Principles of Signals and Systems
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Lecture – 24
Z-Transform – Definition, Region of Convergence (ROC), z-Transform of Unit Impulse and Step Functions

Hello, welcome to another module in this massive open online course. In this module let us start looking at a new topic that is this z transform or the z transform, all right.

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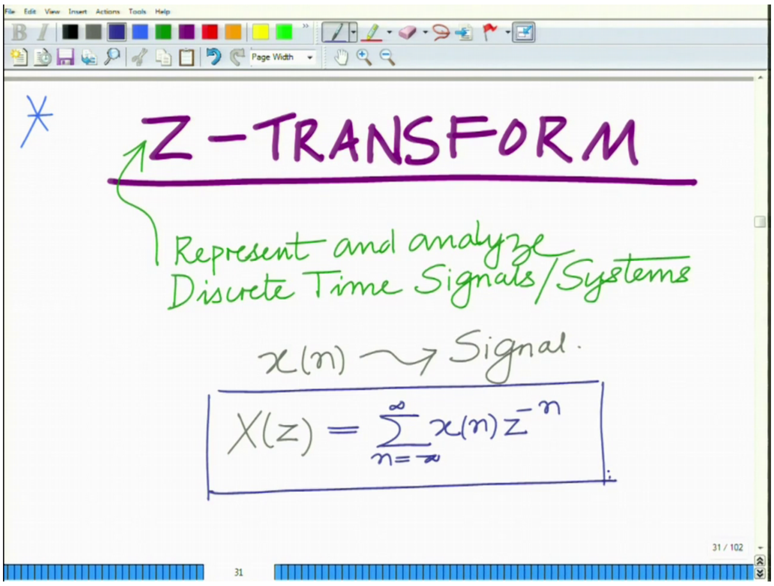

So, we want to start looking at a new concept; that is, the z transform which is basically, this is basically for to analyze represent and analyze discrete time. So, we have seen the s trans the laplace transform which is for analyze for which is for the continuous time signals and systems.

So, this is to represent and analyze discrete this is to represent and analyze discrete time signals and systems. Now for a given signal x n, this is your given signal the z transform x of z is defined as x of z equals summation n equals minus infinity to infinity x n z raise to the power of minus n.

This is the definition of the z transform.

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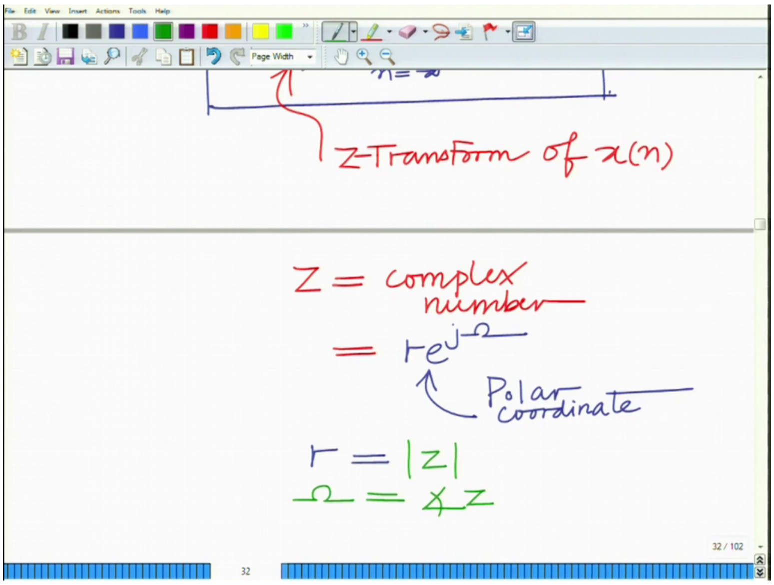

So, this is the definition of the z transform, of a x of n, where z is a complex number, I can be represented as in terms of it is polar coordinates that is magnitude and phase as r e to the power of g omega. This is the polar coordinate representation; where r equals the magnitude, correct?

The magnitude of z and omega equals the phase or the angle of the complex numbers. All right. So, this is the z transform and will be used to represent and analyze discrete time signals and systems, the z transform of the signal x of n is given by summation n equals minus infinity to infinity x n z raise to the power of minus n where z is a complex number, all right.

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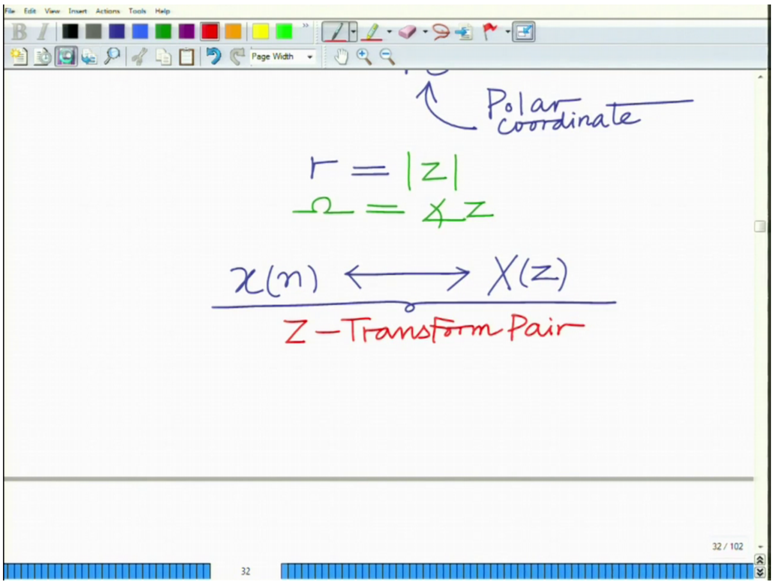

And then this is a and then this is a also presented as follows that is x of n and x of z follow signal z transform pair. So, this form a z transform pair.

