Principles of Signals and Systems
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Lecture – 25
Properties of z -Transform –Linearity, Time Shifting, Time Reversal, Multiplication by n

Hello, welcome to another module in this massive open online course. So, we are looking at the Z transform and its properties. So, let us continue our discussion on this very important transform for discrete time signals and systems termed as Z transform.

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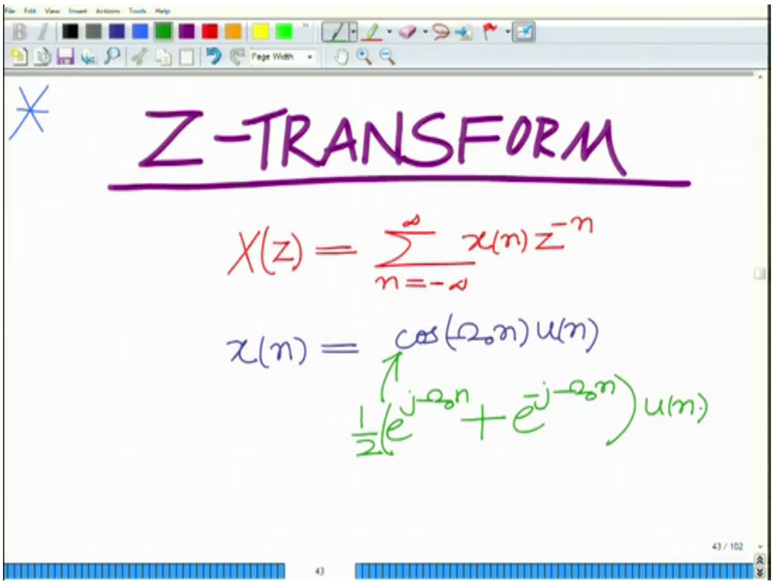

Remember we already seen the definition of the Z transform, X of z equals summation n equals minus infinity to infinity x of n, z raise to minus n and all the Roc the region of convergence of the Z transform is the values of z for which this converges for this is for which this possibly infinite summation converges.

Now, let us look at the Z transform we are looking at the Z transforms of some common signals. We have looked at the Z transform of the unit impulse and the unit step function let us look at the Z transform of a cosine signal that is cosine omega naught n un which as you know cosine omega naught n can be written as. So, this is basically cosine omega naught n can be written as half e raise to j omega naught n, half e raise to j omega naught n plus e raise to minus j omega naught n times u n. And the Z transform of this is X of z which is half summation n equal to minus infinity to 9 mm infinity e raise to j omega naught n plus e raise to minus j omega naught n u n z power minus n.

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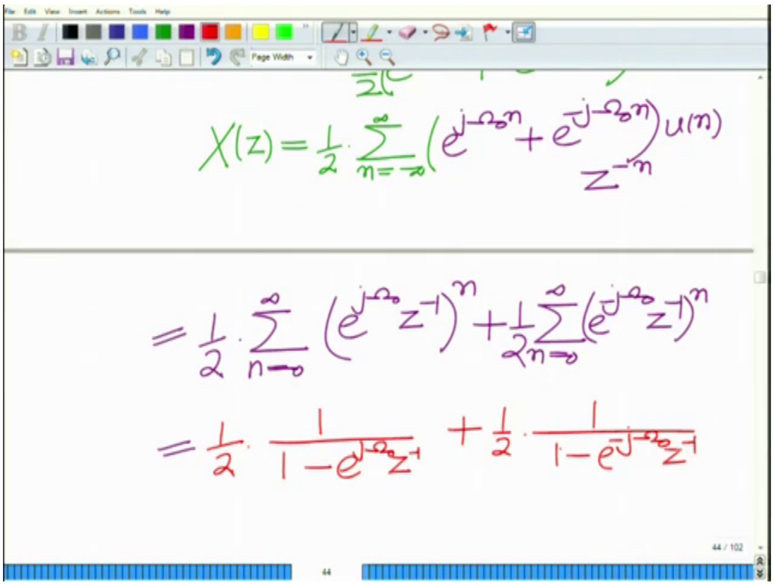

Now, u n is non0 only for n greater than equal to 0. So, this is equal to half summation n equal to 0 n equal to 0 to infinity, e raise to j omega naught n, z power minus n. So, this is e raise to j omega naught z inverse raise to the power of n plus half summation n equal to minus infinity is equal to 0 to infinity, e raise to, e raise to minus j omega naught z inverse raise to the power of n which is equal to well this is 1 over or this is half the factor of half is there 1 over half 1 over 1 minus e raise to j omega naught z inverse plus half 1 over 1 minus e raise to minus j omega naught z inverse, e raise to minus j omega naught z inverse correct and if you sum this, this is equal to half.

