Principles of Signals and Systems
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Lecture – 22
Laplace Transform Example Problems – Inverse LT through Partial Fraction for Poles with Multiplicity Greater than Unity

Hello, welcome to another module in this massive open online course. So, we are looking at example problems to understand the properties and applications of the Laplace transform, let us continue our discussion ok.

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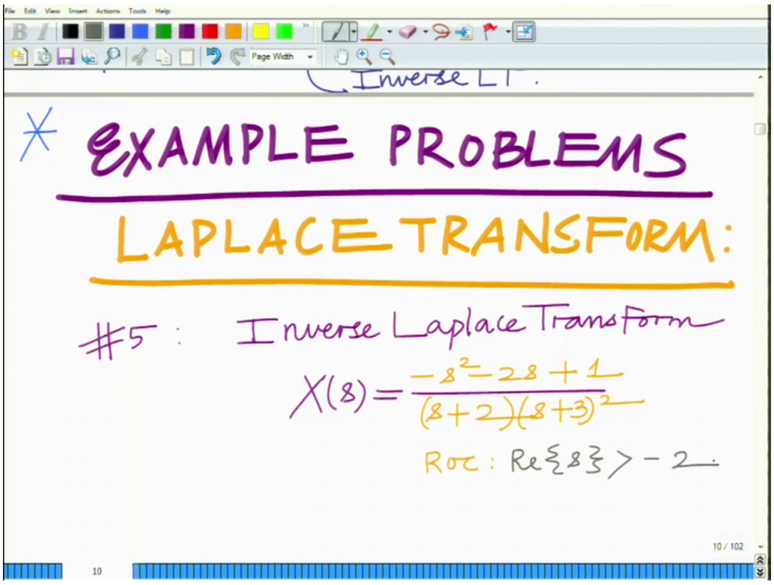

So, we want to keep want to look at some other example problems for the Laplace transform.

And what we have is basically we want to find let us look at problem number 5, we want to find the inverse correct Laplace transform of X s equals Laplace transform given by minus s square minus 2 s plus 1 divided by s plus 2 times s plus 3 whole square. And the ROC is basically real part of s greater than minus 2 that is ROC ok.

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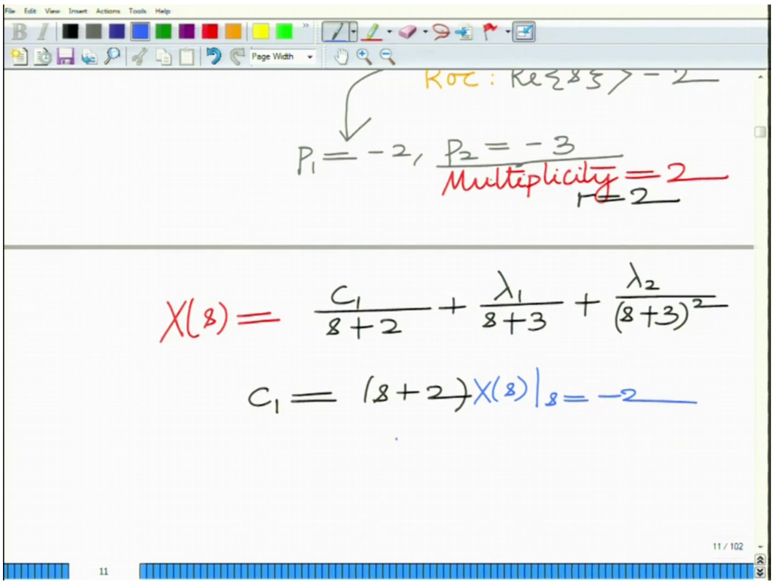

And now remember we can use this so, this is now if you look at this here s plus 3 square focus on this this is a multiple pole. So, there is the poles are basically p 1 equals minus 2 p 2 equals minus 3, but this p 2 equals minus 3 this is a multiple pole of order 2, this is a pole that is this is a repeated pole correct of order 2.

So, pole has multiplicity of order 2, and because you have the term of the form s plus s plus 3 whole square your multiplicity equal to 2 therefore, the partial fraction expansion will be of the form you will have X s is equal to C 1 by s plus 2 plus corresponding to s plus 3, you will have 2 terms 1 is lambda 1 by s plus 3 plus lambda 2 divided by s plus 3 whole square.

Now, the term so, multiplicity equals 2 which means the factor r is equal to 2. Now C 1 the constant C 1 corresponding to s plus 2 in the partial fraction expansion, that can be found as seen before in the usual a that is s plus 2 times s of s evaluated at the pole that is s equal to minus 2.