So, these form a these form a z transform pair. Now similar to the laplace transform one can also define a region of convergence for the z transform region of convergence is basically range of values for which the z transform.

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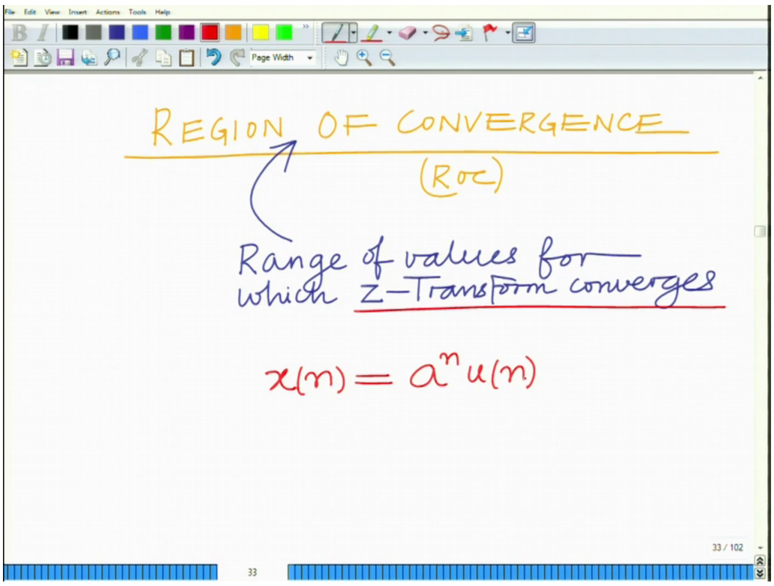

Convergence the region of convergence or the ROC is simply range of a values for which range of values for which the z transform converges. For instance, consider x of n equals a raise to the power of u n where u n is the discreet unit step function; that is, u n equals 1 if n is greater than equal to 0 and equals 0 otherwise. So, there considering signal x n which is a raise to the power of n u n.

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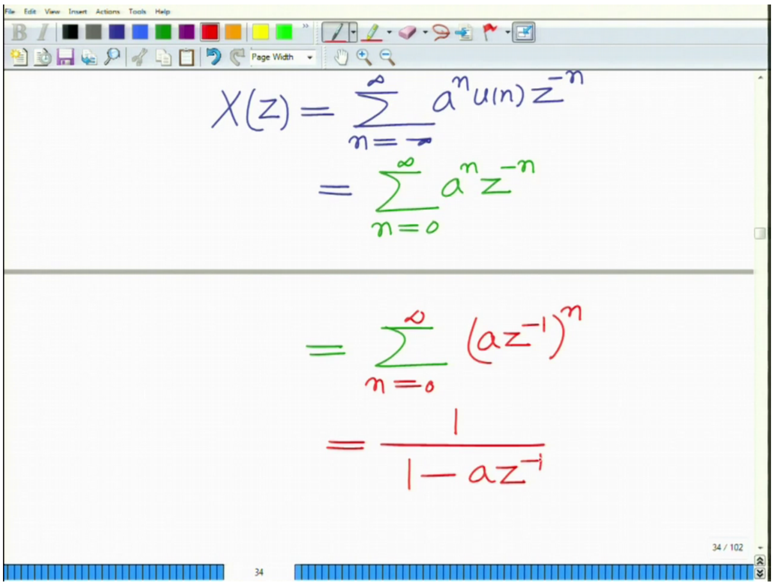

Now the z transform of this signal x of z equals summation m equals minus infinity to infinity a raise to n u n z raise to minus n which is u n is non 0 only for n greater than equal to 0 and this is therefore, equals n equals 0 to infinity a raise to n z raise to the power of minus n; which is summation n equals 0 to infinity a z inverse raise to the power of minus n which is basically 1 over 1 minus a z inverse.

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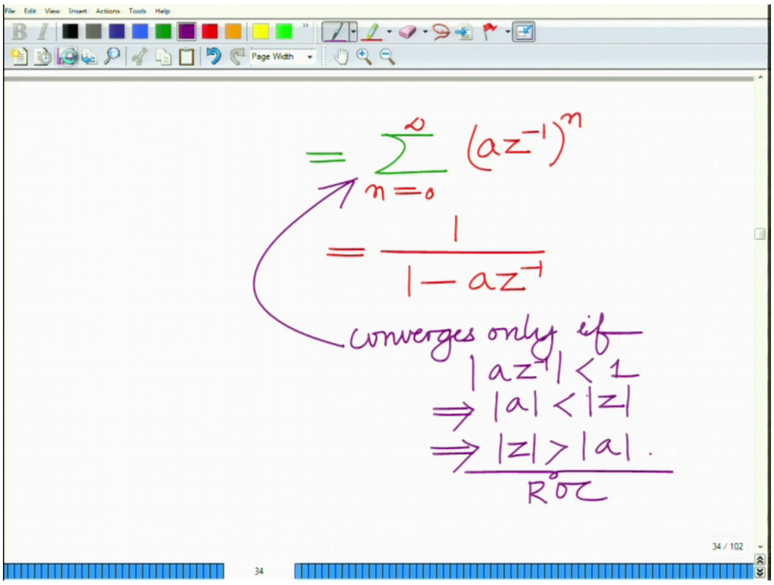

