Principles of Signals and Systems
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Lecture – 26
Properties of z-Transform – Convolution, Inverse z-Transform through Partial Fractions, Properties of ROC

Hello, welcome to another module in this massive open online course. We are looking at the Z transform with properties and applications; let us continue this discussion today. Let us look at the convolution right, how to implement or what is a property of the Z transform with respect to the convolution operation. So, start by looking at convolution.

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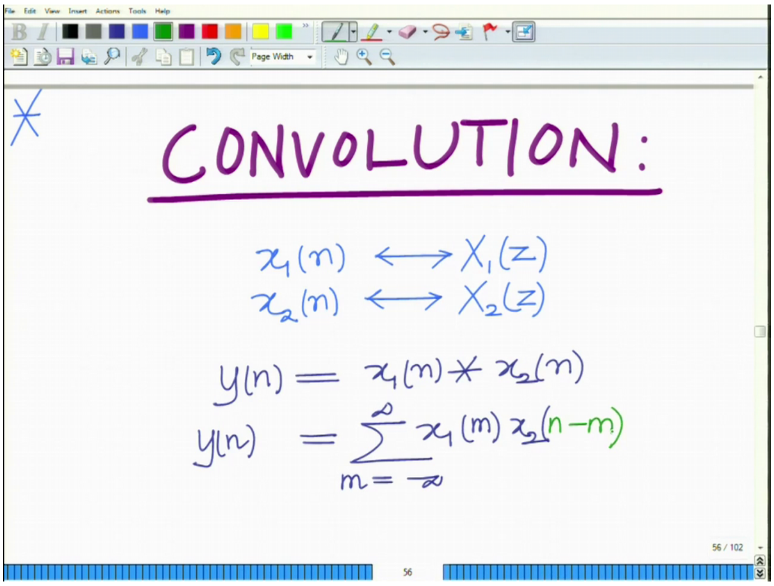

Remember the convolution of two discrete time signals can be defined; that is lets say we have two discrete time signals x 1 n which has Z transform X 1 Z, and we have another signal x 2 n which has Z transform X 2 Z.

What can we say about to yn, which is the convolution of x 1 and x 2 n; that is this is equal to summation, that is yn equals summation m equals minus infinity to infinity x 1 n into or x 1 m rather into x 2 n minus m.

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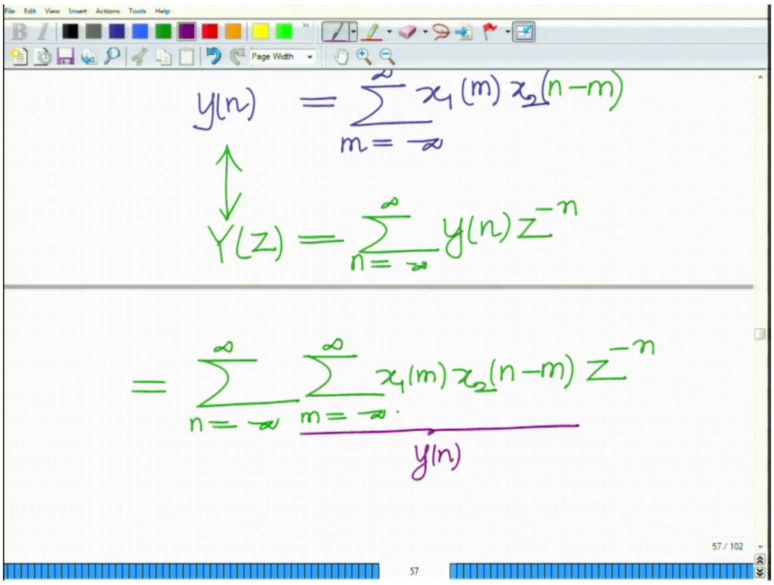

Now we have the Z transform of y n is Y z which is given as summation n equals minus infinity to infinity y n z raised to minus n. This is the Z transform of the sequence yn which is summation from minus infinity to infinity y n z raised to minus n.

We want to find out what is the Z transform of yn, when yn is the convolution; that is convolution of two signals discrete time signals x 1 n and x 2 n, and therefore, now substitute the expression for y n from above. This is the expression for y n correct, and substitute the expression for y n from above. So, I have summation, we just write it clearly through this is equal to summation n equals minus infinity to infinity summation m equals minus infinity to infinity x 1 m x 2 n minus m z raised to minus m. So, this part, if you remember this part is y of n. So, summation n equals minus infinity to infinity y of n z raised to minus n. Now I am going to interchange the order of summation.