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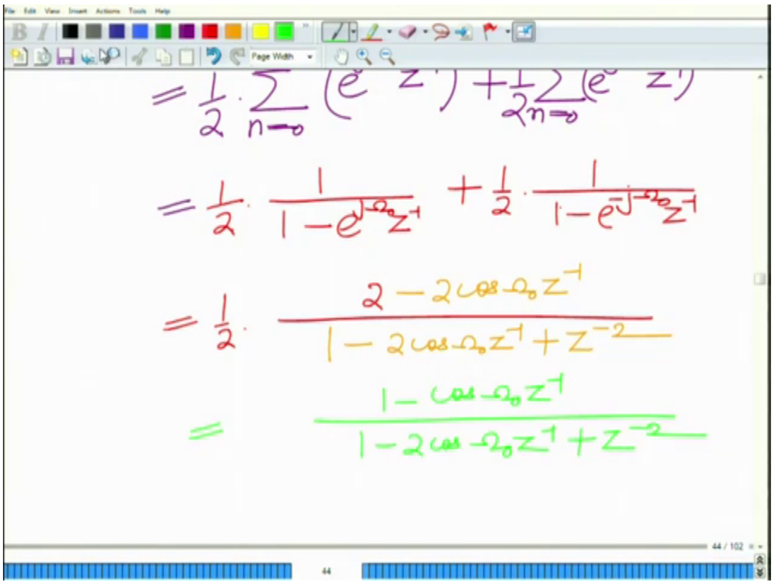

In the numerator you have 1 minus e raise to j omega naught z inverse plus 1 minus e raise to minus j omega naught z inverse. So, 1 plus 1 is 2 minus e raise to j omega naught plus e raise to minus j omega naught is 2, cosine omega naught z inverse divided by 1 1 minus e raise to j omega naught z inverse into 1 minus e raise to minus j omega naught z inverse is 1 minus 2 cosine omega naught z inverse plus e raise to j omega naught into e raise to minus j omega naught is 1 z minus 2. And this is equal to well 1 minus cosine omega naught z inverse by 1 minus 2 cosine omega naught z inverse plus z minus 2 which simplified by multiplying the numerator and denominator by z raise to minus 0 z square. So, that will give you z square minus cosine omega naught z or minus I am sorry there is a factor of 2 over here or there is a factor of this more factor of 2.

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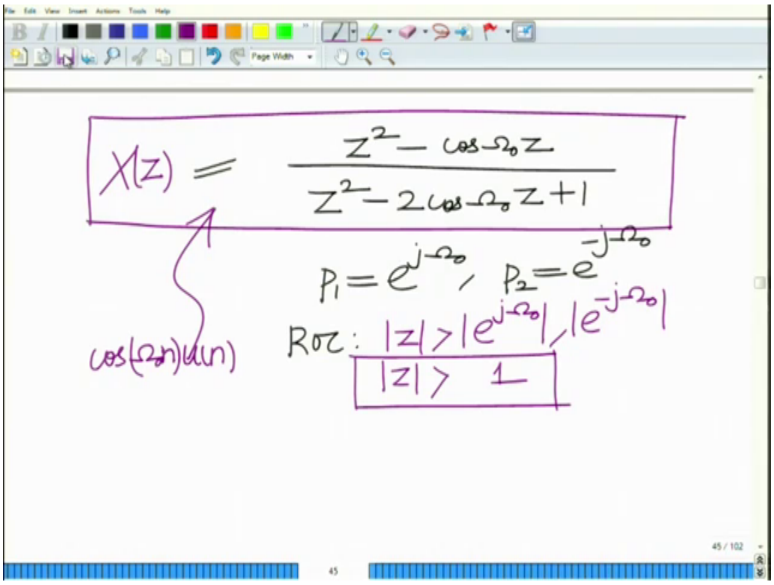

So, this is z square minus cosine omega naught z, minus cosine omega naught z divided by z square minus 2 cosine omega naught z plus 1 and the region of convergence remember this has 2 poles that is e raise to j omega naught and e raise to minus j omega naught. So, the 2 poles are there are 2 poles P 1 equals e raise to j omega naught and p 2 equals e raise to minus j, e raise to minus e raise to minus j omega naught and therefore, the Roc, remember Roc equals magnitude of z greater than magnitude of e raise to j omega naught comma it has to be greater than magnitude of both poles. And magnitude of both poles equals 1 so Roc is magnitude z greater than greater than one this is the Roc and this is the Z transform of cosine. This is the Z transform of cosine, cosine omega naught n times u of n.

So, this is the Z transform of the signal all right and the Roc is magnitude of z greater than 1 all right. And now what we want to do, so we have looked at the Z transform of some of the common signals such as the unit impulse function the unit step function cosine omega naught n where omega naught is the frequency times u n all right and you can derive the Z transform of some other common signals for instance such as sin omega naught n u of u n and so on, etcetera all right.

What we will do now is look at some of the properties. Like the important properties of the Z transform. So, what we want to look at next is the properties of the Z transform.