But this converges only if magnitude this some converges infinite some converges only if magnitude is inverse less than 1 which implies that your magnitude a must be less than magnitude z. Or if magnitude a must be greater magnitude z must be greater than magnitude a. And this is basically or ROC, this is the region of convergence for this z transform. So, a raise to power n u n the signal has z transformed 1 over 1 minus a z inverse; however, the z transform convergence only if magnitude of z is greater than the magnitude of a. Let us consider a to be a real quantity less than 1, consider 0 less than a less than 1, then we can plot the roc.

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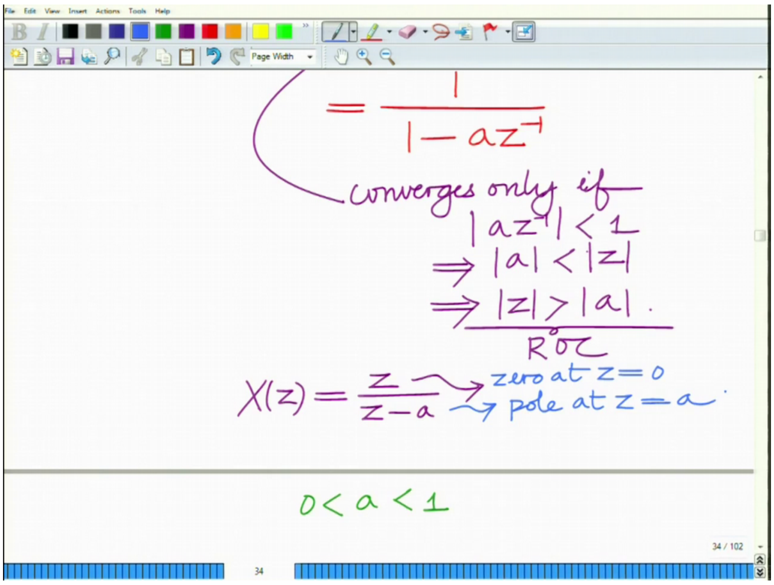

So now, if you look at this z over z minus a look at this, x of z equals z over z minus a this has a 0 that is root of the numerator that z equal to 0, and this has a pole at that is root of the numerator pole at z equal to a.

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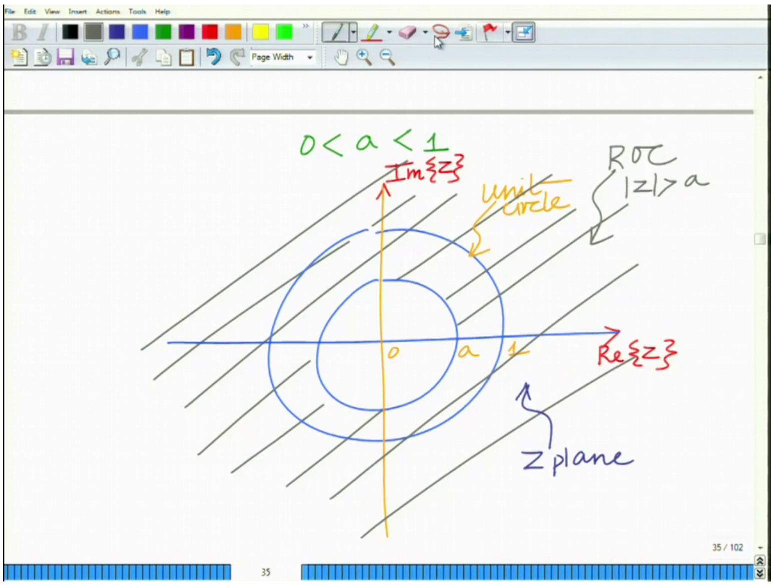

And I can plot this in the z plane I can plot this we are assuming that magnitude of a so, let us this is a, this is the unit circle; that is, with magnet represents quantities of magnitude of 1 this is 0 and the region of convergence is so, this is the z plane think of this as the real part of z, this is the imaginary part of z imaginary part of z and the region of convergence is all values of z such that a magnitude of z is greater than a which means basically covers all these entire circle excluding. So, this is basically our so, this is basically your ROC. That is magnitude z grater than a. And this is termed as your z plane.

On the x axis, we have the real part of z, and in the y axis we have the imaginary part of z. And we are plotting all values of z where the ROC where the z transform converges, this is known as the region of convergence; and that basically includes all values of z such that magnitude of z all values of z such that the magnitude of z is greater than a; that is, basically all values of z in the z plane which outside this circle, correct? The constant modulus circle which represents all quantities which are magnitude of z equal to a. So, that is the region of converges. Consider now as slightly different signal.