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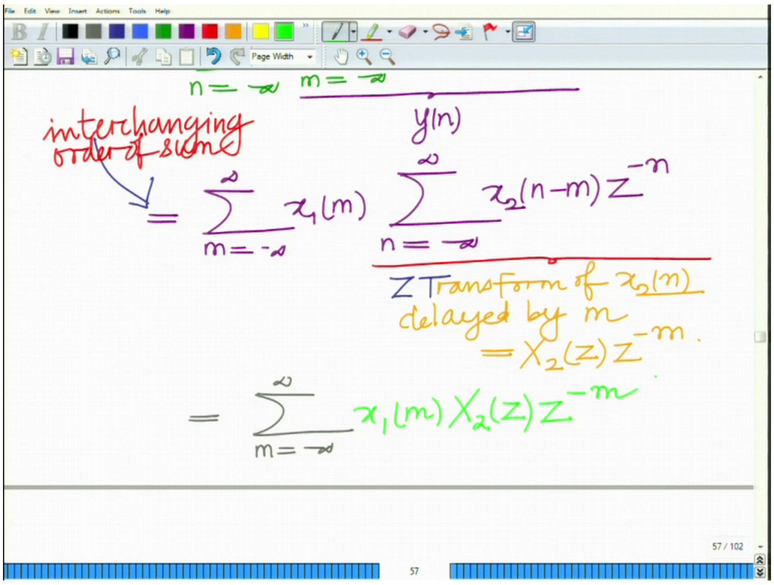

So, I am going to write first m equals minus infinity to infinity x 1 of m summation n equals minus infinity to infinity x 2 of n minus m z raise to minus n. So, this is basically by interchanging the order of summation, interchanging the order of the sum and.

Now, you can see, this is the Z transform of x 2; that is x 2 n minus m that is x 2 delayed by m. So, this is Z transform x 2 n delayed by m, and therefore, this is nothing, but X 2 z into z raised to minus m and therefore, I can now write this as summation m equals minus infinity to infinity x 1 m X 2 z into z raised to minus m. Now the x to z comes out common.

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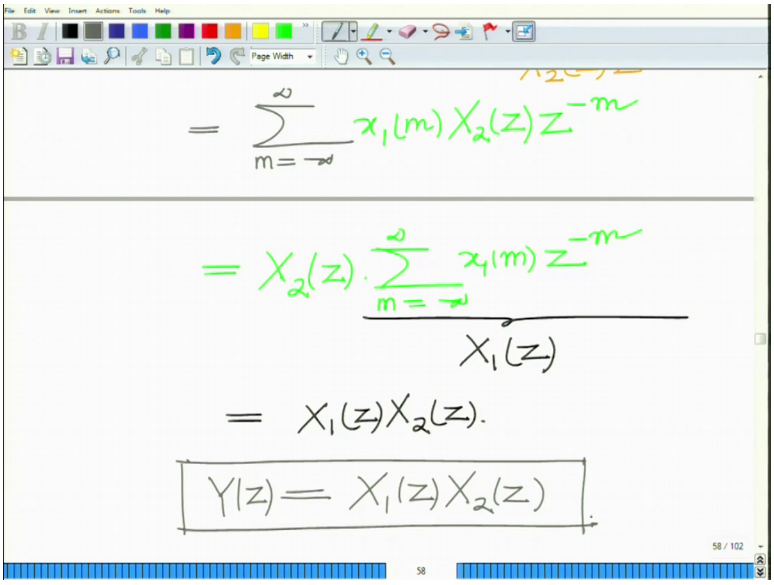

So, this is X 2 z summation m equals minus infinity to infinity x 1 m into z raised to minus m, and this you can clearly see, this is X 1 z.

So, this is basically the product of the Z transforms X 1 z into X 2 z. So, what we have here, is basically Y z equals X 1 z 2 X 2 z. So, the Z transform of the convolution of two signals is basically the product of their Z transforms, and this is similar to what we have seen in the Laplace transform. This is when you convolve two signals in the time domain in the transform domain. There are Laplace transforms are multiplied, there is a Laplace transform of the convolution of two signals, is basically the product of the individual Laplace transforms. Since they four similar to the Laplace transform this property gives us a convenient way to implement convolution, because convolution is typically fairly intricate to evaluate.

So, one can, rather one can otherwise as a trick evaluate the Z transforms of the signal multiply the Z transforms, and take the inverse Z transform to evaluate the convolution. So, this is an interesting property basically which.