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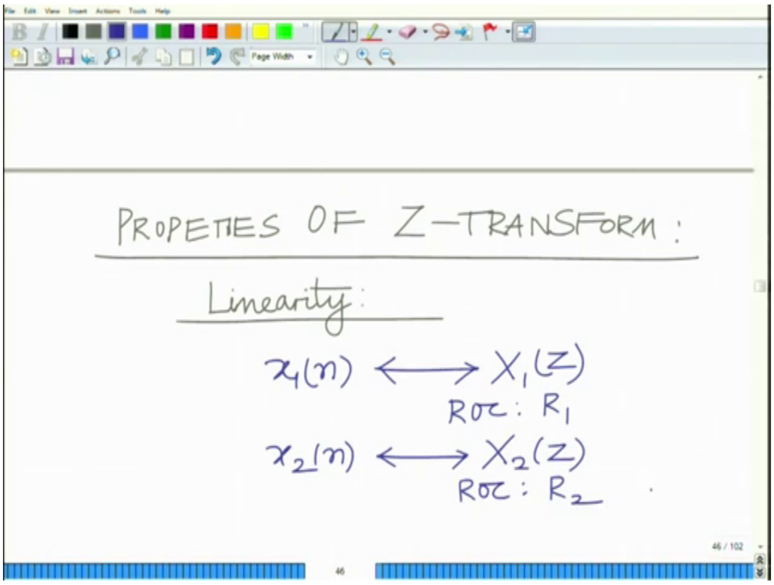

Properties of the Z transform, you know the first property is the Z transform sturm the linearity that the Z transform is a linear transform which means to say that basically if I have x 1 n and this is similar to the Laplace transform x 1 n has Z transform x 1 z and Roc is R 1 x 2 n as Z transform x 2 z Roc is R 2 then this implies that a 1 x 1 n x 2 n will have Z transform a 1 x 1 z plus a 2 x 2 z.

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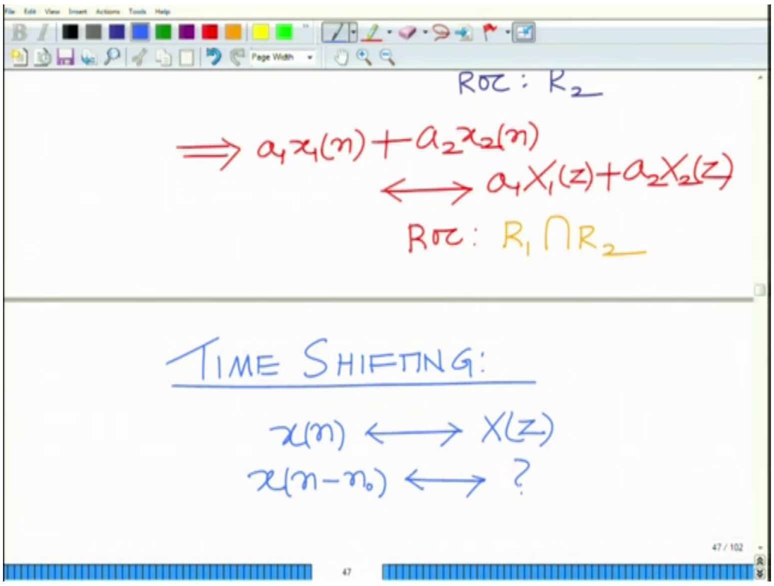

That is a linear combination that is a 1 times x 1 and plus a 2 times x 2 and there is a linear combination of the input signals results in a linear combination of their Z transforms the same linear combination that is a 1 x 1 z plus a 2 x 2 z all right this is the linearity property and the Roc will at least be greater than the intersection of the 2 Roc. So, the Roc will be a superset or you can simply say let us say the Roc is basically the intersection of both R 1 intersection R 2. It will at least be this in practice it can be it is a super set all right. So, it can be more than R 1 intersection R 2.

Let us look at another property the time shifting property that is let us say x of n has the Z transform X of z, then your x of n minus n naught what is the Z transform of x of n minus n naught.

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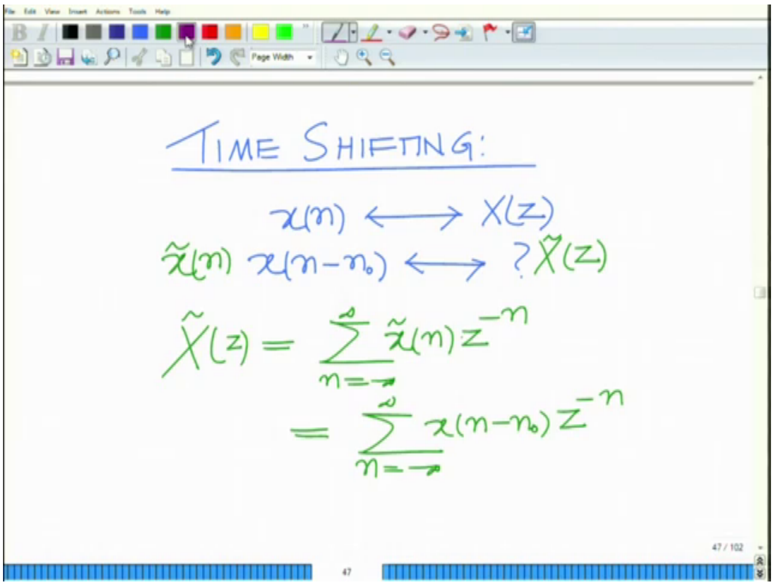

Now, you can see let us consider this as x tilde n with Z transform x tilde z then x tilde z equals summation n equals minus infinity to infinity, x tilde n z to the z raise to minus n which is n equals minus infinity to infinity, x of n minus n naught z raise to minus n.

