Principles of Signals and Systems
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Lecture – 20
Laplace Transform –Partial Fraction Expansion with simple Poles and Poles with Multiplicity Greater than Unity, Laplace Transform of LTI Systems

Hello welcome to another module in this massive open online course all right. So, we are looking at the properties of the region of convergence of the Laplace transform. We will let us just continue this discussion if xt is a 2 sided signal.

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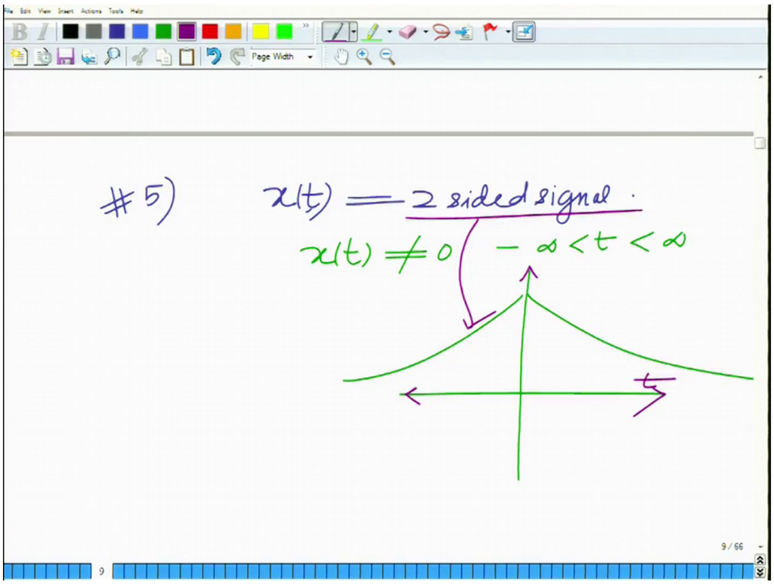

On the other hand, if xt is a 2 sided signal which implies xt is non-zero for entire range; that is minus infinity less than t less than infinity, that is it looks something like it is a 2 sided signal. This is the time axis and this is x of t is non-zero.

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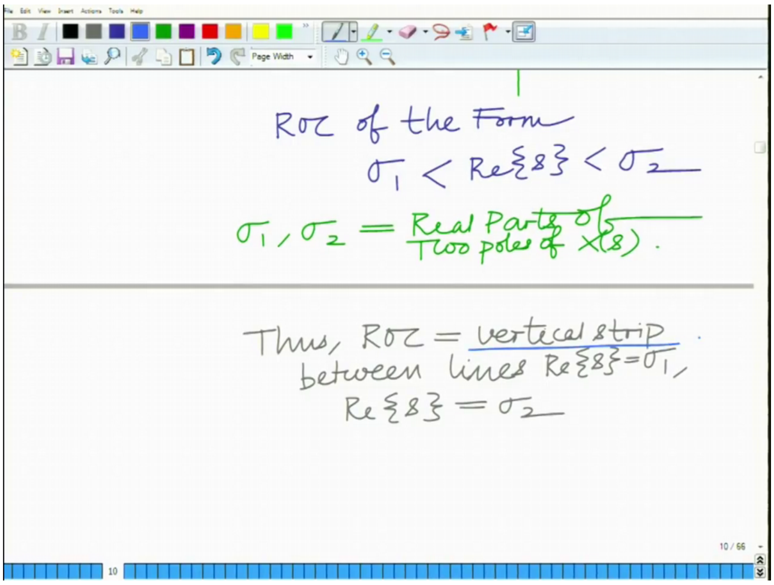

Then the ROC is of the form sigma 1 less than real part of s less than sigma 2, where sigma 1 comma sigma 2 are the real parts of two poles of X of s, where sigma 1 sigma 2 are the real parts of two poles of X of s. Thus the ROC is the vertical strip. This ROC equal to vertical strip between the lines, real part of s equals sigma 1 comma real part of s equals sigma 2. So, the ROC is basically a vertical strip and this, you can check this as follows; that is basically if you look at this.

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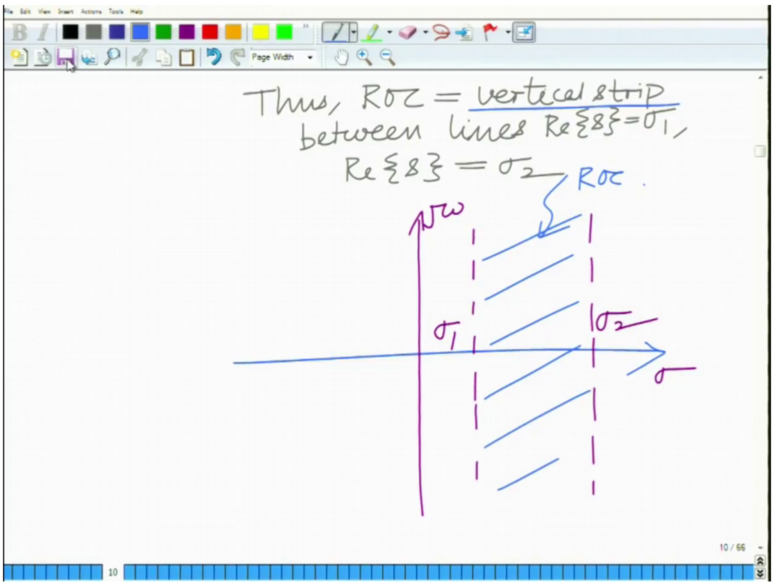

Again let me just draw this. So, you have sigma, you have j omega, you have your sigma 1, you have your sigma 2. The real parts between these two poles, and therefore, the ROC will be this vertical strip. This is your ROC.