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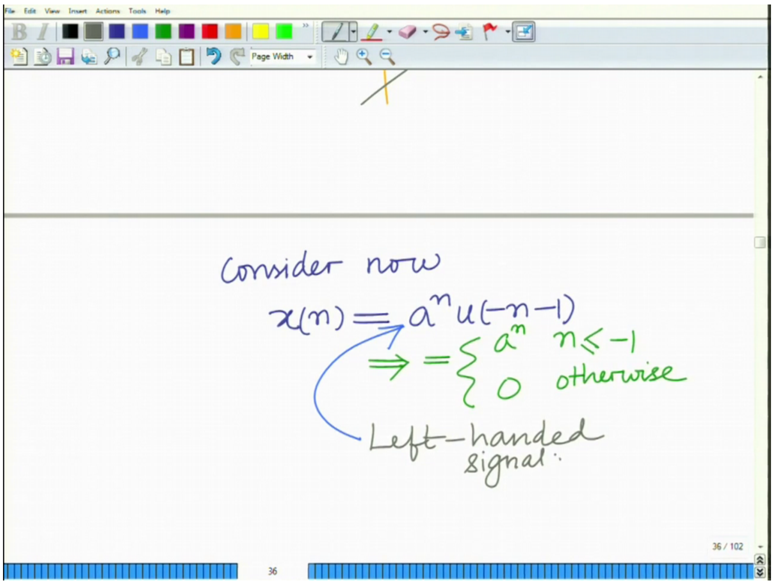

This is what is; this is a n u minus n minus 1; that is, basically implies equal to a n for n plus 1 greater than equal to 0, which means n less than or equal to minus 1 equals 0 otherwise. So, this is what is known as a left-handed signal, remember similar to your continuous time, because it is equal to 0 for t greater that is n greater than n greater than or equal to 0. So, this is basically your left-handed signal. This is a left-handed signal v.

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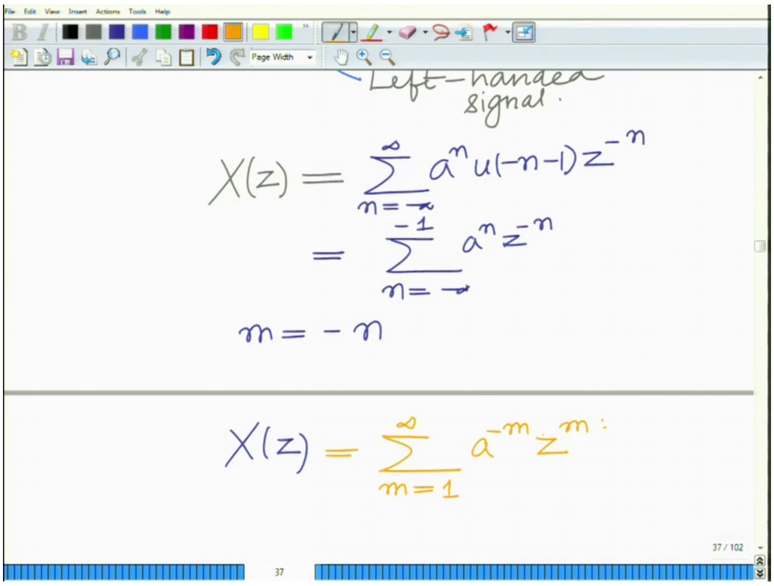

Now f you look at the z transform of this x of z is summation n equal to minus infinity to infinity u n minus u minus n minus 1 0 z raise to minus n. Now this is non-0 only for n less than or equal to minus 1. So, this is n equals minus infinity to minus 1 a raise to n z raise to the power of minus n now set m equals minus n, change the index then the z transform x of z becomes summation, n goes from minus infinity to minus 1 m goes from one to infinity n equals minus m equals minus n. So, this will be a raise to minus m z raise to m which is basically summation m equals minus infinity to infinity a inverse z raised to the power of m.

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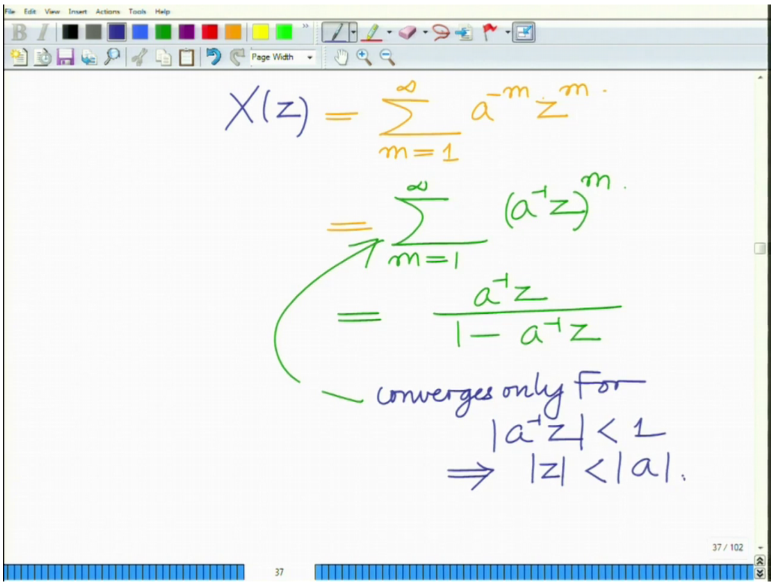

Which is basically a inverse z divided by 1 minus a inverse z, but this converges only if a inverse z this quantity is strictly magnitude is less than 1; this implies magnitude of z less than magnitude of a. So, the x of z, and which is equal to and now you can simplify this you can see that this is clearly equal to; let me just consider a slight modification let us consider this signal minus a raise to minus n u minus n. So, this will be minus I will have a minus sign everywhere.