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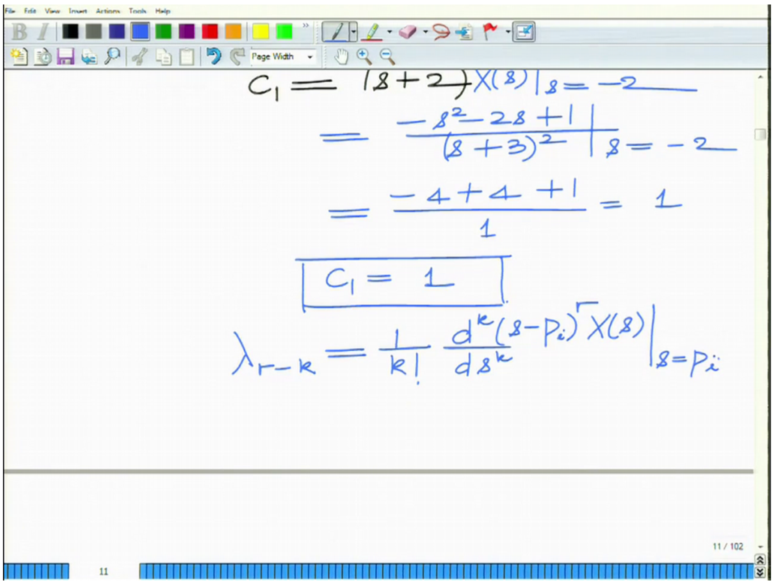

Which is basically you are turns out to be minus s square minus 2 s plus 1 divided by s plus 3 whole square evaluated at s equal to minus 2, which is equal to minus 4 plus 4 plus 1 divided by 1 which is equal to 1 so, C 1 is equal to 1 the coefficient C 1 is equal to 1.

Now, let us look at for lambdas now to evaluate lambdas we know we have the property that lambda r minus k equals 1 over k factorial that is the coefficients corresponding to the repeated pole that is s plus 3 whole square whole minus 3 is just multiplicity 2. That can be evaluated lambda r minus k has is 1 over k factorial d k s minus p i that is a pole i-th pole which has multiplicity r s minus p i raise to the power of r X s divided over d s k that is a k-th derivative evaluated at s equals p i ok.

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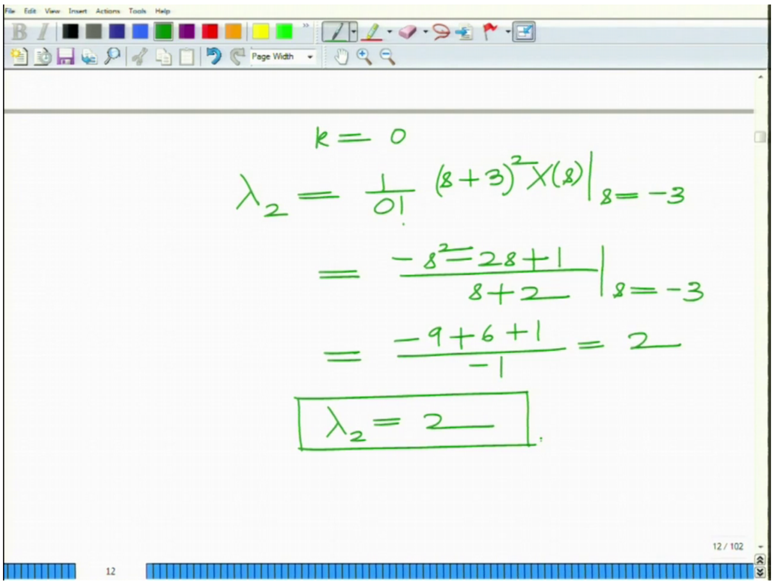

Now, first let us set k equal to 0, now if k equal to 0 we have lambda r and remember we have r equals 2. So, lambda 2 can be evaluated as 1 over 0 factorial times the k-th derivative k equals 0. So, there is no derivative s minus pi is minus 3 so, s plus 3 whole square into X s evaluated at s equal to minus 3 0 factorial is 1. So, this is minus s square minus 2 s plus 1 divided by s plus 2 evaluated at s equals minus 3, and that gives us basically that gives us your minus 9 plus 6 plus 1 divided by 2 minus 3 that is minus 1, which is minus 7 minus 3 plus 2. So, that is minus 2 divided by minus 1 that is 2. So, lambda 2 equals so, we have the coefficient lambda 2 equals 2. So, this is the coefficient lambda 2.

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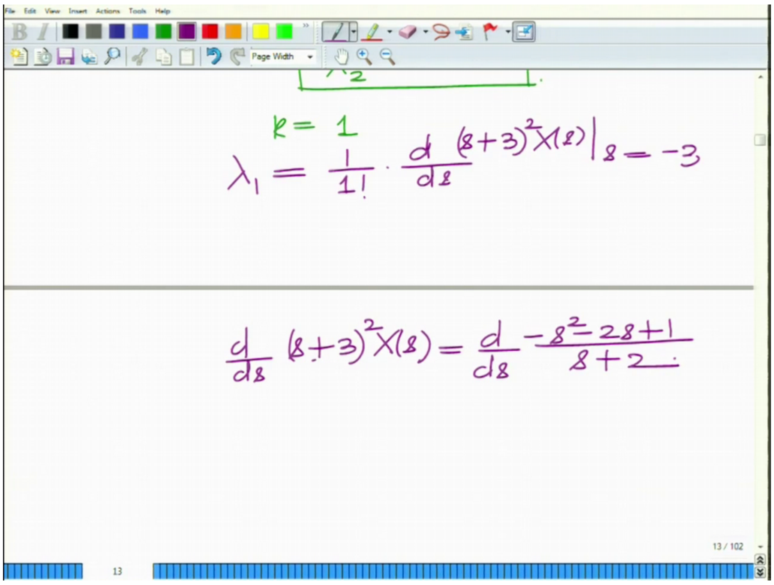

Now, if we set k equals 1, we have lambda 1 lambda r minus k 2 minus 1 is 1 over k factorial that is 1 over 1 factorial times d the derivative first order derivative of X s or I am sorry not X s that is s plus 3 whole square times X s evaluated at s equal to minus 3.