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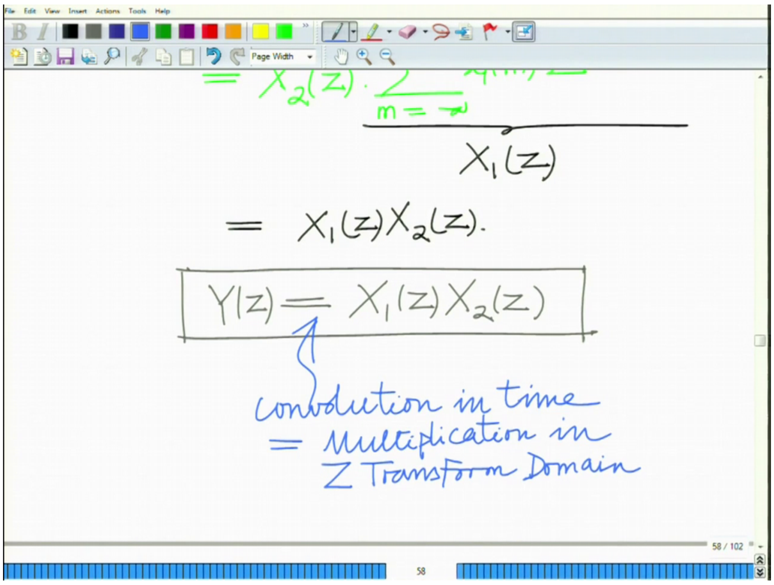

Says that convolution in time, this multiplication in the Z transform domain. So, this can convert two signals; the Z transform resulting signal is basically the multiplication of the Z transforms of the signals being convolved. Let us now look at techniques to evaluate the inverse Z transform and similar to the inverse Laplace transform one can consider, one of the simplest techniques. So, there is by partial fraction expansion ok.

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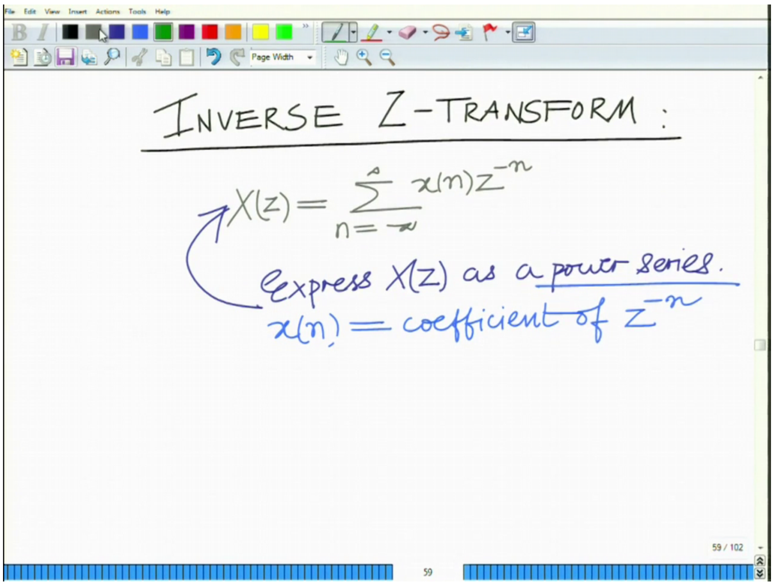

So, let us say we have Xz equals summation n equals minus infinity to infinity xn z raised to power of minus n. Now let X z. Now one thing we can do is, the simplest thing you can do is, express Xz as a power series, simplest way to evaluate the Z transform is to express Xz as a power series. Now once you express Xz as a power series x m is nothing, but the coefficient of z raised to minus n. So, that is one simple way to find the Z transform. So, xn.

So, once you express Xz as a power series, realize that xn is the coefficient of z raised to minus n. Now let us look at another technique that is. So, we can call this as the power series technique.

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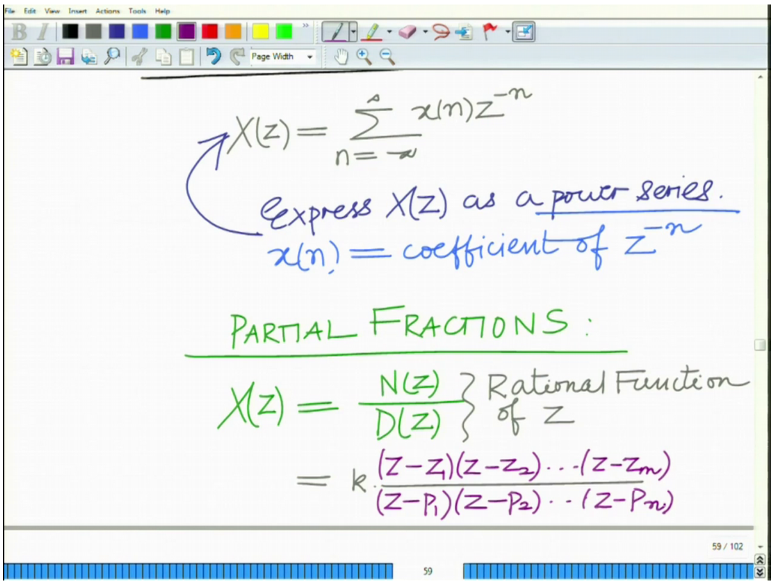

Let us look at another technique that is a partial fraction expansion. So, basically you have, let us say a Z is a rational function; that is in Z divided by Dz, that is this is a rational function of Z, this is a rational function of Z which I can write as some constant k times Z minus Z 1 Z minus Z 2 Z minus Z m divided by Z minus P 1 Z minus P 2 Z minus P n.