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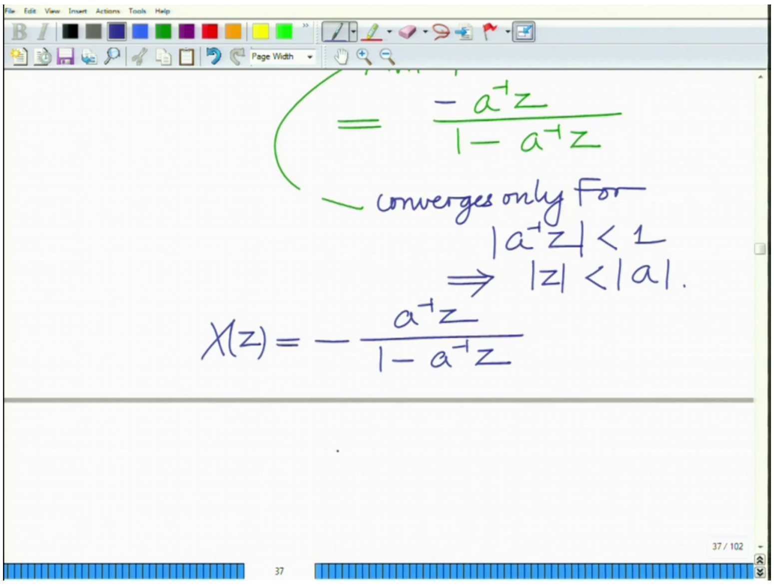

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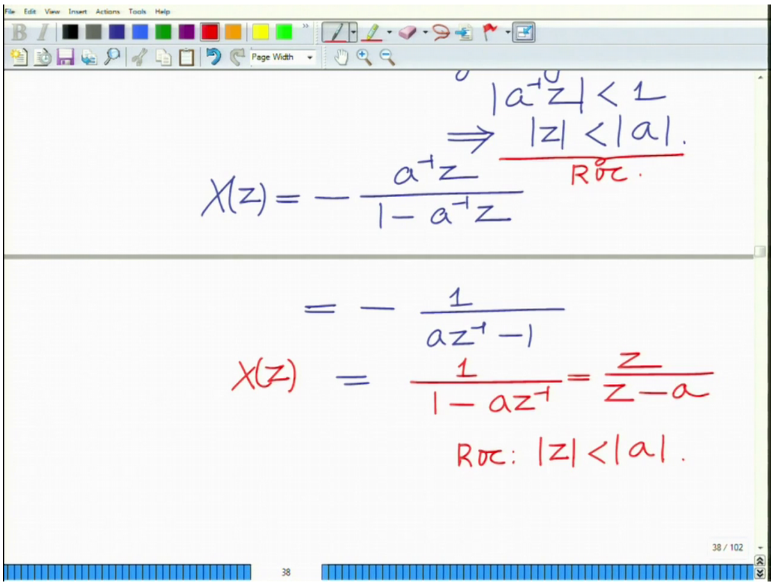

So, this will be minus and this will be again minus sign and this will be minus sign, you can see if you simplify this you will have x of z equals minus a inverse z by 1 minus a inverse z which is equal to minus 1 divided by a z inverse minus 1 which is equal to 1 by 1 minus a z inverse; which is equal to 1 by 1 minus a z inverse and you can also write this as equal to z by z minus a. And in fact, the previous 1 also you can write this as 1 over 1 minus a z inverse Multiplying numerator and denominator by z this becomes z over z minus a.

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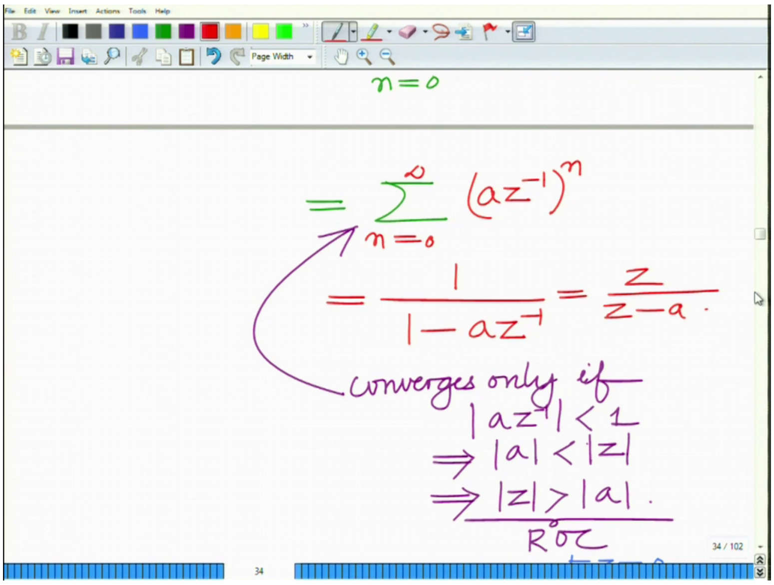

So, you can see both of them are equal to z over z minus a; however, the previous 1 has an ROC magnitude of z greater than magnitude of a; however, this has an or ROC magnitude of z less than magnitude of a. So, we have the same z transform x of z equals 1 over 1 minus a z inverse, but with different ROC s that is magnitude z less than magnitude of a for this right handed signal.