Now, let us look at d by d s of s plus 3 whole square times X s, now this quantity is basically the derivative of minus s square minus 2 s plus 1 divided by s plus 2.

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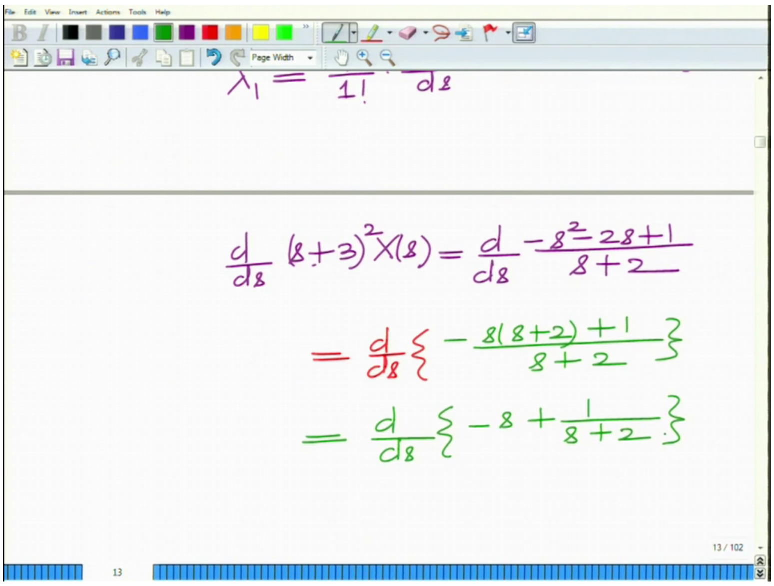

Which is basically equal to the derivative of you can simplify this as minus s into s plus 2 plus 1 over s plus 2, which is the derivative of minus s plus 1 over s plus 2 which is equal to basically.

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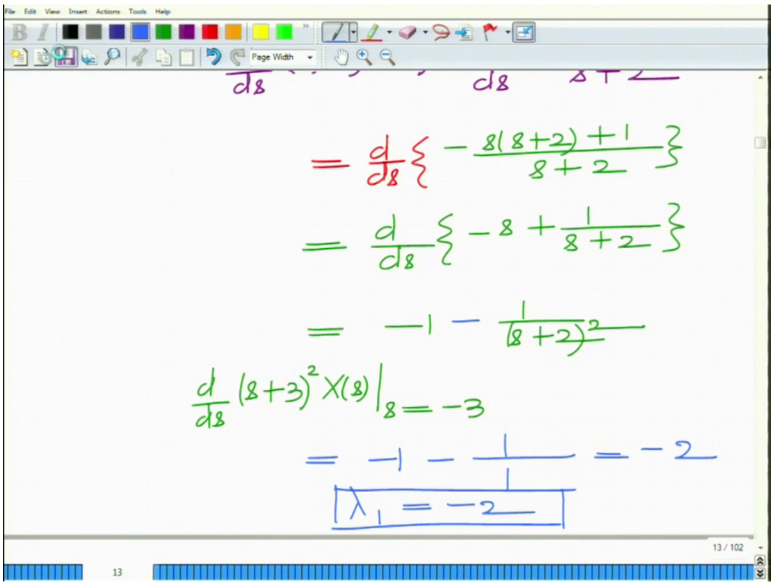

Now, the derivative of minus s plus 1 over s plus 2 this is equal to minus 1 plus 1 over s plus 2 whole square. Now, therefore, we will have the derivative of s plus 3 whole square times X s evaluated at the pole s equal to minus 3, this will be minus 1 plus 1 over 2 minus 3, 2 minus 3 is minus 1 whole square which is 1 I am sorry. This will be minus 1 minus 1 over s plus 2 so, the minus 1 which is equal to minus 2.

So, basically what we have is lambda 2 equal so, I am sorry lambda 1 equals minus 2. So, we have shown basically that we have evaluated the coefficient lambda 2 as 2 lambda 2 the coefficient corresponding to s plus 3 square is 2 and lambda 1 the coefficient corresponding to s plus 3 is minus 2. And therefore, the partial fraction expansion of X s can now be written as X s .

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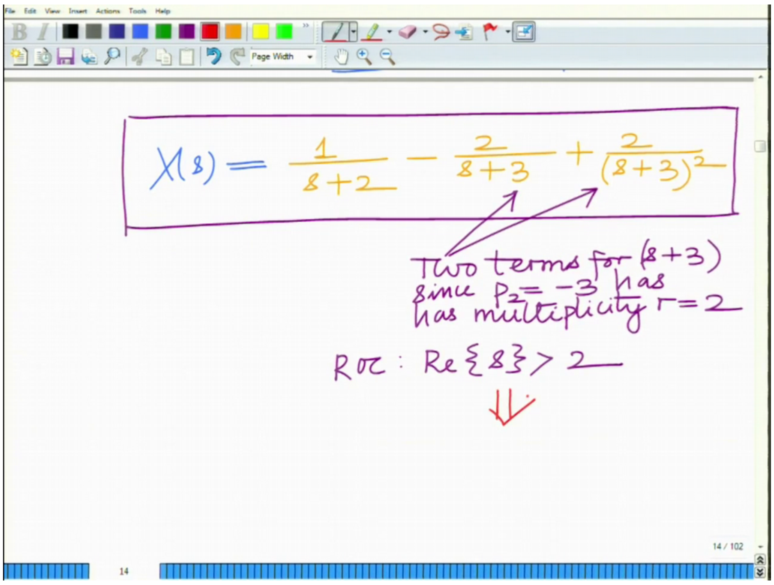