For instance, let us take again a simple example; this I think is best illustrated using an example.

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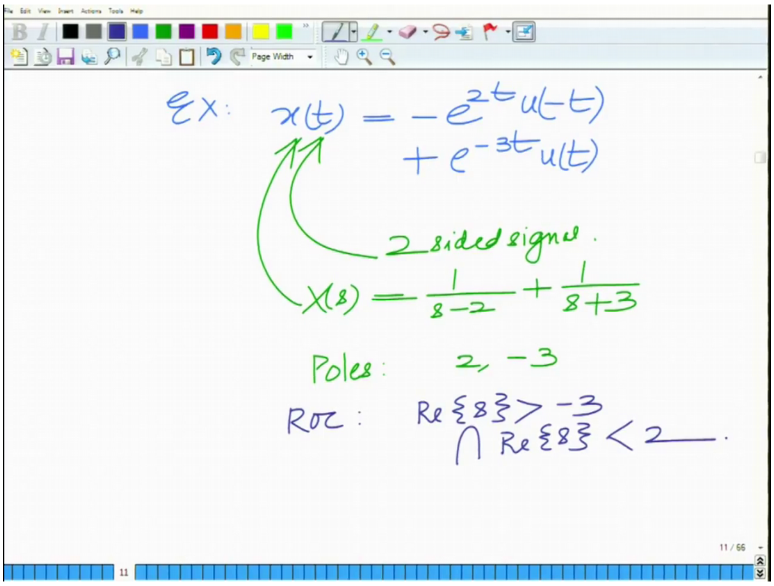

For instance if you take xt equals minus e raise to 2 t, you mind u u minus t minus e raise to 2 t u minus t correct, and plus e raise to minus 3 t u t, then the Laplace transform. Now look at this, this is a 2 sided signal. So, this you can click, this you can check, because u minus t is non 0 for t less than 0, u t is non 0 for t greater than 0.

So, this is a 2 sided signal this and the Laplace transform Xs. You can verify that is given as 1 over s minus 2 plus 1 over s plus 3, and the poles are 2 comma minus 3, and the ROC will be. Basically you can see it is the intersection of the ROCs of the both signals; that is real part of s greater than minus 3 intersection real part of s less than 2; that is the intersection, and which implies the ROC is basically minus 3 less than real part of s less than.

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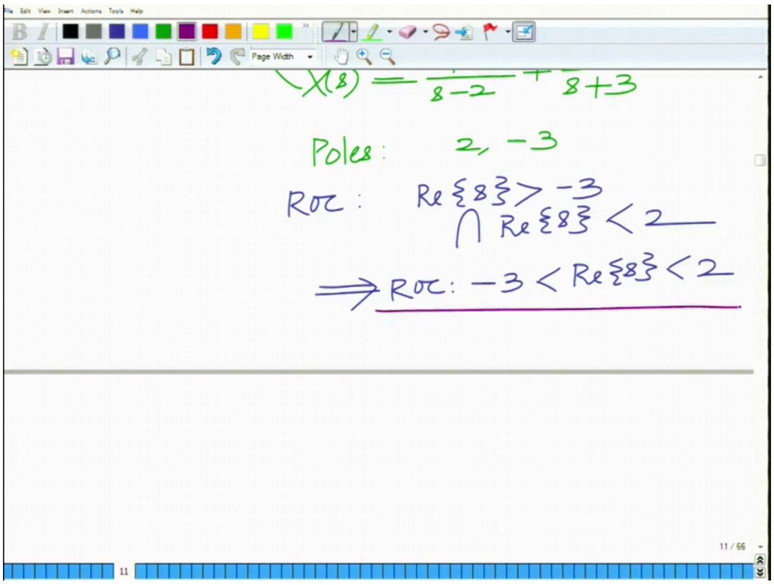

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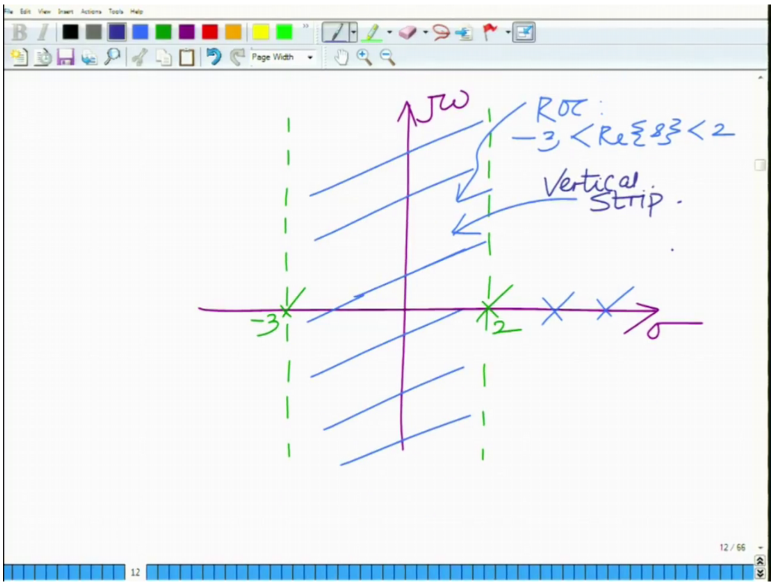

So, this is basically your final ROC and therefore, it includes the strip between minus 3 and 2, and this can be simply illustrated as follows; this is sigma, this is j omega and you have minus 3 sigma equals minus 3. Here this is one pole and you have sigma equals 2. Here this is another pole, this is the other pole and ROC is the strip between this two. ROC is minus 3 less than real part of s less than 2, and any other poles can only lie outside the ROC. So, you can have poles which are outside only, outside the ROC ok.