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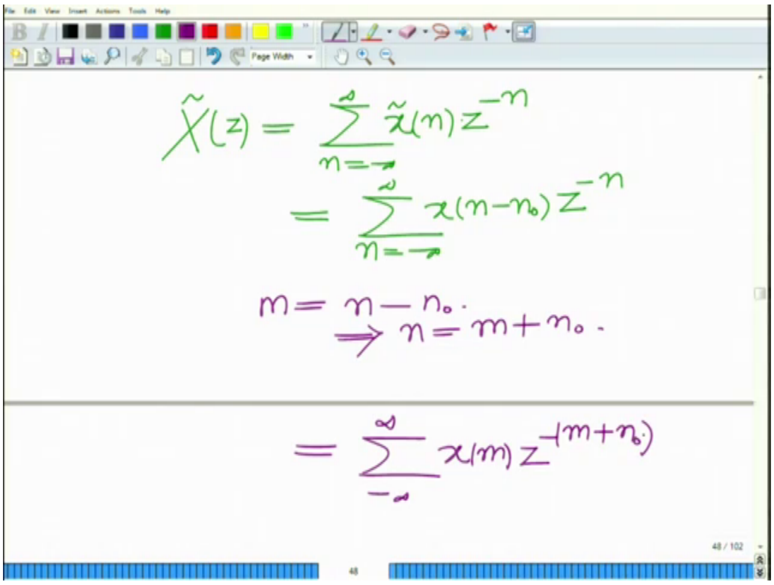

Now, set m equals n minus, now this implies n equals m plus n naught and the limits will still be infinity minus infinity minus n naught which is minus infinity to infinity minus n naught which is infinity x of n minus n naught is x of n z raise to minus n is m plus n naught which is equal to.

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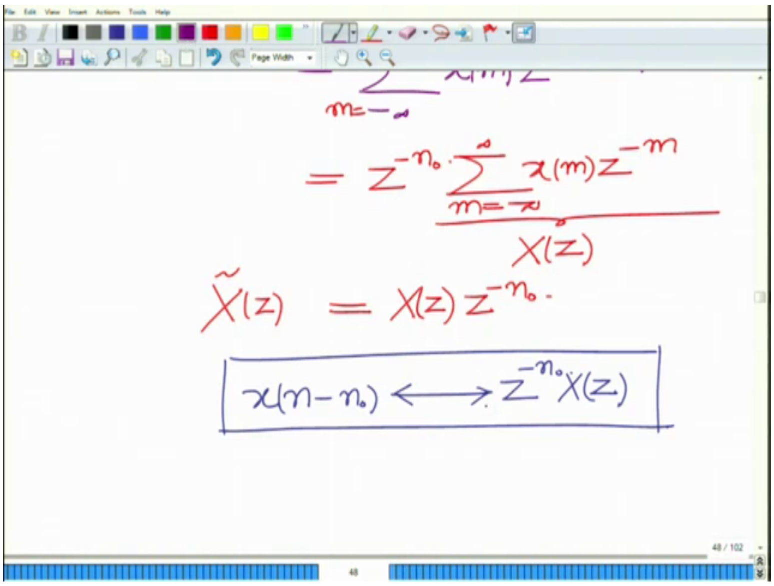

Now if you take the z raise to minus n naught outside, z raise to minus n naught this is m whole to the power minus infinity to infinity x of m z raise to minus m and remember this is X of z, so this is x tilde of z equals X of z into z raise to minus n naught all right or basically to say your delay x of n minus n naught signal delayed by n naught has Z transforms z raise to minus n naught X of z raise to minus n naught X of z. So, this is the delayed signal.

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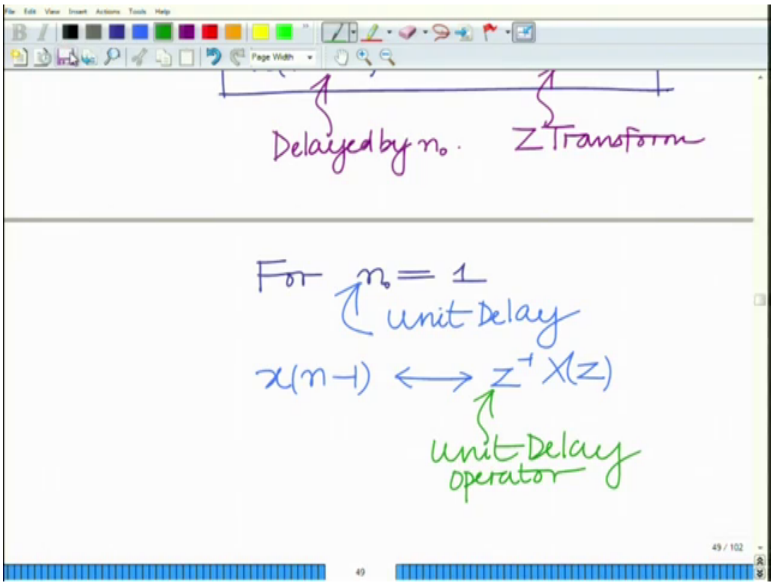

This is a signal delayed by n naught and this is the Z transform, not the signal that is it is multivariate original Z transform multiplied by z raise to minus n naught.