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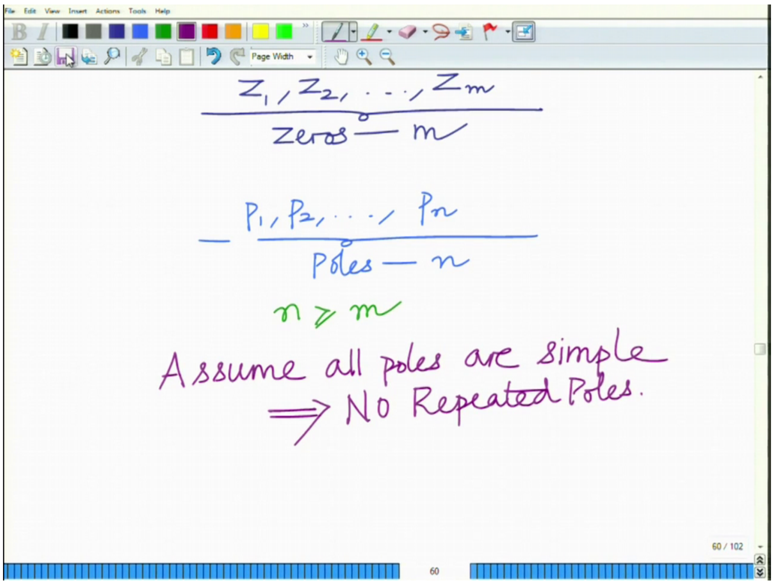

Now, we know that Z 1 Z 2 Z m are the zeros that is denominator roots of the numerator polynomial Zs are the zeros, and you have m such zeros Z 1 Z 2 Z m are the zeros, which are basically nothing, but the roots of the numerator polynomial. Remember we are assuming a rational function; that is Z transform is given by ri is it given as a rational function of Z, and that is its of the form of a numerator polynomial divided by a denominator polynomial and P 1 P 2 P n. These are; obviously, the poles, and we have these are the n poles.

And let us assume that n greater than or equal to m and further assume, all poles are simple; that is there are no repeated poles noise. This implies there are no repeated poles in which case I can express Xz over z, the partial fraction expansion of Xz over (Refer Time: 11:52) is given as C naught over z plus C1 over z minus p 1 plus c 2 over z minus p 2 plus.

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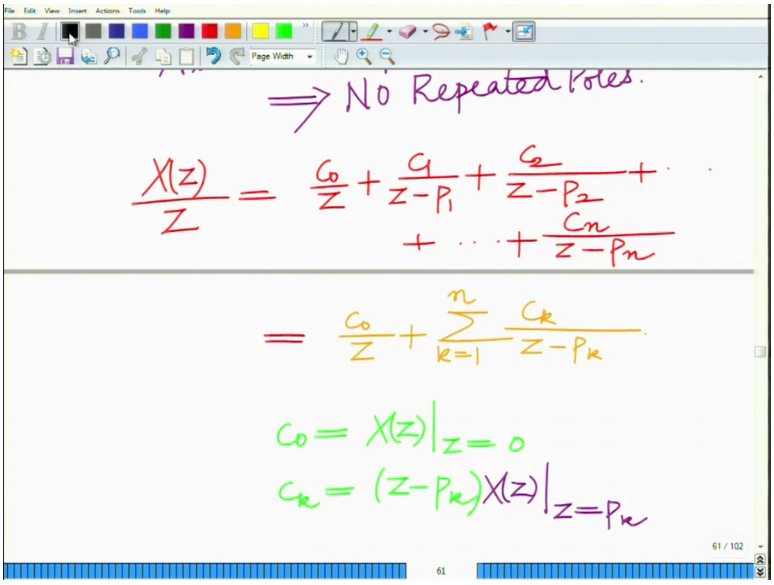

So, on Cn over z minus Pn which can be simplified as c naught over z plus the summation k equals 1 to n Ck over z minus Pk, where the coefficient C naught can be found as C naught equals simply evaluate Xz at z equal to 0, and Ck is basically given as z minus Pk times Xz evaluated at z equals Pk.