And this is basically a vertical strip. So, this is the ROC; that is sigma 1 less than. So, the form sigma 1 less than real part of s less than sigma 2. This is basically the ROC, and the ROC in this example is minus 3 less than real part of s less than, and so let us look at the partial fraction expansion.

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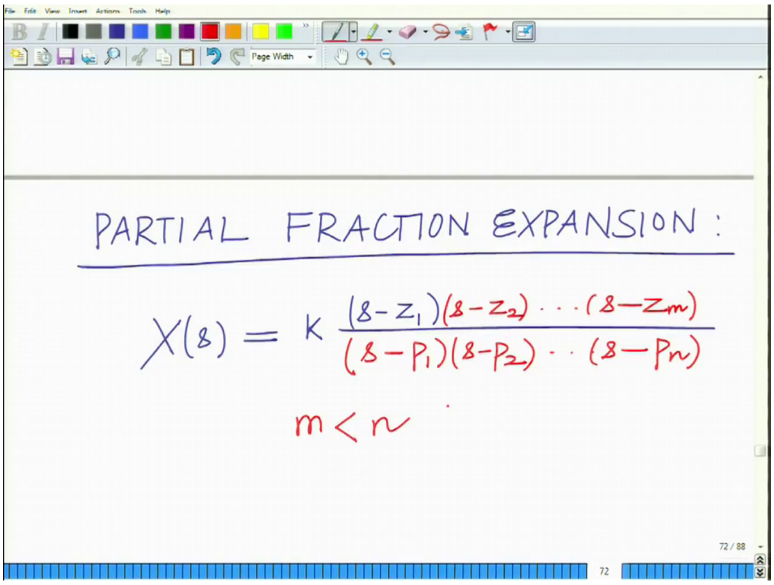

The partial fraction of expansion that is, if X we have X s equals some constant K times s minus Z 1 times s minus Z 2 into s minus Z m divided by s minus P 1; that is Z 1 Z 2 Z m are the 0s s minus P 2 into s minus P n, where m P 1 P 2 P n are the poles and m is less than, m is less than. And remember when m is less than n, this is known as a proper rational expression or the proper rational function.

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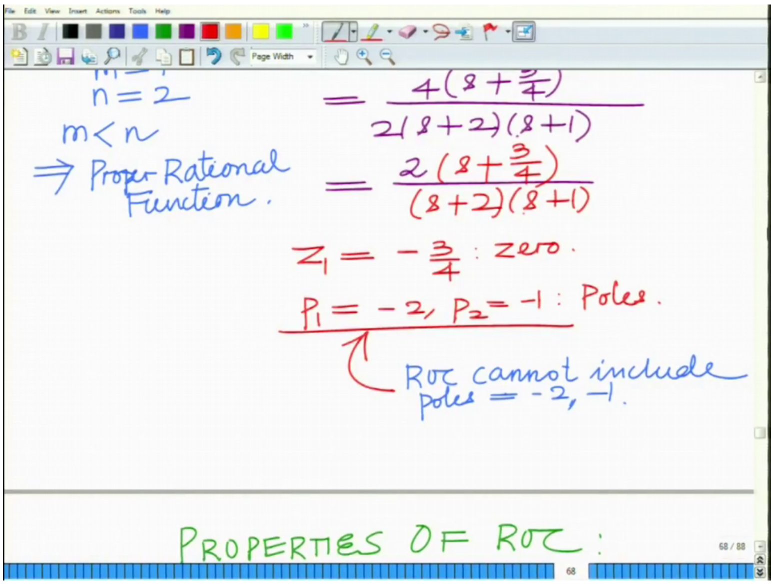

So, if it is a proper rational function, I can express, that is implies this is a proper rational function. If this is a proper rational function then I can express X s as the sum, and simple pole case. We are assuming all the poles are distinct; that is we have simple poles.

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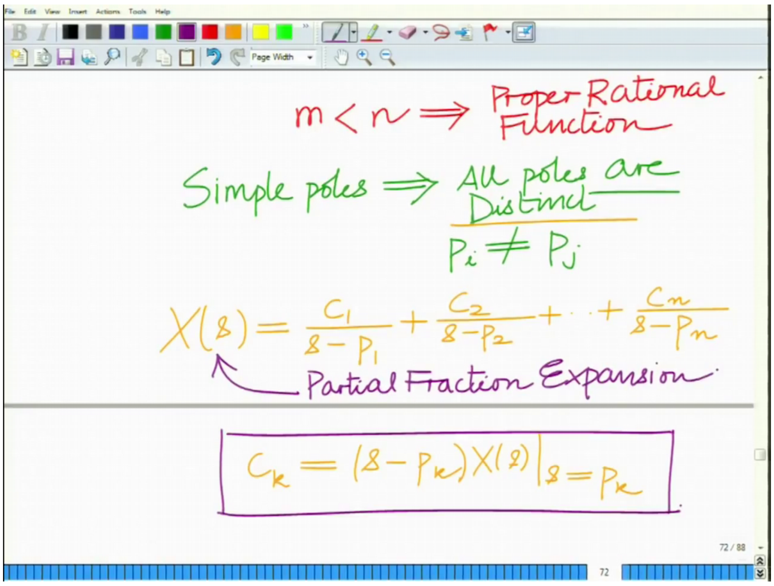

That is no two poles are such that P i equals P j; that is all poles P i not equals P j.