Now, this for the special case where n naught equal to 1 that is for a unit delay this will simply become z inverse that is z raise to minus 1 times X of z therefore, this operator z inverse that is e raise to minus 1 is also termed as the unit delay operator frequently you will represent you will see it using 2 represented used to represent the structure all right to represent a filter correct in which a signal or a signal sample x n is delayed by a unit delay to derive x n minus 1 is represented using the operator z inverse or using the block z inverse which represents a unit delay for a or a unit sample delay.

So, for n equal to for the special case of n naught equal to 1 that is a unit delay we have x of n minus 1 has the Z transform z raise to minus 1 X of z. So, this is also termed as the unit, this is also termed as the unit delay operator and frequently you will see this in systems when you represent a system typically you have a block which looks like this which means you have a block which means if the input is x n then the output is x n minus 1.

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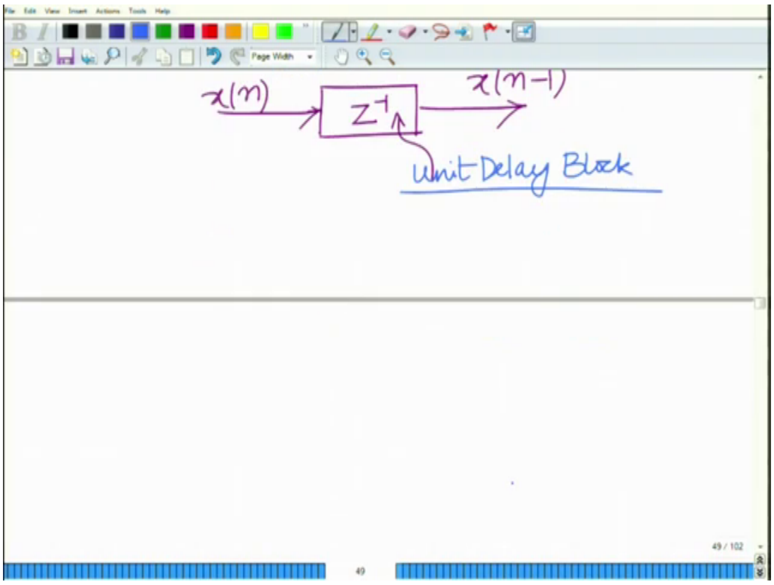

So, this is simply our delay block or unit delay block. So, this is a delay block that he used to represent a delay a sample delay and even consider the constructs or even link several such blocks to basically delay it delayed the signal by an arbitrary amount for instance if you kind of connect to such blocks in series the net signal will be delayed by 2 times 2, so x n, x n minus 1. And now it would not be x n minus 2 and then you can do processing on x n x n minus 1 x n minus 2 and so on all right.

So, let us go on to the next aspect that is basically your time reversal. So, let us look at the time reversal.

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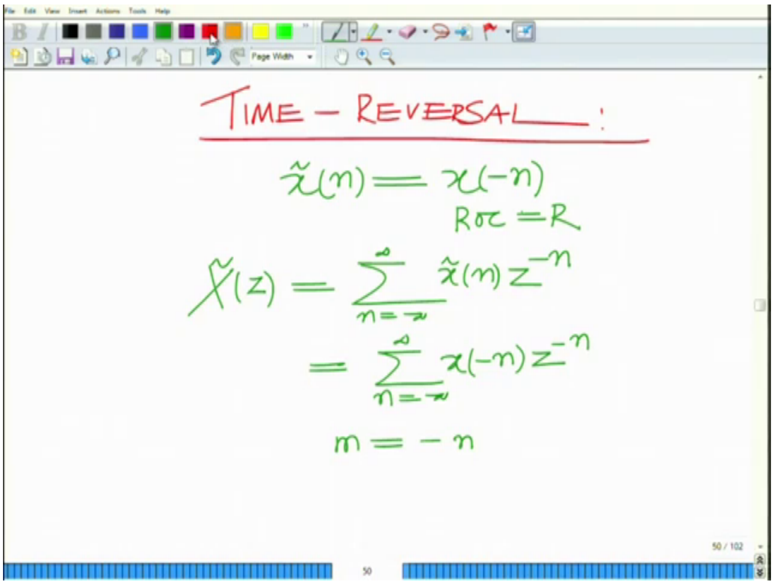

In time reversal what we have is x tilde n equals x of x of minus and let us say the Roc of the Z transform is R. So, you are raising reversing the time for instance if x n is not that is basically you are reflecting it about the about the y axis for instance x n is 0 or n less than 0 then this x tilde of n which is the time reflected time reverse version of x n will be 0 for n greater than 0.