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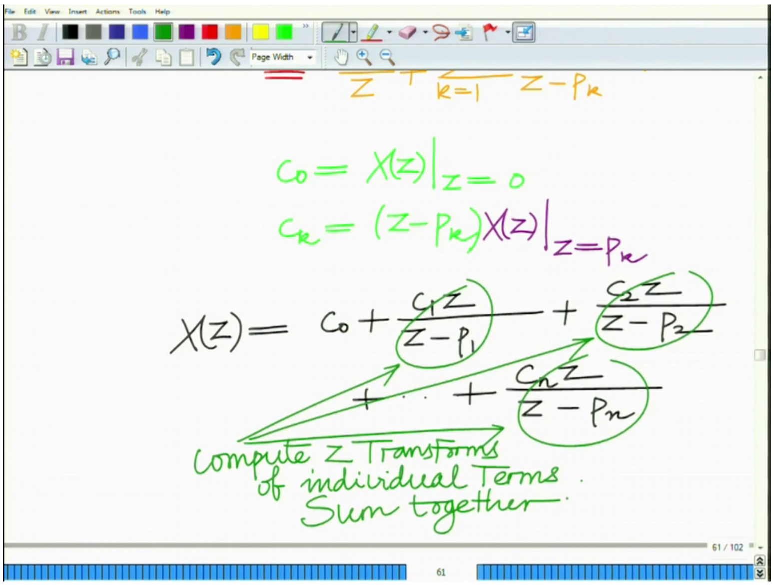

And finally, we have substituting the values of C naught C 1 up to C n, where we have n simple poles we have, and taking the Z in the denominator on to the other side, we have

X equals C naught plus C1 Z divided by Z minus P 1 plus C 2 Z divided by Z minus P 2 plus C n Z divided by Z minus P n.

And now the inverse transform can be found, because remember we know the inverse transform of each factor, which is of the form Z minus Z over z minus P i. We know the inverse transform of every such term, and therefore, now forming the Z transforms of these individual partial fractions and then forming the sum that gives us the inverse Z transform of this, of this rational function of Z. So, we know the Z transform. So, for instance 0 minor or Z minus P 1 Z over Z minus P 2 Z or Z; so, put these together. So, compute Z transforms of these terms, compute Z transforms of the individual terms, and then followed by their sum ok.

So, that gives us the inverse Z transform. Now if Xz has multiple poles.

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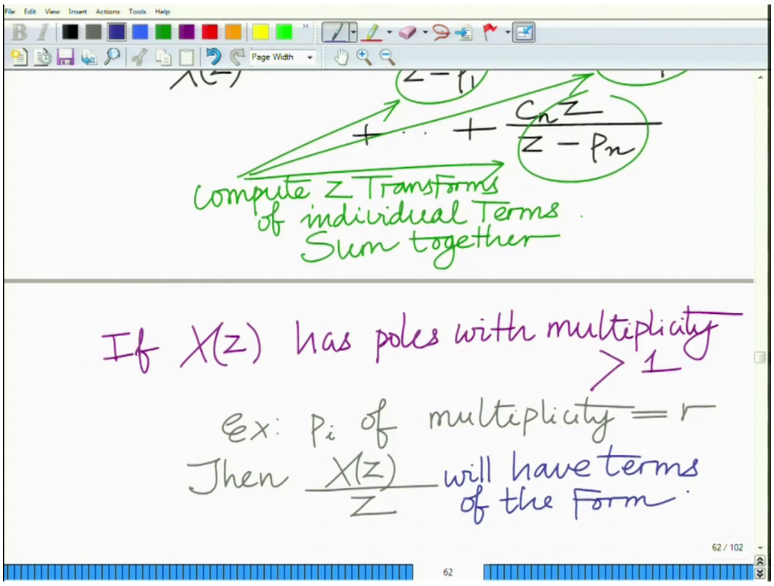

If Xz, if fz has poles with multiplicity greater than 1; that is imply, lets say example P i of pole P i of multiplicity equals r correct multiple multiplicity means multiplicity greater than 1, means repeated poles that is your terms of the form Z minus P i square multiplicity of 2 or multiplicity of 3 implies Z you have a term of the form Z minus P i cube; that is pole Pi has multiplicity of 3. So, in that scenario.

If you, if let us say pole P i has multiplicity r then Xz over Z will have terms of the form.