So, we have all the poles are distinct for. So, for the simple pole case, all poles are simple, which means all poles are basically distinct, and for that scenario Xs can be expressed as C 1 by s minus P 1 plus C 2 by s minus P 2 plus C n by s minus P n, where the coefficient C k is given as s minus Pk times Xs evaluated at s equals P k. So, this is the partial fraction expansion of Xs, and C k is the coefficient k th coefficient all right.

In the partial fraction this is a coefficient in the partial fraction expansion of Xs. This is a coefficient in the partial fraction expansion of Xs.

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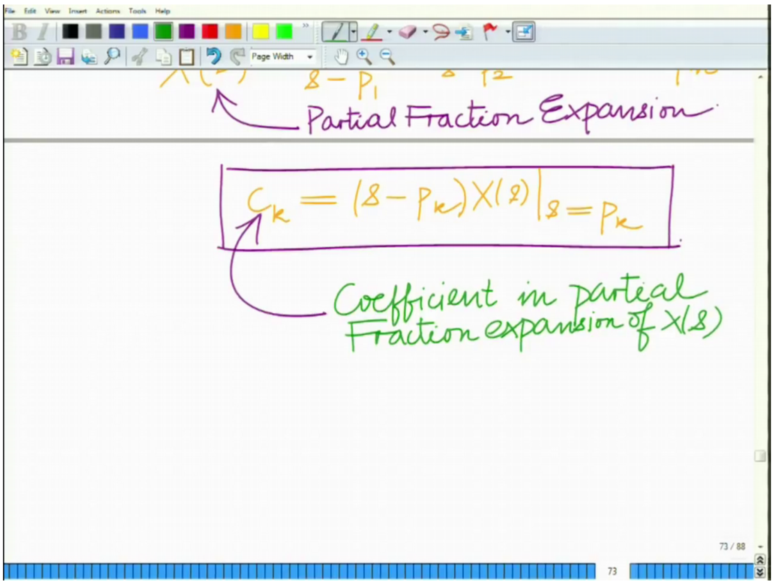

So, far; so, what we have, is a partial fraction expansion for the case where m is less than n all right; that is the degree of the numerator polynomial is strictly lower than the degree of the denominator polynomial. This is a proper rational function, Xs is a proper rational function all right.

In this case we express using its partial fraction expansion; that is the sum C 1 by s minus P 1 plus C 2 by s minus P 2 so on, and Ck, the coefficient of the k th term is given as s minus Pk times Xs evaluated at the pole s equals P k. And all the poles are assumed to be simple; that is there are no repeated poles. So, all the poles are distinct P i not equals P j ok.

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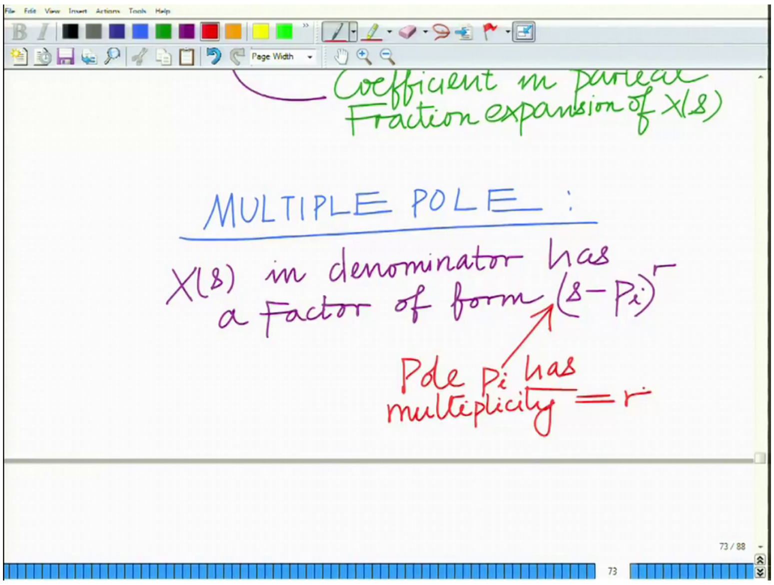

Now, for the multiple pole case, now for the multiple pole case Xs in denominator has a factor of form. It has a factor of form s minus P i raise to the power of r and then in this scenario, we say that the pole P i has a multiplicity of r. So, this implies pole P i, pole Pi has multiplicity equal to r.

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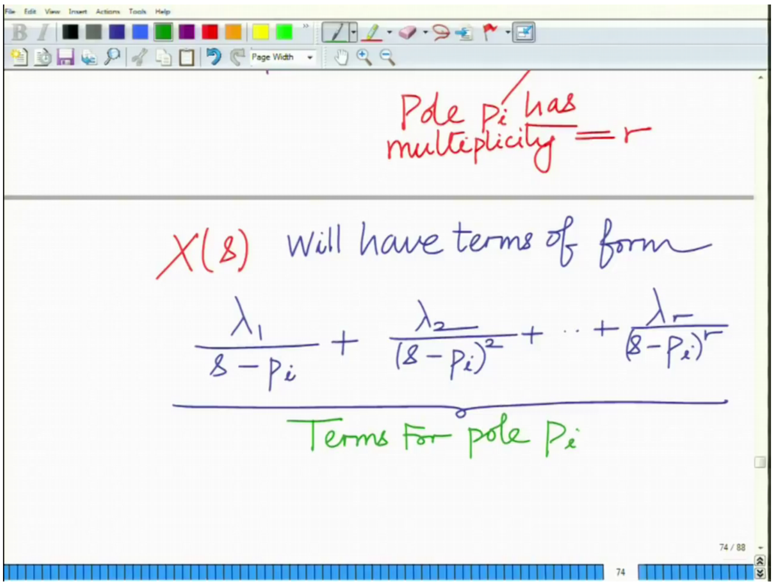

And therefore, Xs will have terms of the form the partial fraction expansion of Xs will have terms of the form. We will have terms of the form lambda 1 divided by s minus P i plus lambda 2 divided by s minus P i square plus lambda r divided by s minus P i raise to the power of r.