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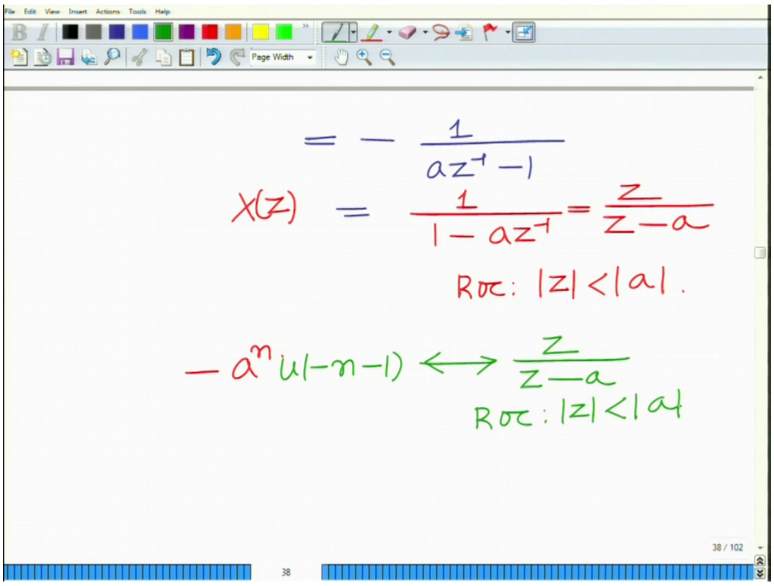

So, for this signal x of n equals, let me just write this again x of n equals minus a raise to the power of n minus u of minus n minus 1 also has the z transform z over minus a, but the ROC is magnitude z less than magnitude a. And therefore, the ROC of this left-handed signal is magnitude z is less than magnitude a and therefore, both the signals have the same z transform; that is, a raise to n u n and minus a raise to n u of minus n minus 1. Have the same z transform z over z minus a, but the ROC Is the regions of convergence are completely difference. So, therefore, 2 different signals can have the same see transform, but different regions of convergence. So, to specify the signal accurately, correct? I have to specify the signal uniquely along with the z transform the ROC also has to be specified, otherwise the z transform is incomplete. So, the ROC of this signal looks like again let us consider 0 less than a less than 1 that is the circle of amplitude a lies inside the unit circle.

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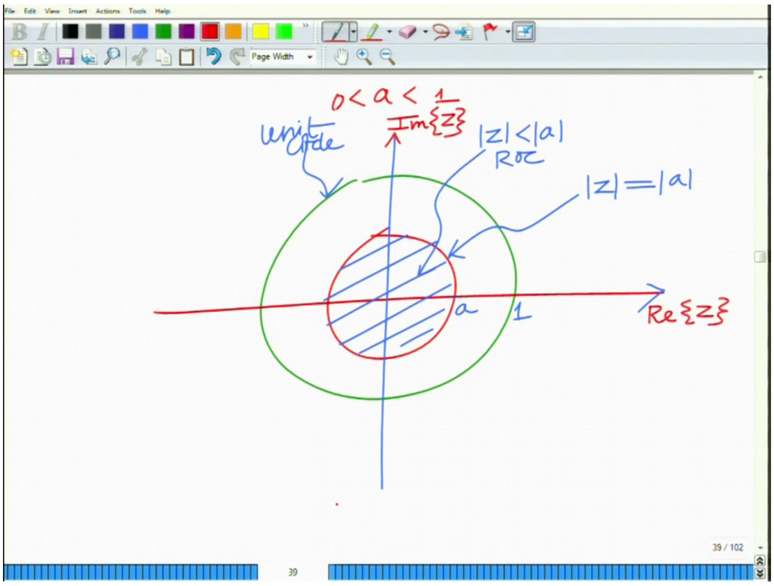

So, this and the region of convergence is this the region laying inside the interior of the circle magnitude of z equal to so, this circle is magnitude of z equal to magnitude of n this is the interior of the circle, this is the unit circle. So, this is also termed as the unit circle and this is of course, the z plane. This is the real part of z, and this is the imaginary part of z. Real part of z and the imaginary part of z. There for what you are what you can see is therefore, what you can see is x n equals a n u n has the z transform z over z minus a, and x n equals minus a n that is the left-handed signal u minus n minus 1 also has the same z transform z over z minus a.