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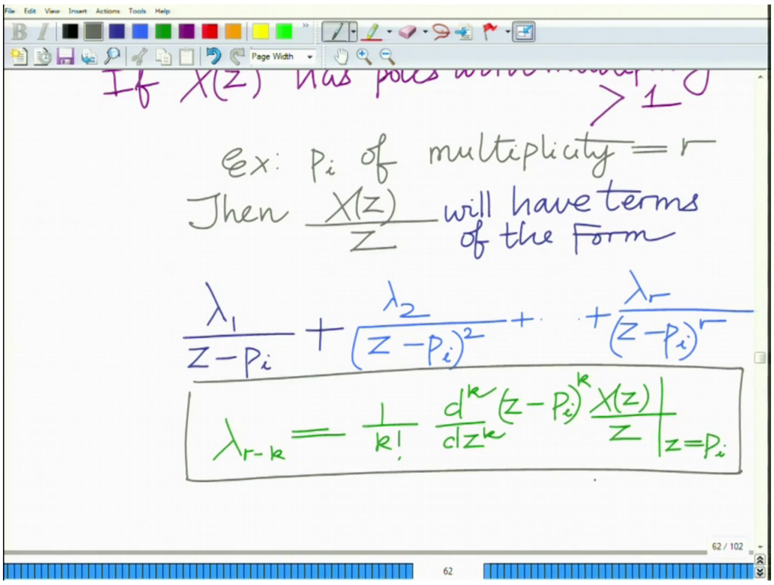

each such pole Pi with multiplicity greater than 1 will contribute to two terms of the form lambda 1 by Z minus P i over lambda 2 by Z minus P i square plus lambda r over Z minus P i raise to the power of r, where this quantity lambda r minus k, k equals. Well 0 to r minus 1, this will be given as 1 over k factorial d, d raise to k over d z to the k z minus P i raised to the k Xz by z evaluated at; that is the kth derivative of Z minus Pr raised to the power of k times Xz over Z evaluated at Z equals Pi.

So, this is the expression for. So, this is the expression for lambda r minus k. So, this is the expression hold the term lambda r minus k, which basically there is a coefficient of the term Z minus P i raised to the power of r minus k alright. So, we have lambda 1 or Z minus P i lambda 2 or Z minus P i square so on and so forth. Lambda r over Z minus P raised to the power of r the coefficient lambda r minus k is 1 over k factorial d to the k; that is d raised to the power k over dz k z minus p i to the power of k Xz over z evaluated at Z equal to P r; that is a kth derivative of Z minus P i raised to k into Xz over Z evaluated at is equal to Pi. And from this, once again one can find the inverse Z transform of the given function.

Let us now look at the properties of the region of convergence.

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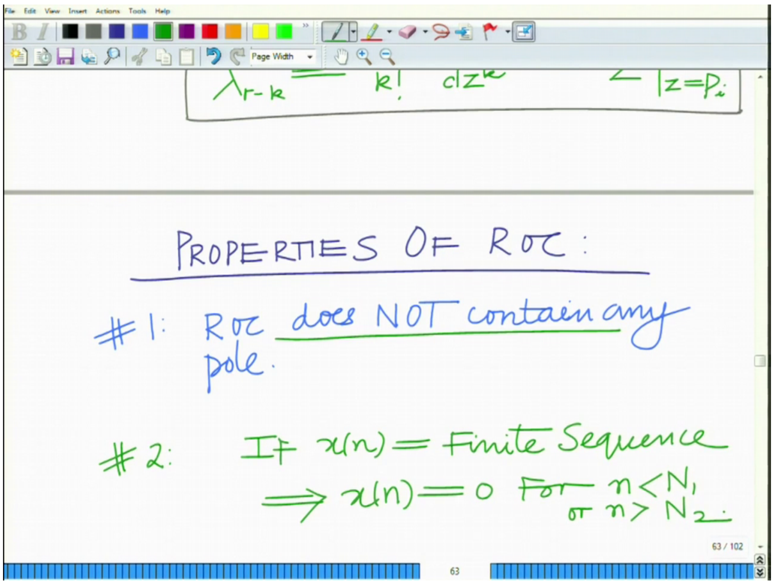

The properties of the ROC, the properties of the ROC are as follows; number 1, ROC does not, the ROC does not contain any port 2, if xn equals of finite implies that xn equal to 0, for either xn equals 0 for n less than n 1 or n greater than n 2; that is it is nonzero, only in the interval n 1 to n 2 and Xz converges for some value of Z Xz converges for some value of Z alright. Then this implies that ROC equals the entire Z plane.

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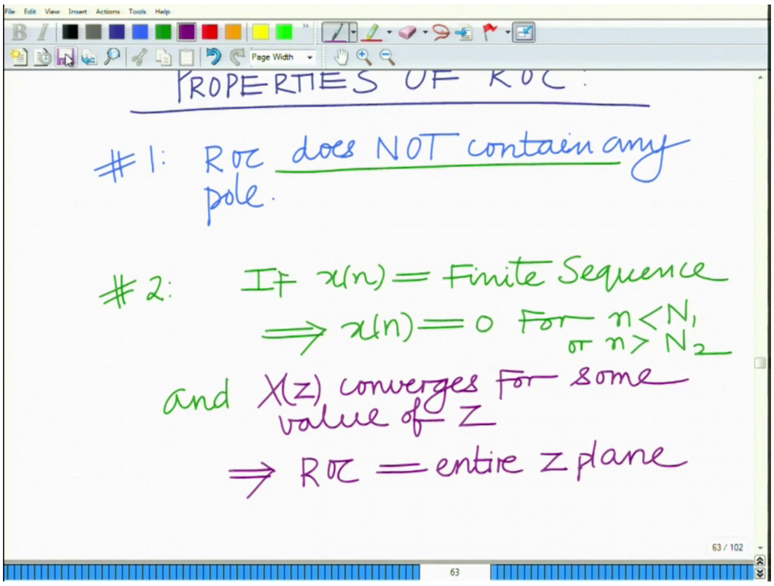