So, these are the terms corresponding to s minus P corresponding. These are the terms corresponding to pole P i and you will have the rest of the terms corresponding to other poles, terms for pole P i which has multiplicity r. So, these are the terms. So, there will be r terms corresponding to the pole P i all right. Lambda 1 by s minus P 1 P i lambda 2 by s minus P i square so on up to lambda r divided by s minus P i raise to the power of r, where r is a multiple t multiplicity of the pole here, and the coefficient lambda, lambda r or lambda r minus k is evaluated as follows

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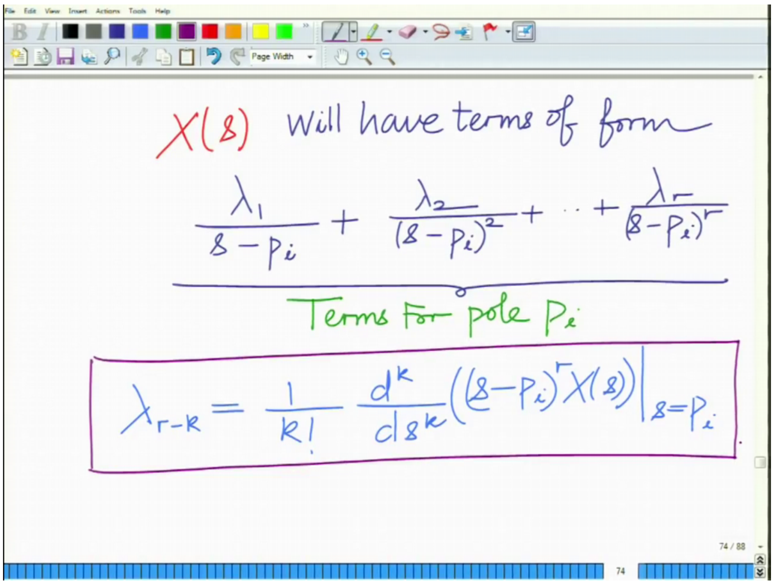

We have the expression for lambda r minus k that will be equal to 1 over k factorial. So, this expression will be equal to 1 over k factorial d k raise to power k s minus P i to the power raise to the power of r into Xs evaluated at s equals P i. So, this lambda r minus k. Let me just write it again. Once again clearly lambda r minus k equals 1 over k factorial terms be to the k ds to the k s minus Pi raise to the power of r X to the power of X s evaluated at s equal to P i evaluated at s equal to P i ok.

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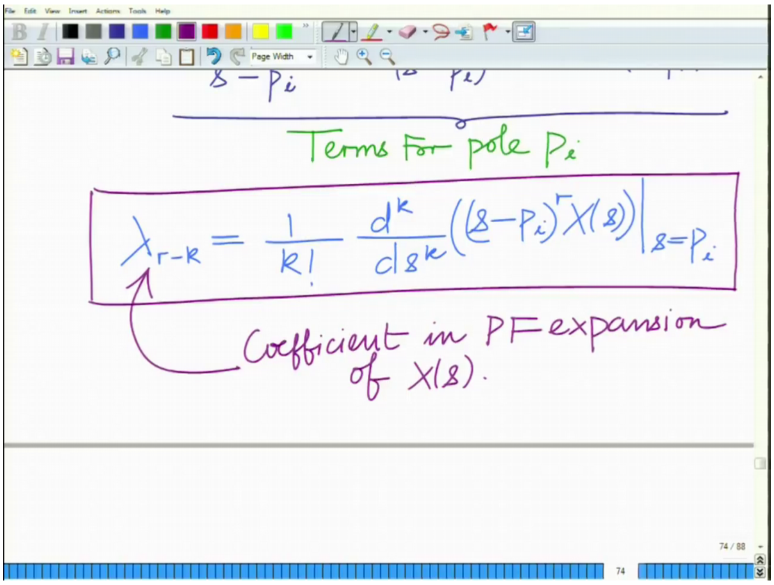

So, this is basically the expression coefficient of. Remember coefficient in the partial fraction expansion, coefficient in partial fraction expansion of Xs. This is the coefficient in the partial fraction expansion of the rational function, the rational for Xs which is the rational function of s only. So, that takes care of various scenarios; that is when you have simple poles as well as multiple poles.

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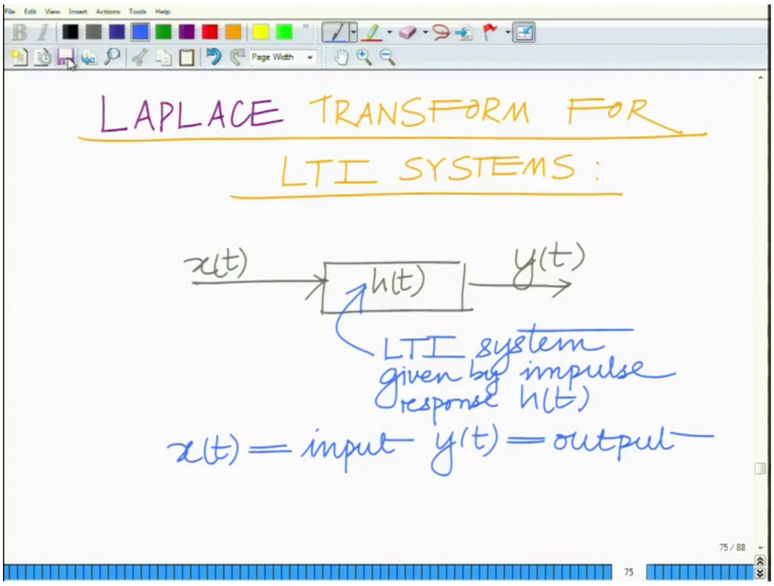

Now, let us look at the Laplace transform properties for LTI systems; that is the next aspect, the Laplace transform and its application for the for LTI systems. So, let us look at Laplace transform for LTI systems. Now for an LTI system, consider an LTI system given by impulse response ht. So, this is LTI system xt, x t equals the input and yt equals the output of the LTI system.