I hope you can see that X s can now be written as C 1 over s plus 2, but C 1 has been shown to be 1. So, 1 over s plus 2 minus 2 over s plus 3 plus 2 over s plus 3 whole square. So, this is a partial fraction expansion of X s considering the pole minus 3 which has a multiplicity of 2. So, therefore, we have 2 terms correct pole multiplicity of 2 we have 2 terms in the partial fraction expansion. Since the pole p 2 equals minus 3 as we have already seen p 2 equals minus 3 has multiplicity r equal to 2 as the multiplicity r is equal to 2.

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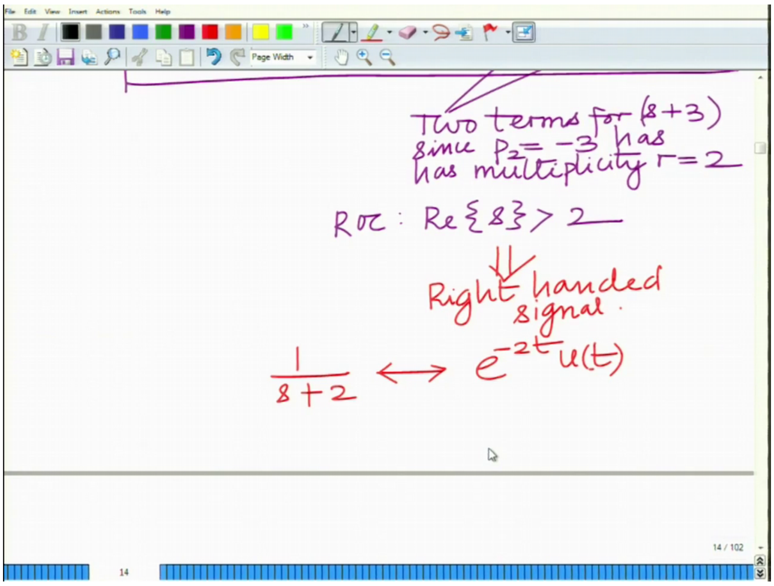

Now, the ROC we have seen is real part of s greater than 2 which implies this is basically a right handed signal. So, implies x t x t is basically a right handed signal. Now 1 over s plus 2 the inverse Laplace transform for right handed signal is e raise to minus 2 t u t 1 over s plus 2. So, we have the real part of s greater than minus 2 therefore, x of t will be a right handed signal all right. So, 1 over s plus 2 that has the inverse Laplace transform e raise to minus 2 t u t.

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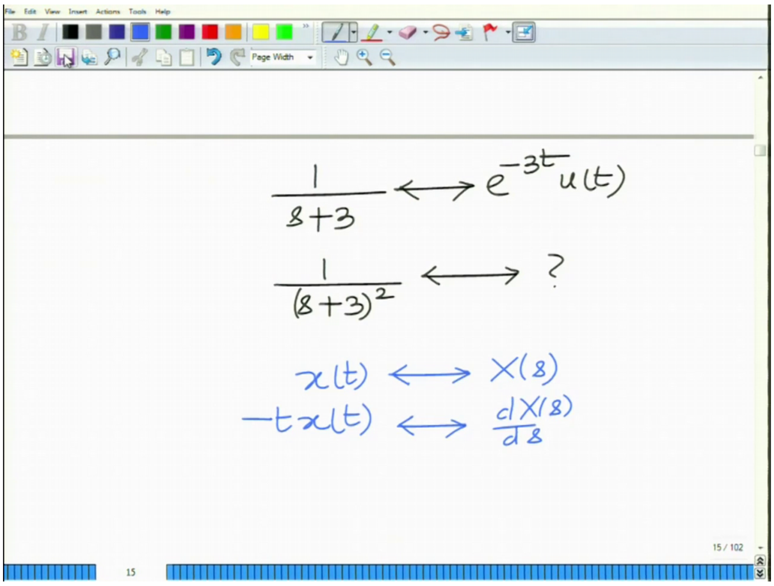

Similarly, 1 over s plus 3 will have the inverse Laplace transform e raise to minus 3 t u t. So, 1 over s plus 3 will have the inverse Laplace transform e raise to the power minus 3 t u t, now what about 1 over s plus 3 whole square, what is the inverse Laplace transform of this.

Now, for the inverse Laplace transform of this we will use the property, that d x s over d s dx s d that is derivative of x s that is let us write the complete property of x t has Laplace transform x s then dx s d s has the Laplace transform minus t x t this is the property that we are going to use.