So, the point is that basically if you have a finite sequence; that is if the sequence is not infinite, it is a finite sequence. It exists only in a finite region; that is it exists only in the finite interval n 1 to n 2; that is 0. If n is less than n 1 or 0, if n is greater than n 2 and it is nonzero, only in this interval n 1 to n 2, then basically the Z transform of that signal that is a finite signal exists everywhere on the Z plane, if it converges for a single point, if it converges for any single point it exists everywhere on the Z plane; that is the entire Z plane now for a right handed signal.

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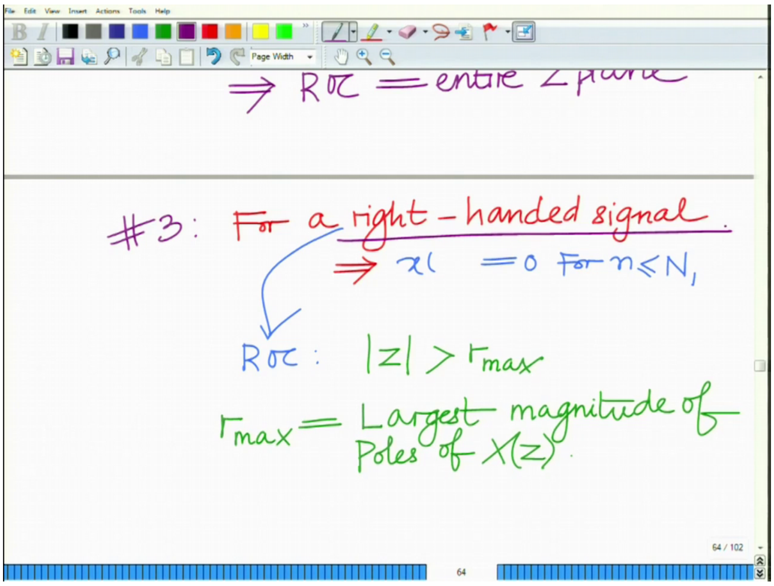

Again this is something similar to the Laplace transform for right handed signal. Now right handed signal implies that xn equals 0; that is there exist some n 1 that says our x and equals 0 for n less than equal to n 1. Now this implies for a right handed signal ROC is of the form magnitude of Z; that is ROC is of the form magnitude of Z greater than r max, where r max equals the magnitude largest magnitude of the largest or largest. This is the largest magnitude of the poles of poles of x of z.

So, r max is the largest amongst the magnitudes of the poles of Xz. And if it is a right handed signal, then the ROC has to be. If it is a right handed signal then the ROC has to be of the form magnitude of Z greater than r max. Now similarly for a left-handed signal.

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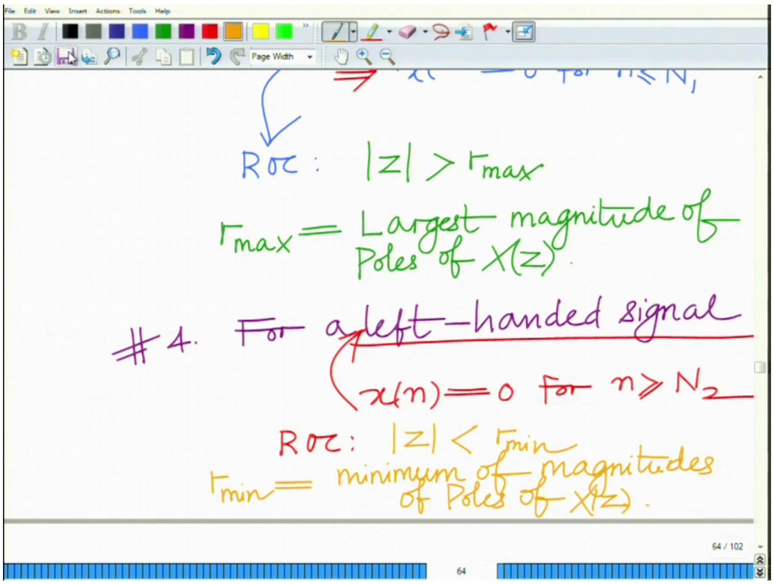

Now what is the left handed signal? once again left handed signal is, if xn equals 0 for some n greater than or equal to n 2; that is there exist an n 2 such that xn equals 0 for n greater than equal to n 2. This implies that the ROC is of the form magnitude of Z is less than r min, r min equals smallest magnitudes; that is the minimum of the magnitude, minimum of the magnitude, minimum of the magnitudes of polls of X of c; that is magnitude Z is less than that is for a left-handed signal. You have the ROC is of the form magnitude Z less than r min, that r min is the minimum of the magnitudes of the poles of X of c. So, your ROC will be of the form.