Then we have, we know that the output yt of an LTI system is a convolution of the input xt with the output, with the it is a convolution of the input xt, with the impulse response ht of the LTI system. Therefore, the output impulse response. Remember convolution in the time domain is a product in the transform domain, there is a Laplace domain. Therefore, since we have yt equals xt convolved with ht.

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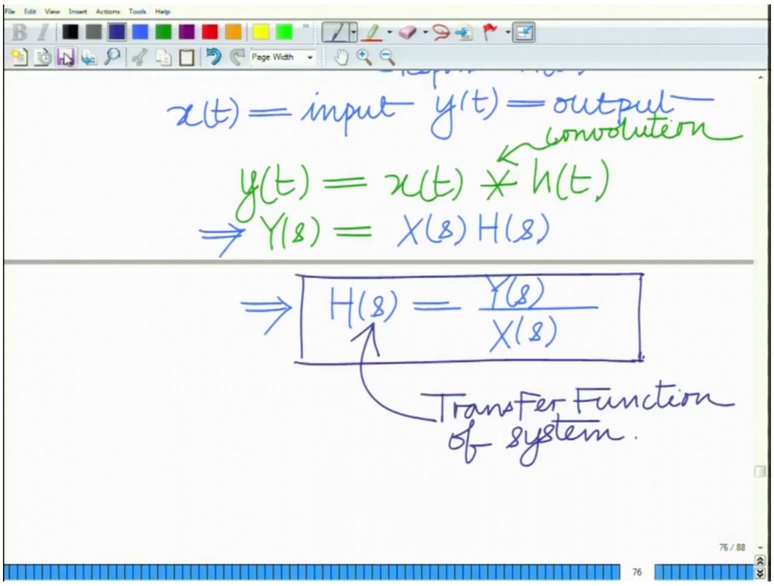

Remember this is your convolution and therefore, when we take the transform, the Laplace transform Ys will be the product of the Laplace transforms of Xs term Hs.

And. In fact, this also implies this, implies that also, implies that Hs equals Y s divided by Xs and Hs. This quantity is known as. This is a very important role in analysis of LTI systems. This is known as the transfer function, this is known as the transfer function, this is known as the transfer function of the system. So, H s which is the Laplace transform of the impulse response. Remember H s is the Laplace transform of the impulse response ht. This is known as the transfer function, and this characterizes, this is a fundamental aspect property of the system, it characterizes the behavior of this LTI system in a very fundamental fashion ok.

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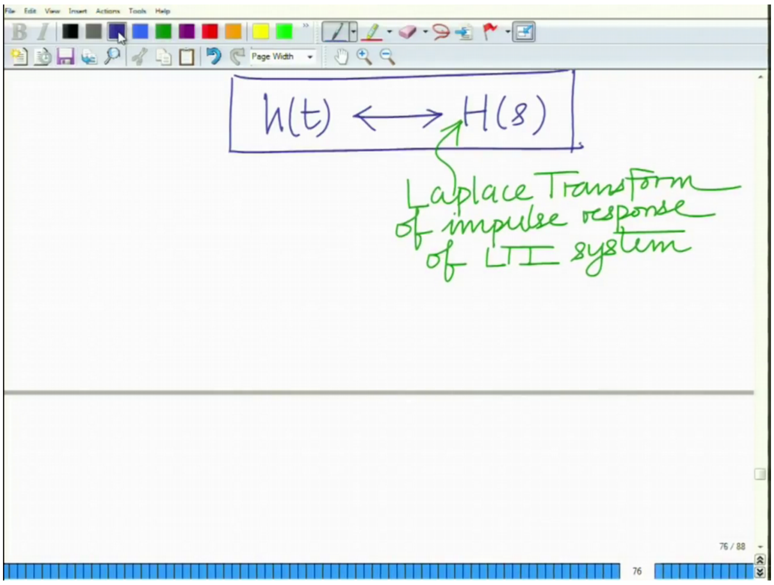

So, ht remember ht has Laplace transform. So, Hs is the Laplace transform of the impulse response. So, this is. So, this is the Laplace transform of the impulse response of the LTI system under consideration further. Now let us look at other properties of the LTI systems and from the perspective of the Laplace transform.