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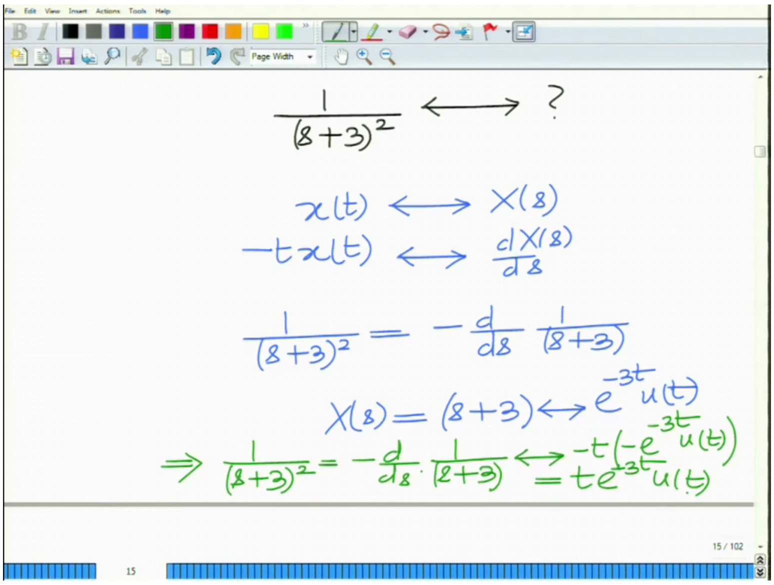

And therefore now, you can see that 1 over s plus 3 whole square is nothing but, minus the derivative of minus d over d s of 1 over s plus 3 and therefore,, but we know what is the signal corresponding to X s equals s plus 3 basically has inverse Laplace transform we have already seen that e raise to minus t 3 t u t which means 1 over s plus 3 whole square, which means basically this implies 1 over s plus 3 whole square. Which is minus d over d s of 1 over s plus 3 will have inverse Laplace transform minus t times minus t correct, 1 over s plus 3 whole square has inverse Laplace transform this will have inverse Laplace transform.

So, minus so, this will have inverse Laplace transform minus of minus t. So, minus this thing which is t times e raise to minus 3 t p raise to minus so, this will be let me write it minus of t times minus e raise to minus 3 t u t which will be t e raise to the power minus 3 t u t. And therefore, finally, what will a so, 1 over s plus t whole square has the inverse Laplace transform t e raise to minus 3 t u t and therefore now, we will put these 3 things together to find the inverse Laplace transform of the original signal remember.

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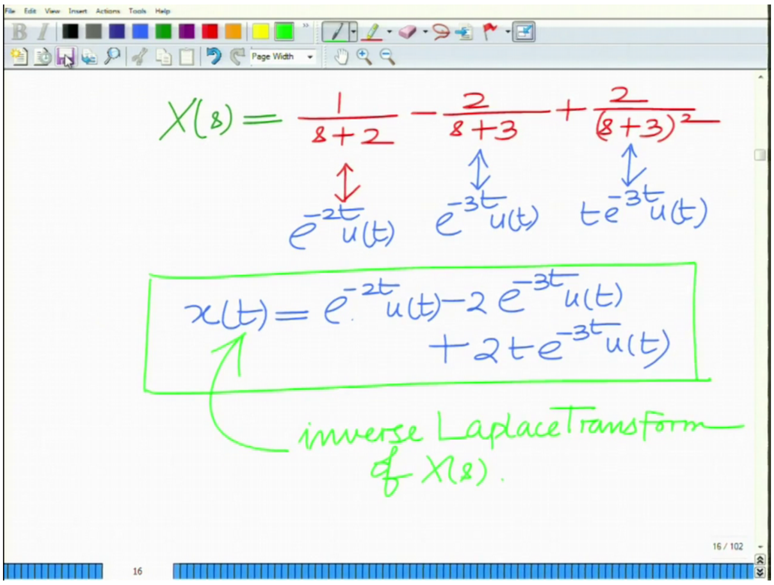

The original signal the original Laplace transform the partial fraction expansion is 1 over s plus 2 minus 2 over s plus 3 plus 2 over s plus 3 whole square 1 over s plus 2 has the inverse Laplace transform e raise to minus 2 t u t this has the inverse Laplace transform e raise to minus 3 t u t, and this has the inverse Laplace transform well this has the inverse Laplace transform t e raise to minus 3 t u t.

So, therefore, the inverse Laplace transform of X s will be e raise to minus 2 t u t plus e raise to minus or minus 2 times e raised to minus 3 t u t plus 2 e raise to plus 2 t e raise to minus 3 t u t. So, that is basically the final answer that we so, this is x t which is the inverse Laplace transform of this is x t which is an inverse Laplace transform of X s. So, this is the x this is the final solution which is the inverse Laplace transform of the original there is a Laplace transform that is given which is X s which is a rational function rational function with the pole at s equal to minus 3 of multiplicity 2 all right.