So, it is a reflector or it is a time reversed version of a reversed version of x n and the Z transform of this it is easy to see this is given as x tilde z equals summation n equals minus infinity to infinity x tilde n z raise to minus n that is equal to n equals minus infinity to infinity x of minus n z raise to minus n which is now set m equals minus n implies this will be equal to limit infinity minus infinity becomes infinity, infinity becomes minus infinity.

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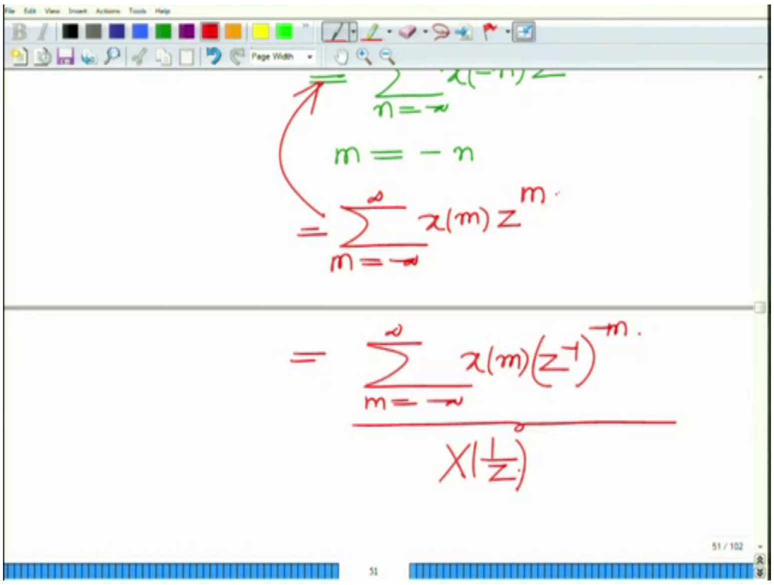

So, the stumped still goes from minus infinity to infinity x of m 0 raise to minus of n, but minus of n is m z x of m z raise to minus m which is m equals minus infinity to infinity x of m z inverse raise to minus m and this is nothing, but the Z transform of x evaluated at z inverse or evaluated at 1 over z. So, x tilde of z or basically if you look at the time reversed version time reversible version of x n that has the Z transform 1 over z.

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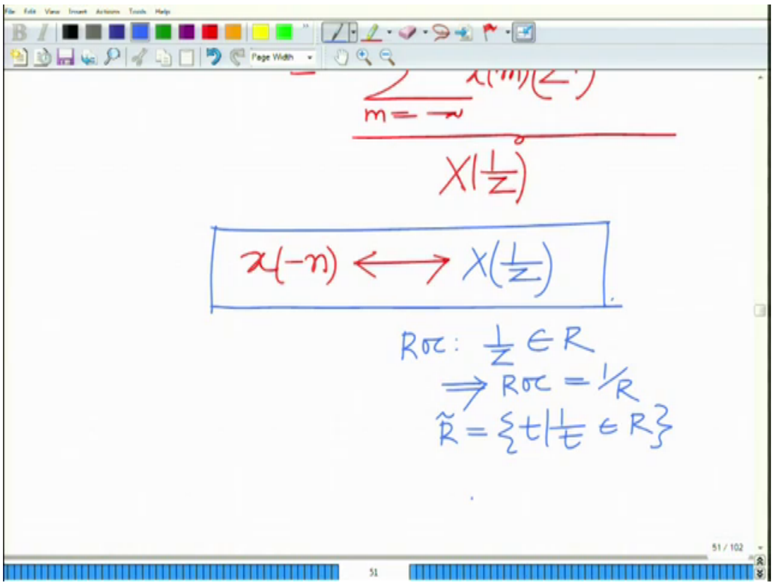

And naturally the Roc will be 1 over z element of R. So, for the Roc, for the Z transform it would converge we have we must have 1 over z belongs it converges remember the Z transform xoc converges where z belongs to R. So, since this x of 1 over z you must have for convergence 1 over z belongs to R which means the Roc must be equal to 1 over R. That is the inverse of any element that is Roc R tilde equals basically all elements see t such that 1 over t belongs to R that is the inverse of all the elements in R, inverse of all the elements in R. The Z transform possibly with the exception of 0 because inverse of 0 does not exist all right.

So, that is the Z transform of the time reverse signal x of minus n which is x of capital x or the x of 1 over z all right. Let us look at another property. Let us see what happens when you multiply the signal by n.

