. Therefore, Z equals e raise to j omega for every omega is in the region of convergence, which means the unit circle must be in the ROC for a BIBO stable LTI system.

So, this implies that for all omega which implies e raise to j omega belongs to ROC element of the ROC e raise to j omega belong set notation belongs to ROC for all omega for every value of omega which implies that the unit circle belongs to the ROC for a BIBO stable system. So, the ROC for a BIBO stable system the system transfer function H of Z must include the unit circle in its ROC region of convergence. Now, let us look at the system function for an LTI system described by a difference equations.

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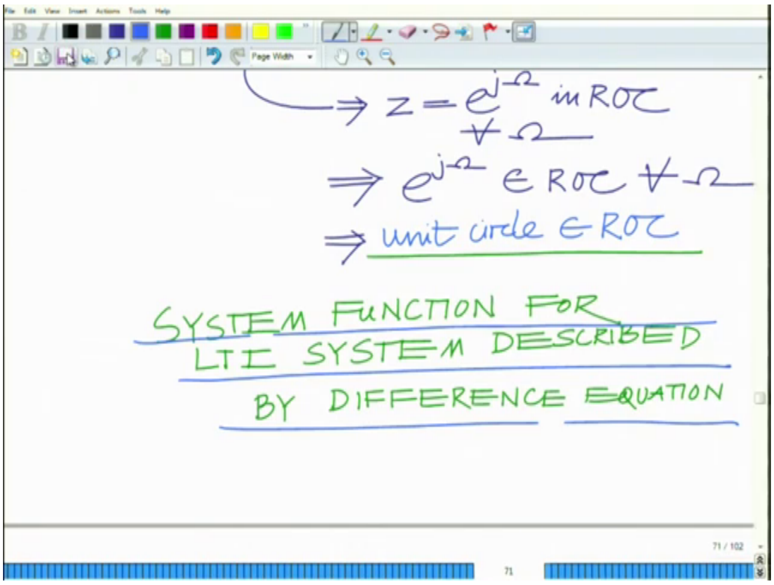

So, what we want to do. And this is again fairly simple for an LTI system described by a difference equation.

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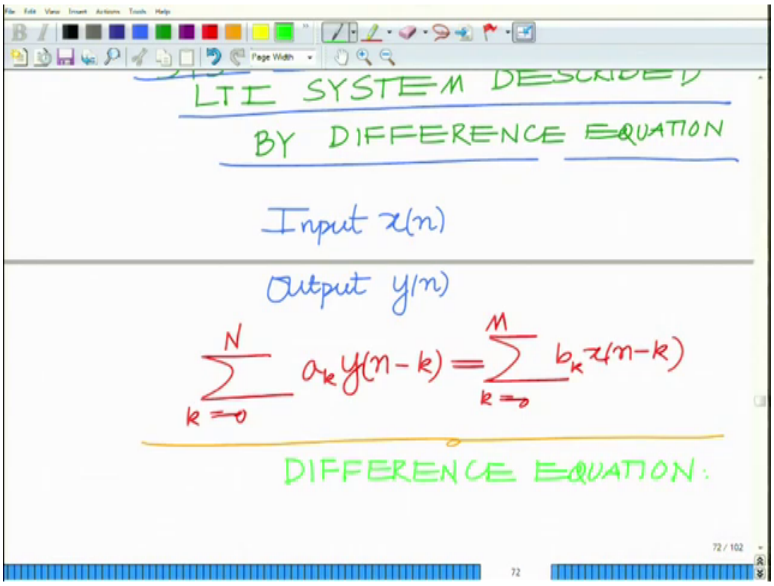

So, this is the system function for an LTI system described by the difference equation consider the input x n and output y n, then the difference equation representation is given as follows summation k equals 0 to n a k equals y n minus k summation k equals 0 to m b k n minus k. Now, this is known as a difference equation. Now, typically for a continuous time LTI system, we have a differential equation correct. We have seen summation a k d raise to k that is the kth derivative d of d raise to k yt over dt k equals summation over k from 0 to m b k d raise to k xt divided by dt that is a dt to raise to k that is the k th derivative of x that is a differential equation correct. This is a difference equation for discrete time signals. This is a difference equation.