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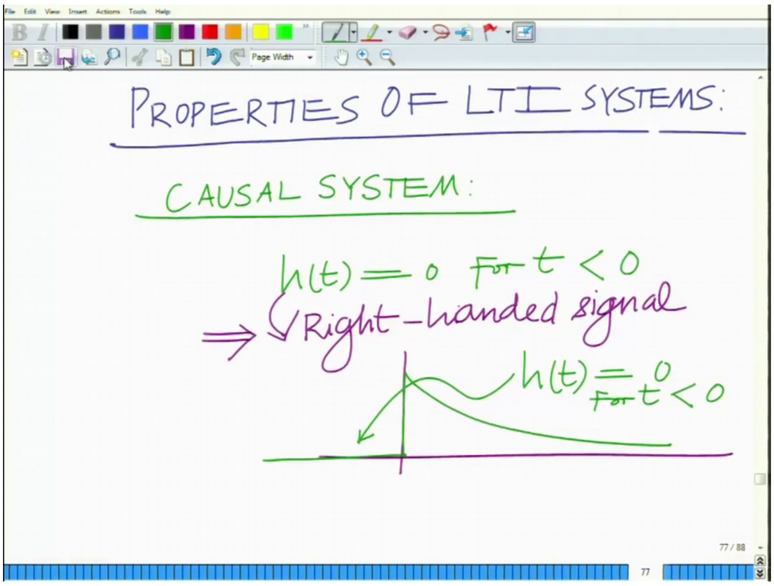

So, other properties of the LTI systems. Let us look at some other properties of the LTI systems. Now let us in particular, let us look at causality. Now for a causal system, remember we have ht equal to 0 for t less than 0, which implies basically if you look at this, this basically implies that h ht is a right handed signal. This basically implies ht is a. Remember we have ht of the form correct, we have ht equal to 0 correct ht equal to 0, for the impulse response has to be 0 for t less than 0 correct; that is the property of the causal, that is the property of the causal, that is the property of the causality a system that is ht equals 0 for t less than 0, which implies ht is a right handed signal, which implies the ROC of ht.

Now, from the properties of the ROC. Therefore, in the Laplace transform Hs must have ROC, which is of the form real part of s is greater than the sigma max.

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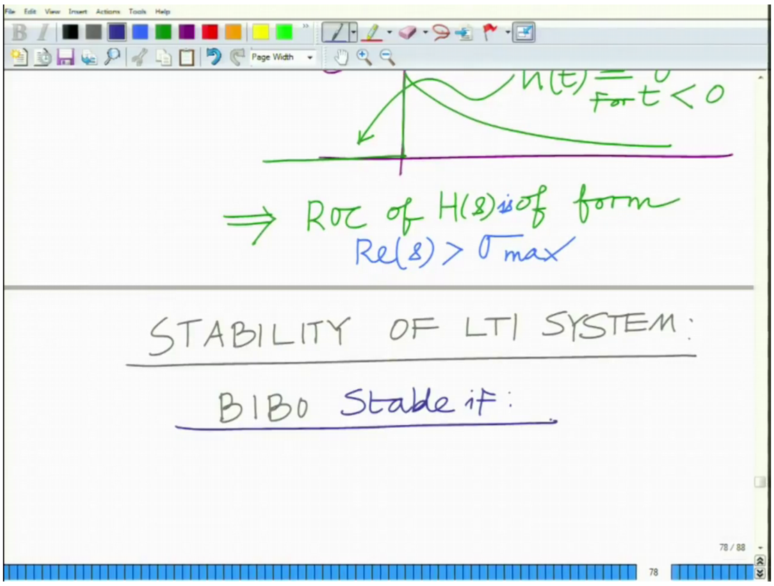

So, the ROC this implies in the ROC of Hs is a form, is a form. The real part of s is greater than sigma max. So, remember this follows from the properties of the ROC; that is since ht is a causal system, if the LTI system is a causal system, it must be the case that the impulse response is non-zero for t less than 0, which is implies that it is a right handed signal, and therefore, the ROC of H s which is impulse response of ht, which is the Laplace transform of ht must be of the form, real part of s greater than sigma max ok.

Now, let us look at stability of the LTI system. Now let us look at the stability of the LTI system. Now it is BIBO stable; remember bounded input bounded output stable.

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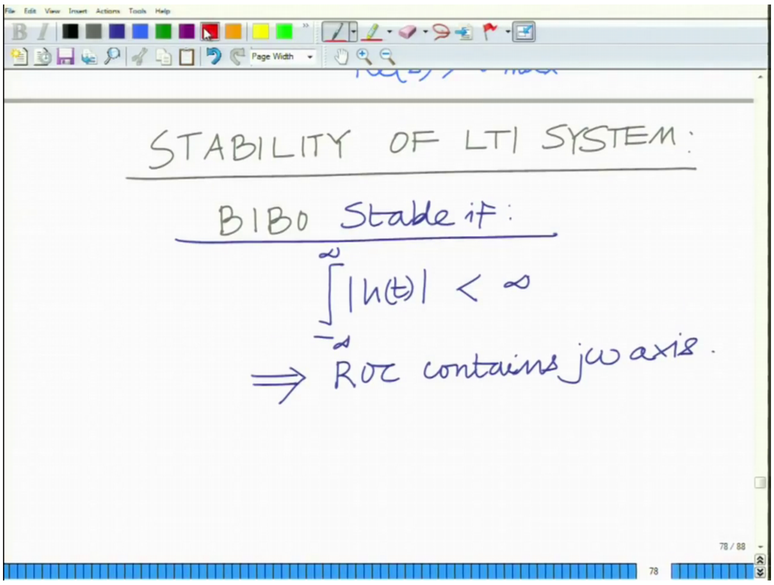

If integral minus infinity to infinity magnitude h of t less than infinity. This implies that ROC contains the j omega axis; that is if you look at the s plane y axis is the j omega. This is the. So, this is basically your s plane.