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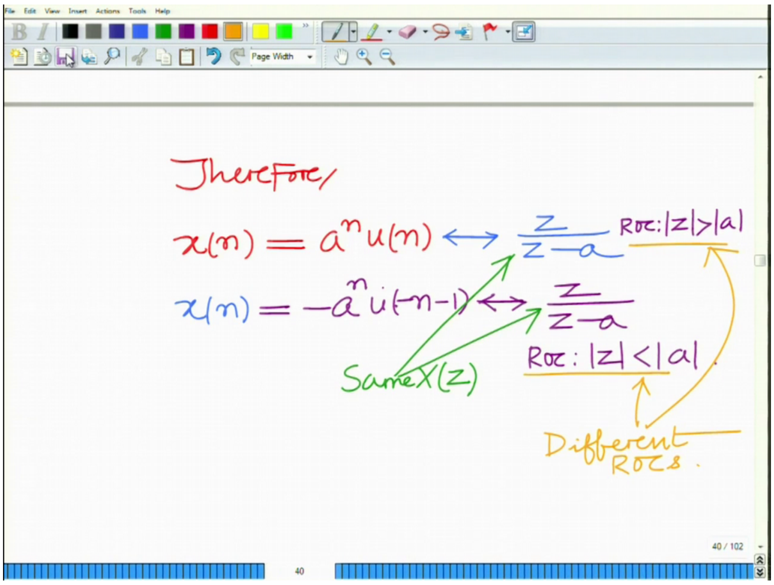

But the ROC of this ROC of the first one is magnitude of z greater than magnitude of a ROC of the second one is magnitude of z, less than the magnitude of a. So, same z transform different signals have same z the same expression for the z transform, but different ROC s, different regions of convergence implies that to uniquely characterize the fully characterized, uniquely or fully specify the signal implies to uniquely specify the signal one has to also specify the region of convergence. Otherwise the z transform is in complete.

Similar to the laplace transform, all right.

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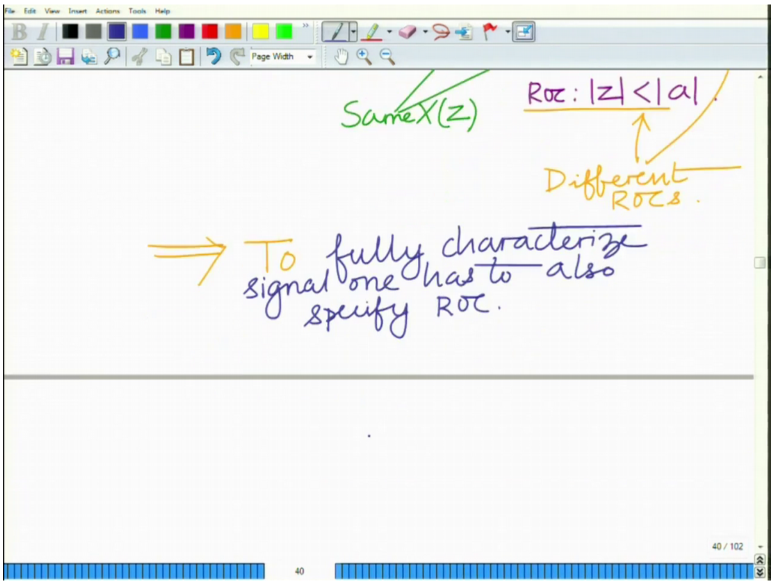

So, fully characterized the signal one has to also specify the ROC. All right, let us look at the z transforms of some common sequences.

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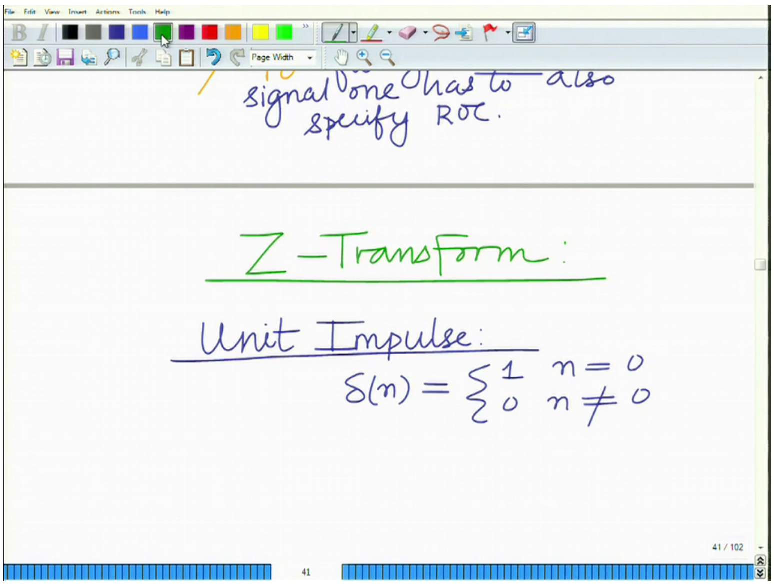

Z transform, let us start with the most common signal or the most fundamental signal which is the unit impulse also the discrete impulse is you are looking at the conical delta or discrete impulse which is equal to 1 if n equals 0 n is 0 if n is not equal to 0, then x of z equals summation n equals minus infinity to infinity delta n z raise to minus n equal z raise to power minus, because this is 0.