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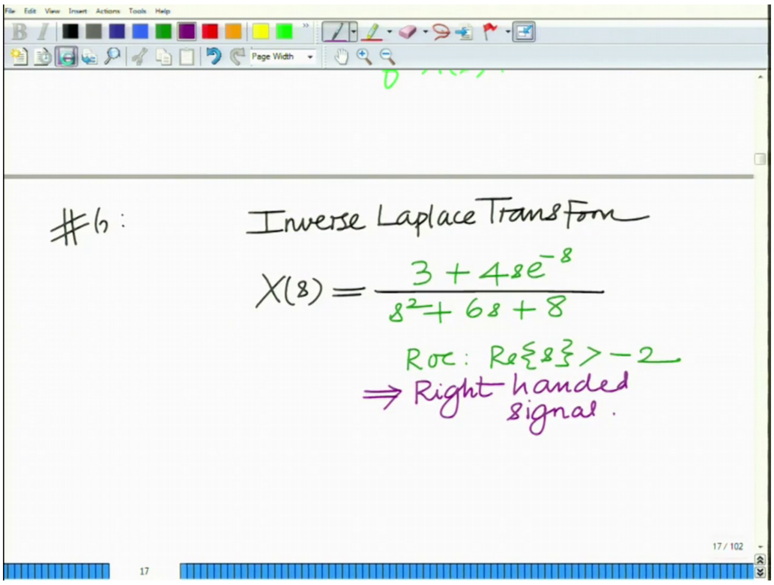

Let us look at another example which is inverse Laplace transform of the signal X s equals 3 plus 4 s e raise to minus s divided by s square plus 6 s plus 8 the ROC is real part of s greater than minus 2. So, this implies there is a right handed signal again this implies once again this is a right handed signal ok.

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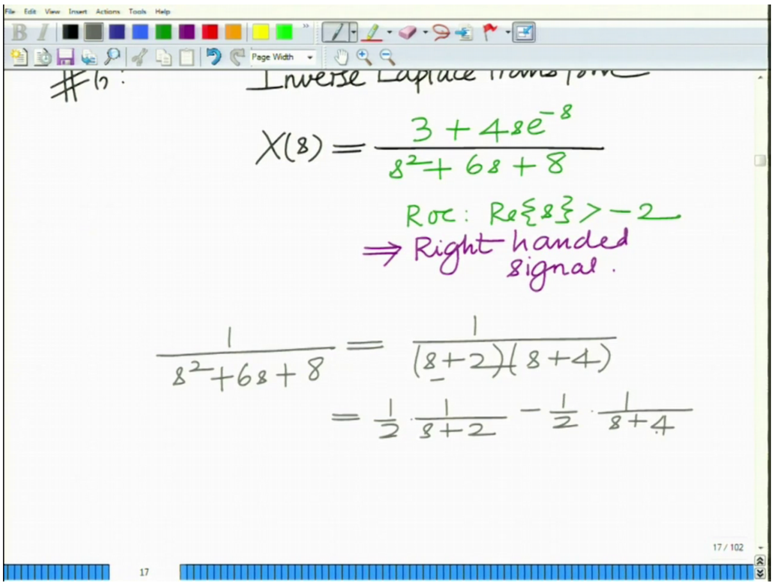

Now, we start by considering 1 over a square plus 6 s plus 8 this is equal to 1 over s plus 2 times s plus 4, which is equal to half 1 over s plus 2 minus half 1 over s plus 4.

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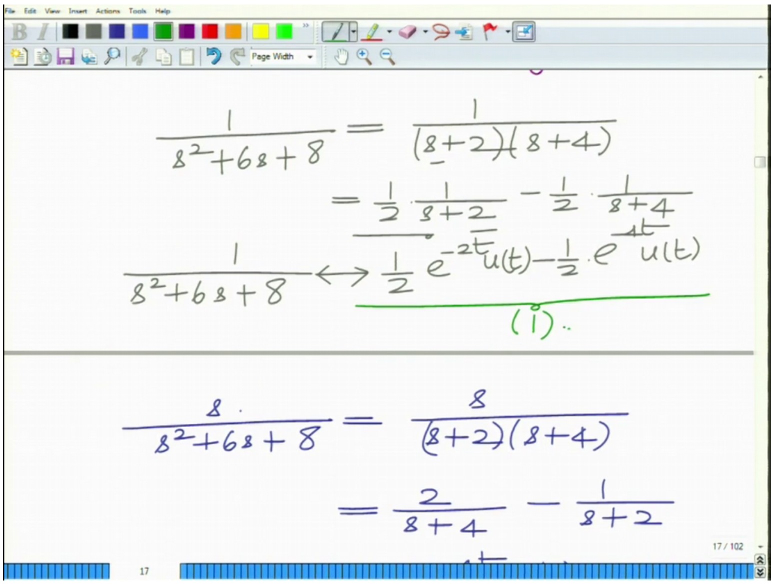

Now, remember for the right handed signal half 1 over s plus 2 is half, well e raise to minus 2 t u t minus half 1 over s plus 4 corresponds to has the inverse Laplace transform e raise to minus 4 t u t. So, this is the inverse Laplace transforms correct this is the inverse Laplace transforms of 1 over s square plus 6 s plus 8 ok.

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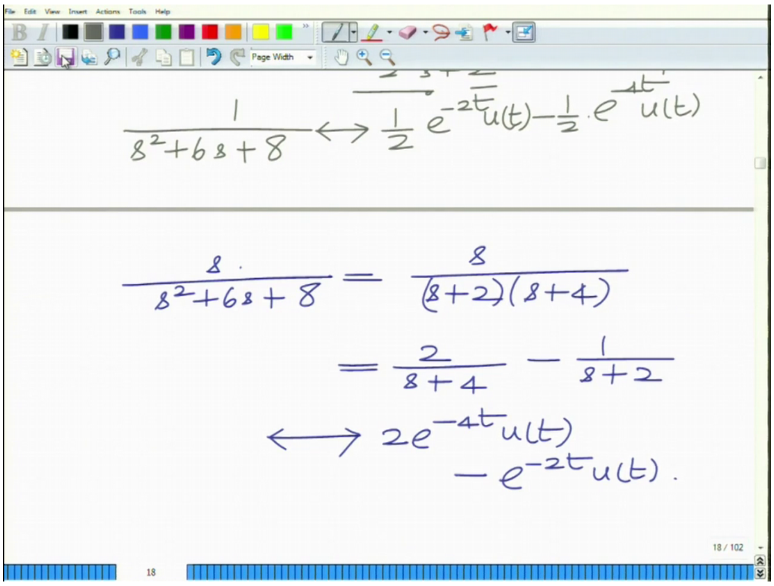