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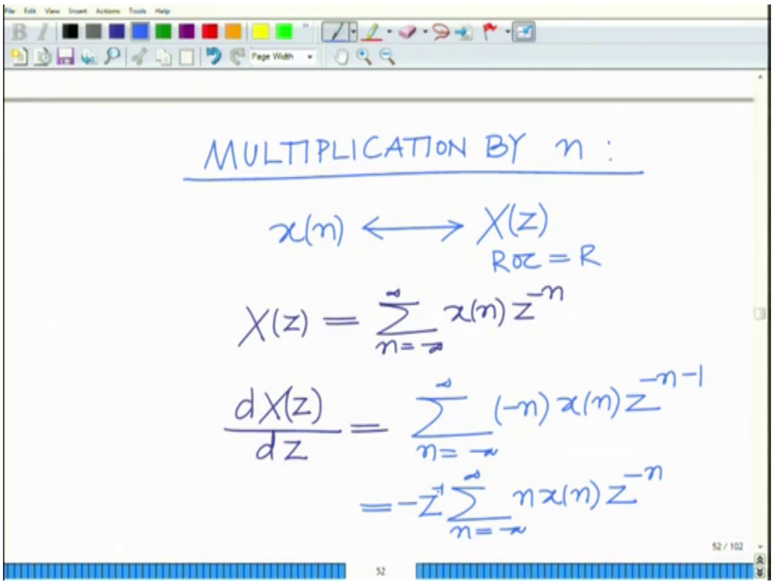

When you multiply the signal by n, now consider the signal x n which has Z transform x or z with the Roc given by R then X of z will be summation n equal to minus infinity to infinity x n z raise to minus n.

Now, if you differentiate this dx z over dz if you differentiate this we have summation n equal to minus infinity to infinity minus n x of n z raise to minus 1 equals minus n equals minus infinity n x of n z raise to minus n you can bring the z inverse outside. So, this will be minus z inverse which means minus z derivative of X of z equals n equals minus infinity to infinity summation n x n z raise to minus n and this is remember we can look at this Z transform of n x n.

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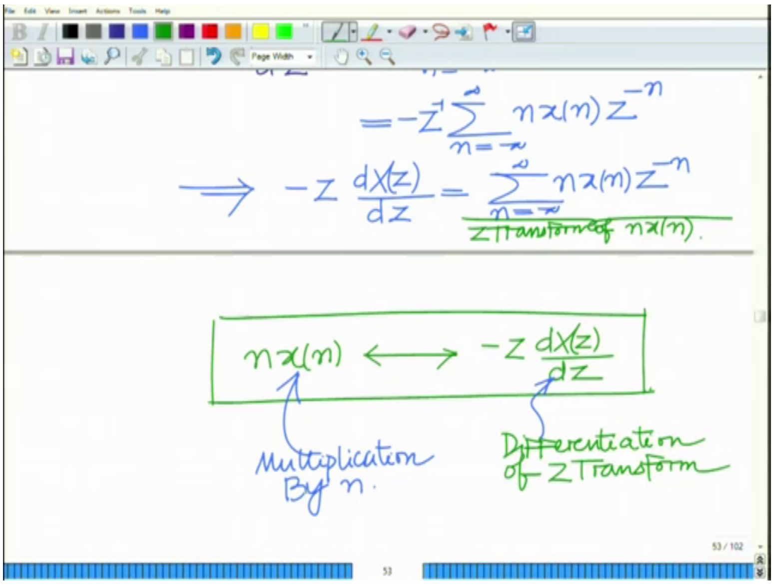

So, the signal that is time multiplication by n that is the signal n x n has Z transform minus z, d x z divided by. So, multiplication in time is basically differentiation in the z domain. So, this is multiplication differentiation of, differentiation of the Z transform. Let us look at one final property that is the accumulation which is similar to integration in time.

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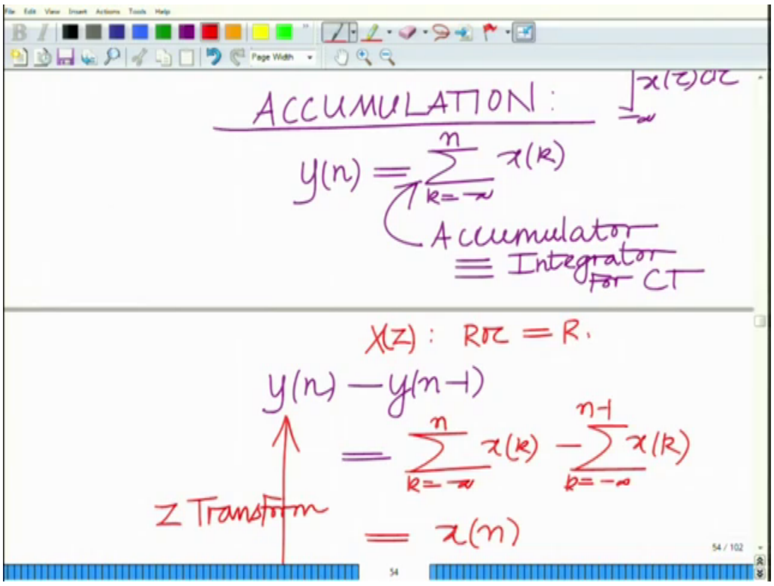

So, we have the accumulation property, we have the accumulation property where we have y n equals summation k equals minus infinity to infinity x of k or k equals minus infinity to n x of k. So, this is termed as an accumulator which is similar to n which is similar to an integrator for continuous time.

Remember we can have an integrator which is of the form integral minus infinity to t x of tau d tau this is an accumulator or basically an integrator for continuous time and now the Z transform, so of this can be obtained as follows now you can see that y of m minus y of n minus 1. If you consider y of n minus y of n minus 1, well this is you can simplify it as summation k equals minus infinity to n x of k minus summation k equals minus infinity to n minus 1 x of k which is basically again you can see the only term that survives is x of n.