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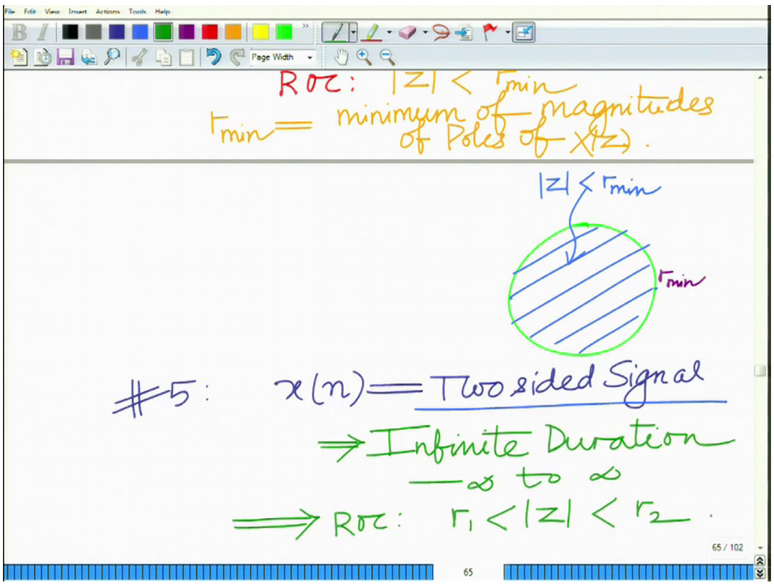

So, if this is r min minimum the magnitude of the poles, your ROC will inside, be inside this circle that is magnitude Z less than r min, where r min is. So, ROC will be interior of the circle or the circular region. So, this is the ROC, is of the form magnitude Z less than magnitude Z less than r min. And finally, if xn is a 2 sided signal. Now if xn equals a 2 sided signal; that is this infinite duration from minus infinity to infinity minus infinity to infinity. Then this implies that the ROC is of the form r 1 less than magnitude of Z is less than r 2.

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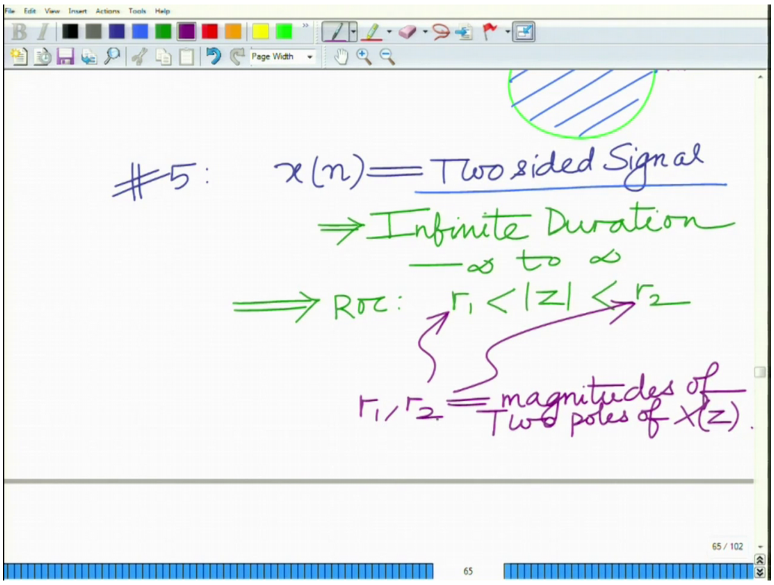

Now, r 1 and r 2 these are the magnitudes of two poles, the form r 1 less than magnitude z less than r 2, where r 1 r 2 are the magnitudes of two poles of X of z. So, these are the properties of the ROC. Remember the ROC stands for region of convergence once again. I think we have seen that enough times before. So, I do need to go or that again. So, ROC stands for the region of convergence, and we have seen the properties of the ROC for several signals; that is what happens to the ROC for a finite signal of finite duration, ROC for a left-handed signal right handed signal and then a 2 sided signal all right. So, these are useful properties of the ROC. One has to bear in mind while, and this helps us better construct, reconstruct as well as analyze the properties of a signal by looking at this as Z transform this alright.

So, we will stop here and continue with other aspects in the subsequent modules.

Thank you very much.