Now let us consider another term, let us consider s divided by s square plus 6 s plus 8 that is your s divided by s plus 2 times s plus 4 which is 2 divided by s plus 4 minus 1 divided by s plus 2, which has the inverse Laplace transform 2 e raise to minus 4 t u t minus s plus 2 will be e raise to minus 2 t u t ok.

But now we remember we have this factor e raise to minus 2 s, and to take care of that to consider that we will use the following property we have e raise to minus.

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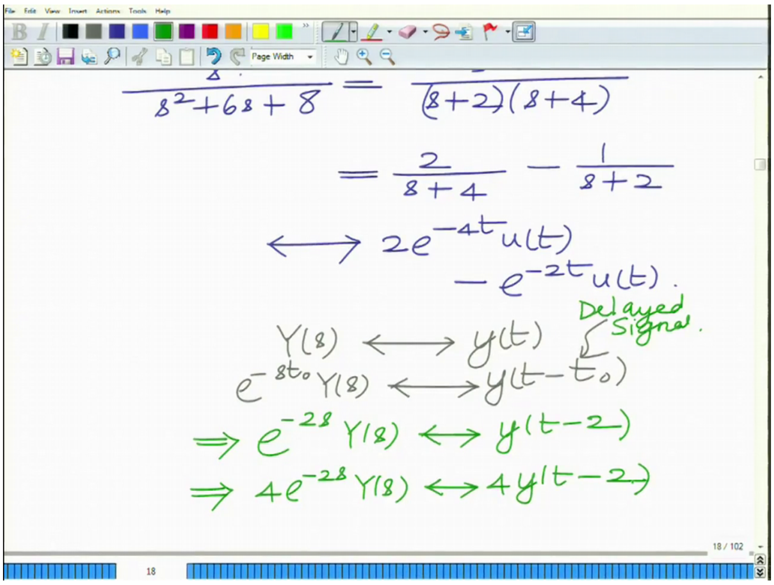

So, let us say we have a signal we consider a signal u s any signal us or let us say we consider any signal Y s which has Laplace transform inverse Laplace transform y t e raise to minus s t naught of Y s has inverse Laplace transform y t minus t naught.

Remember we know this property this is a delayed version of the signal, this is the delayed signal Y t minus t naught which implies. So, this implies that e raise to minus 2 s Y s has universe Laplace transform y t minus 2 this implies 4 e raise to minus 2 s Y s has inverse Laplace transform 4 y t minus 2.

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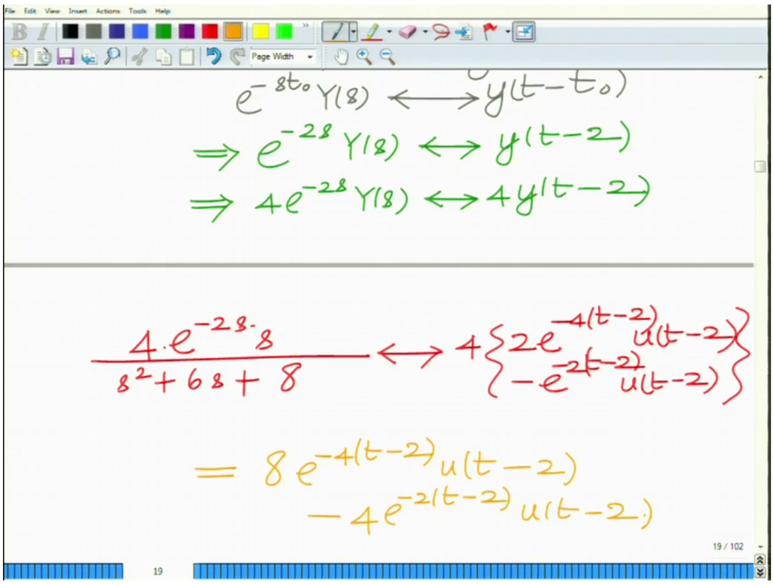

Therefore now using this property we have 4 e raise to minus 2 s s divided by s square plus 6 s plus 8 has the inverse Laplace transform 4 times, well the previously divided signal delayed by 2 that is e raise to minus 4 t minus 2 u t minus 2 minus e raise to minus 2 t minus 2 u t minus 2, which basically equals 8 e raise to minus 4 t minus 2 u t minus 2 minus 4 e raise to minus 2 t minus 2 u t minus 2. So, this corresponds to the second part.

Therefore the final inverse Laplace transform that is a given corresponding to the original signal that is you put these 2 things together remember, we have this which is let us say we call this 1 as result number 2 we have our original result number or original result 1.