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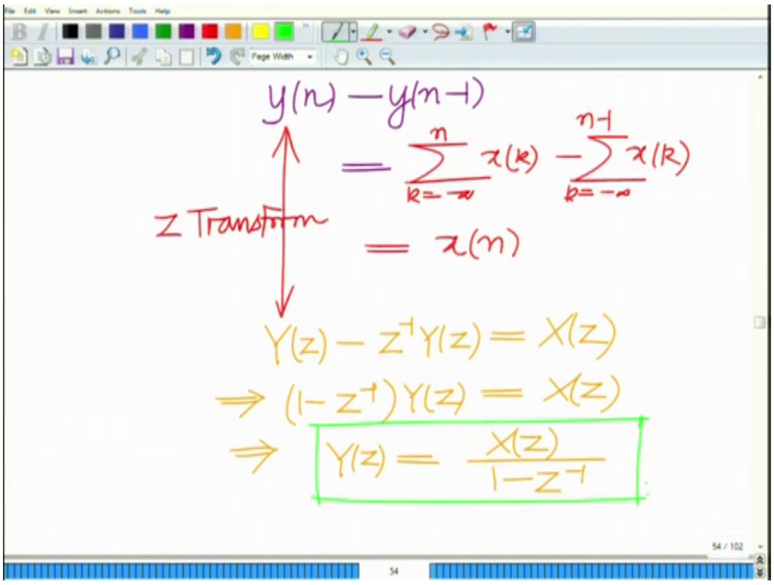

Now, taking the Z transform, now take the Z transform and what you observe is that Y of Z minus Z transform of Y of n minus 1 is Z inverse Y of Z this is equal to X of Z which implies 1 minus z inverse of Y of Z equals X of Z which implies Y of Z, X of Z divided by 1 minus Z inverse.

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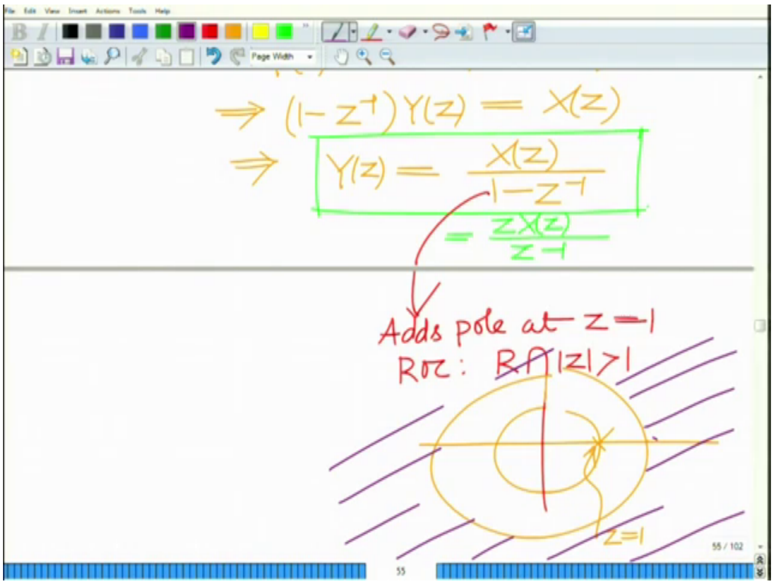

So, or you can simply also write this as Z X of Z over Z minus 1. So, you can see this adds a pole in addition to poles in X Z adds adds pole at Z equals 1 in addition to the poles in X Z, so, Roc has to be. So, Roc of course, since we have a pole at Z equal to 1, if there is a right handed signal you remember you are accumulating it assuming this is a right handed signal.

So, the Roc has to be magnitude of z greater than 1 correct. And previously the signal x of n has an Roc of. So, the Roc must be intersection of this that is R and the region magnitude z greater than 1. So, the Roc if the Roc of x n let us say if x z has Roc equals R, since you are adding a pole at magnitude at z equal to 1 the net Roc will be the previous Roc intersection magnitude z greater than 1, for instance if the previous Roc the previous Roc for instance does not include a part of the region for instance does not include for the let us say the previous Roc let us say simple scenario, let us say we have the previous Roc which already includes.

For instance, let us say this is the previous Roc corresponding to this let us say this is a pole and this is z equals 1 and let us say that this is the unit circle and this is let us say z is equal to 1. So, the previous Roc includes already includes the region magnitude z great. So, this is magnitude z greater than has to be greater than 1 and the previous Roc let us we will just write it slightly this is the unit circle and, this is your pole z equals 1 and this is let us say your this is a unit circle this is the pole and let us say this is the Roc previous Roc.

Now, the previous Roc already has already is the region in which magnitude z is greater than 1. So, the previous Roc is modified otherwise it has to be modified appropriately. So, the Roc will be the, previous Roc R intersection with the region magnitude z greater than all right. So, we will stop here and look at other aspects in the subsequent modules.

Thank you very much.