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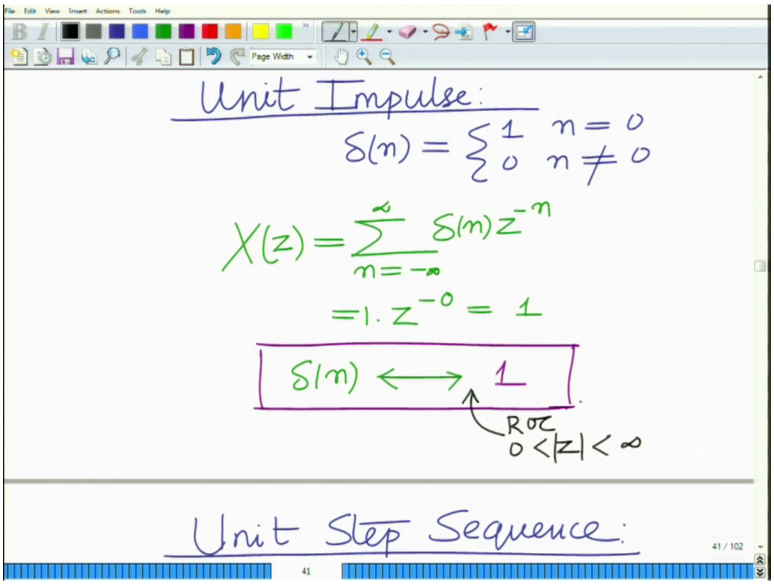

This is non-0 only when n is equal to 0 at n equal to 0 this is 1. So, this is 1 times z raise to minus 0 which is equal to 1. So, delta n the z transform is basically 1. This is unity. Z transform of delta one is basically unity. Let us quickly look at another common signal.

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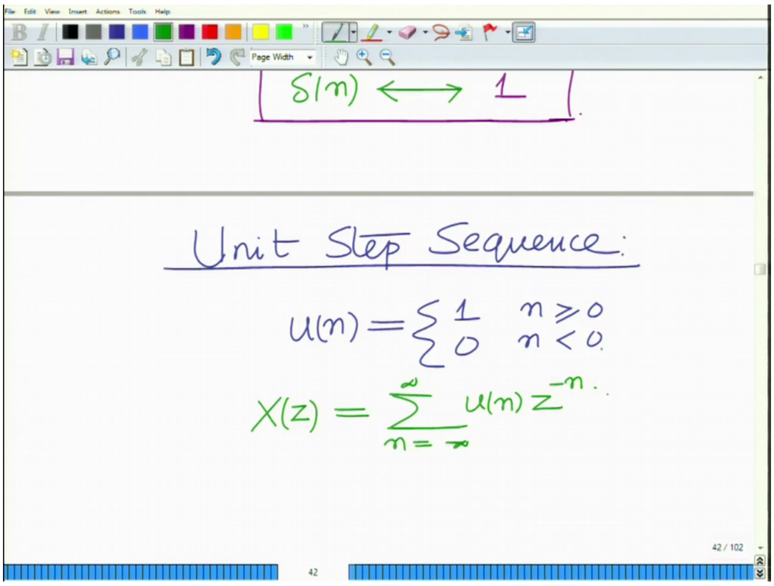

That is the unit step sequence discreet time the unit step sequence u n equals 1 for n greater than or equal to 0, 0 for n less than or equal to n less than 0. So, x of z equals summation n equals minus infinity to infinity u n z raise to the power of minus n which is equal to this is the u signal unit step is non-0 only for n greater than equal to 0, and for n greater than equal to 0, this is one this is summation n equals 0 to infinity z raise to minus n which is 1 over 1 minus z inverse.

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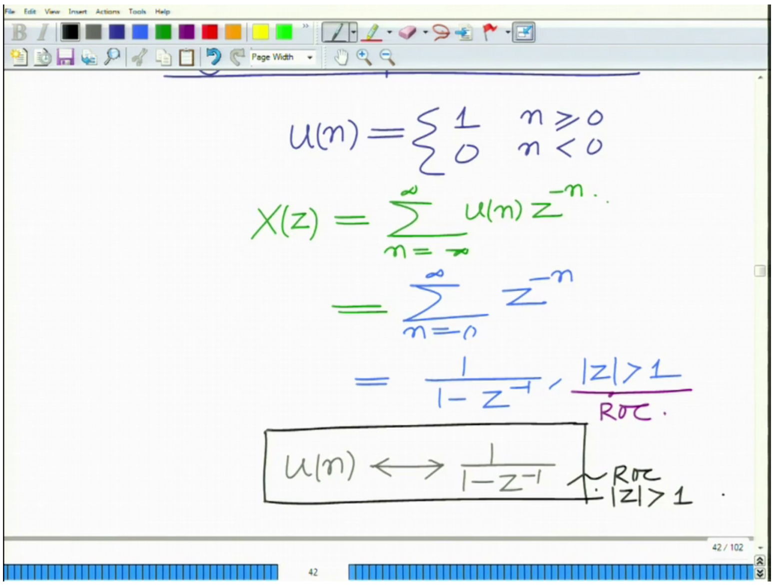

Which you could always obtain by setting a equal to 1 in the earlier expression a raised to n u n, and the ROC is magnitude of z greater than is magnitude of a which is equal to 1. So, magnitude of z has to be strictly greater than 1. This is the ROC of the region of convergence. So, the u unit step signal u n unit step signal u n has the z transform u n is 1 over 1 minus z inverse, all right. So, this is z root and remember the ROC one has to also specify the ROC, ROC is magnitude of z greater than 1. And here the roc in fact, if you look at the unit impulse the ROC is in fact, all values of the 0 probably with the exception of z equal to 0 and z equals to infinity. In fact, the ROC is 0 less than any value of z.

So, this is the ro this is ROC, that is any value of z converges for any value of z; except x equal to 0 or z equal to z equal to infinity. So, that was equal to infinity, this converges all values of z all right. So, with this we will stop this module and continue our discussion on the z transform further in the subsequent models.

Thank you very much.