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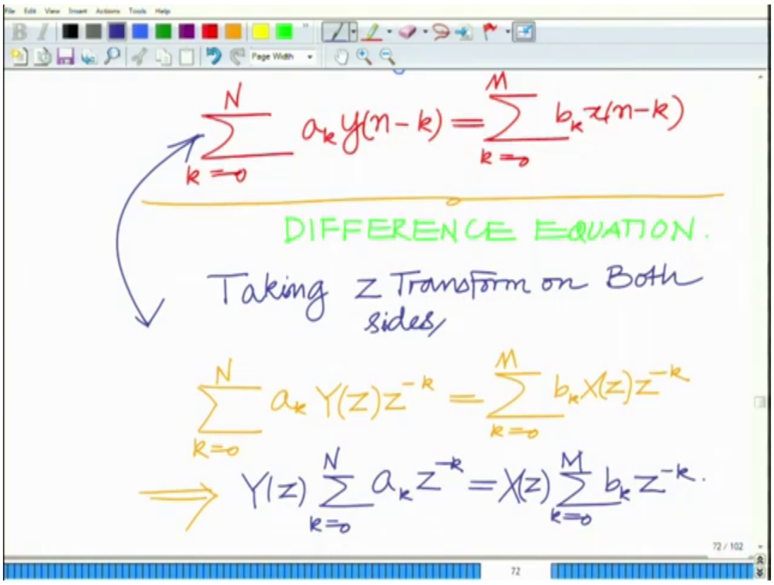

Now, in a difference equation, typically we have now taking the Z transform on both sides, we have summation k equal to 0 a k remember the Z transform of y n minus k that is y n delayed by k is Y Z Z raise to minus k equals k equals 0 to m b k X Z Z transforms of x n minus k is x Z Z raise to minus k which means if you take Y Z common on the left that is summation k equals zero to n a k Z raise to minus k, this is equal to take X Z common X Z summation k equals 0 to m b k Z raise to Z raise minus k.

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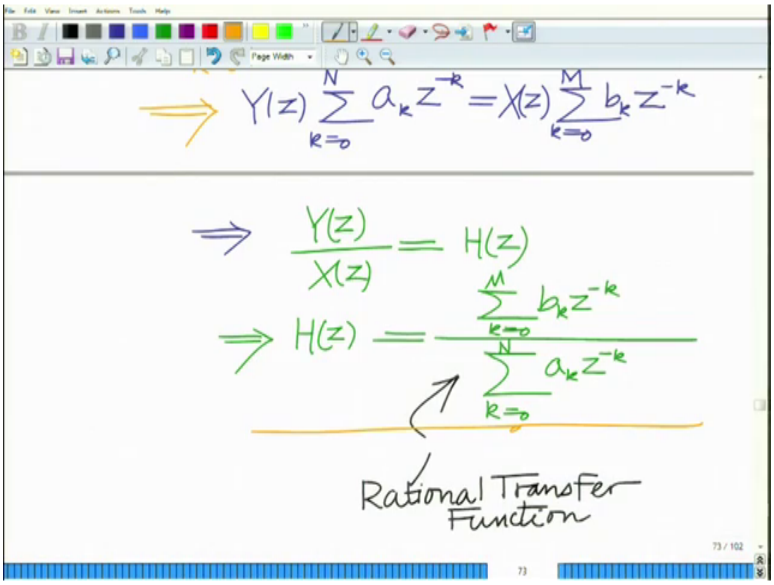

Which implies that Y Z over X Z equals H Z this is well this is summation k equals 0 to m b k Z raise to minus k divided by summation k equals 0 to m or k 0 to n a k Z raise to minus k; so this is basically your H Z which is in the form of a rational transfer function you can see this, this is in the form of a rational function in b. So, it is a rational. So, this is a rational transfer function. And taking the inverse Z transform, one can obtain the impulse response of the system of the system described by this difference equation. So, we have the Z transform, we have the Z transform of the system described by the difference equation all right; and you can see that it is given by that by a rational transfer function.

So, this is basically your Z transform which is given by rational transfer function all right, so that completes our discussion on the definition properties the definition and the properties of the Z transform as well as the analysis all right. So, subsequently when then the modules that follow we look at several problems for the application of the Z transform to understand the various properties and its application all right. So, we will look at other aspects in the subsequent modules.

Thank you very much.