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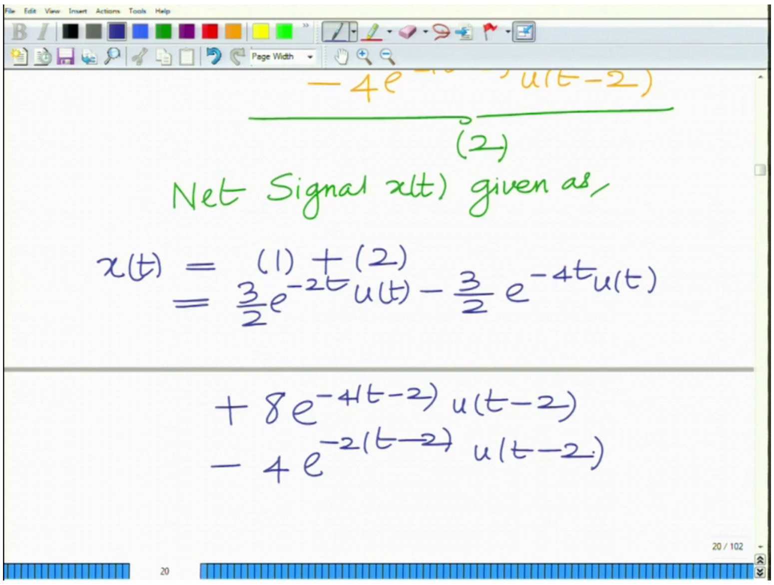

Therefore the final inverse Laplace transform Net signal x t the net signal x t is given as, well we have X s equals remember X s equals or let me just write x t we have x t equals signal 1 solution in 1 plus 2 which is equal to well, this is equal to well it is not exactly signal in 1 because the signal in 1 has to be multiplied by the factor of 3, because we have we are finding 1 over s square plus 6 s plus 8.

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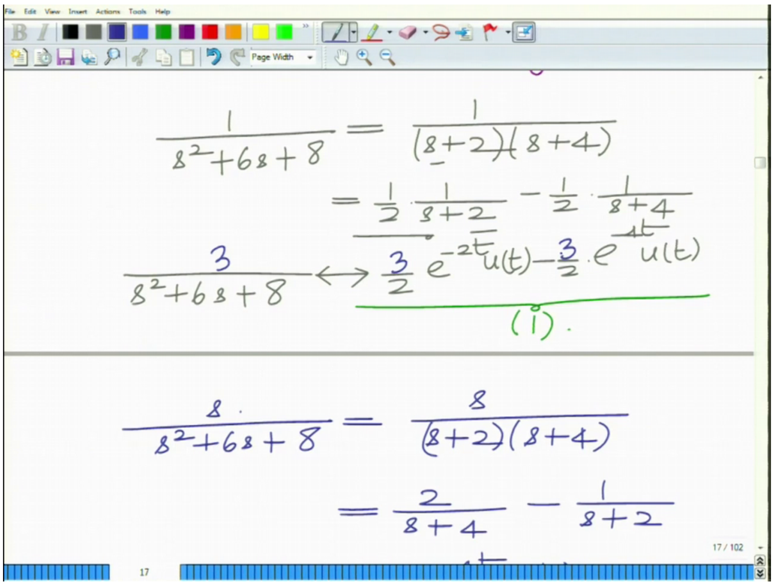

So, 3 over s square plus 6 s plus 8 will be 3 over 2 minus 3 over two. So, it will be now 1 plus 2 which will correspond to which will correspond to 3 over 2 e raise to minus 2 t u t u t minus 3 over 2 e raise to minus 4 t u t plus 8 e raise to minus 4 t minus 2 u t minus 2 minus 4 e raise to minus 2 t minus 2 u of u t minus 2 and this is your final solution ok.

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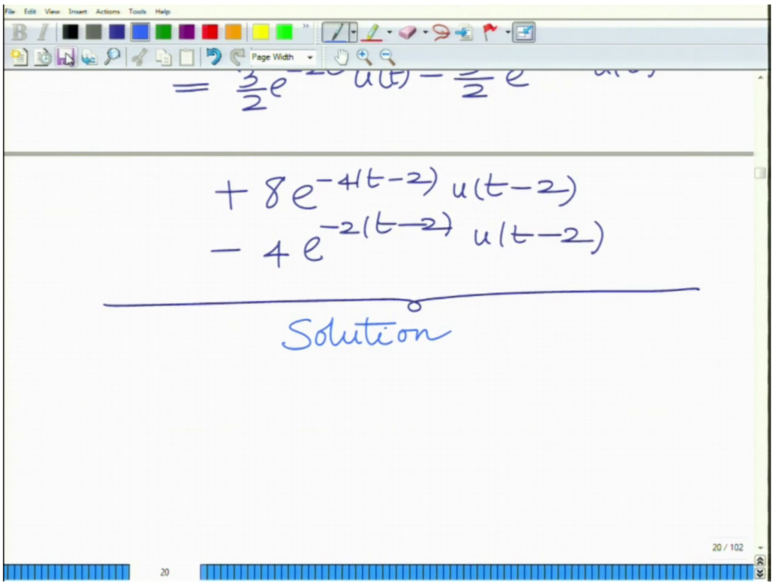

So, basically what we have seen in this module is we have seen some more slightly more advanced or challenging problems of evaluating the inverse Laplace transform all right. So, try to go through this again and understand the entire process. So, we will stop here and look at other aspects in the subsequent modules.

Thank you very much.