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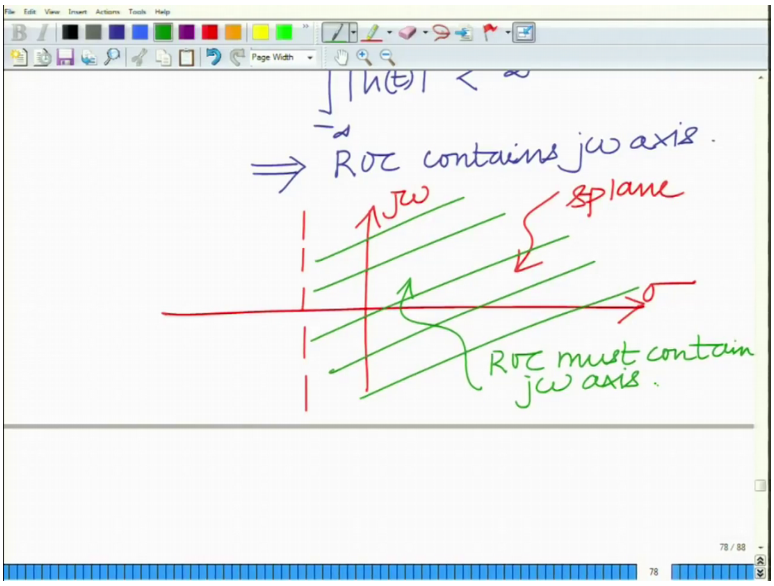

Now, whatever is ROC must contain the j omega plane. If the system is stable ROC must contain the j omega axis; that is this can be seen as follows.

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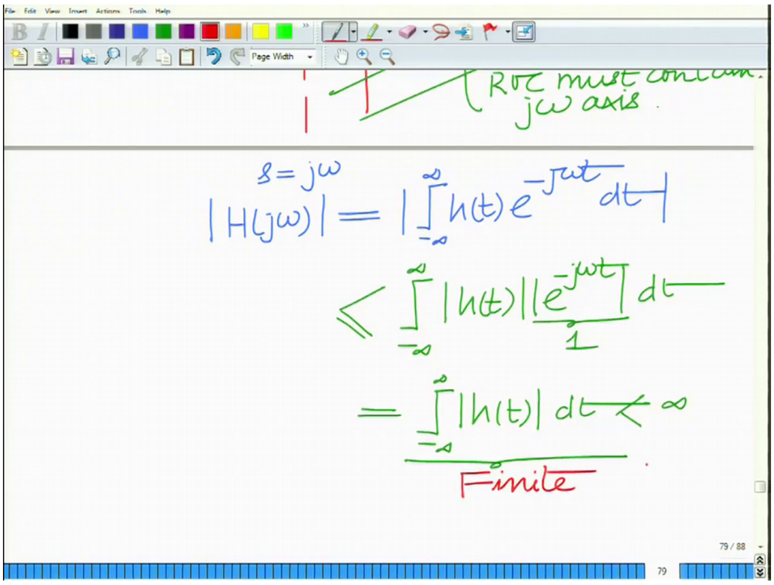

If you look at magnitude of H of j omega; that is equal to integral minus infinity to infinity h of t e raise to minus; that is if you substitute s equals j omega magnitude f of H of j omega is integral minus infinity to infinity h of t e raise to power minus j omega t dt, which is less than. That is a magnitude of the integral is less than or equal to the integral of the magnitude, which is less than or equal to. Therefore, integral ht integral e power minus j omega naught omega t dt.

But magnitude of e power raise to minus j omega t is 1 which means this is equal to integral e power minus minus infinity to infinity h of t dt. So, magnitude of h of j omega is a less than or equal to this, which means. Therefore, if this is finite, if this quantity is finite which implies, if this is finite implies, if this quantity, if integral magnitude integral minus infinity to infinity magnitude ht dt is finite, this implies

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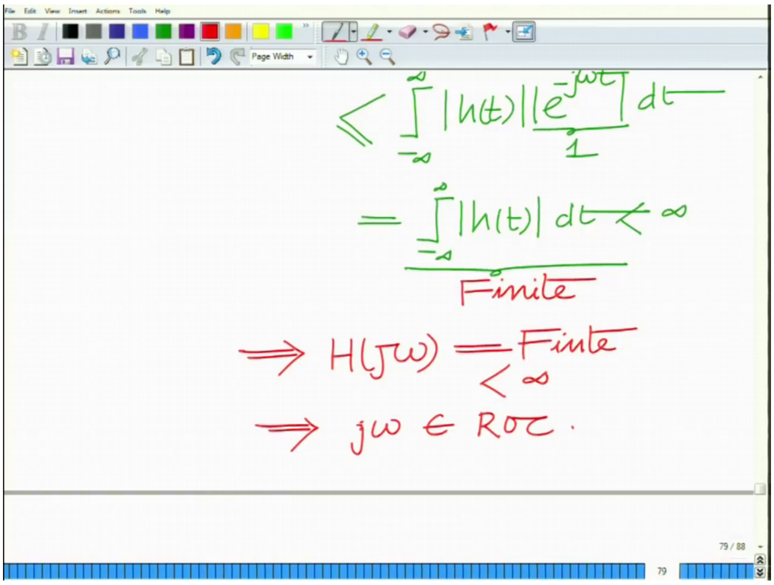

H of j omega equals for finite quantity less than infinity which implies j omega is element of j omega belongs to the ROC.

So, argument is the following; that is h of the magnitude of H of j omega is strictly is less than or equal to integral minus infinity to infinity magnitude h of t dt, which is a finite quantity. Therefore, or magnitude of H of the omega must be a finite quantity, which implies the Laplace transform must converge for s equal to j omega, which implies that j omega belongs to the ROC of the Laplace transform. So, implies j omega Xs belongs to ROC of the Laplace transform that is H of s ok.

And finally, Laplace transform; that is a transfer function of the LTI system described by the differential equation.

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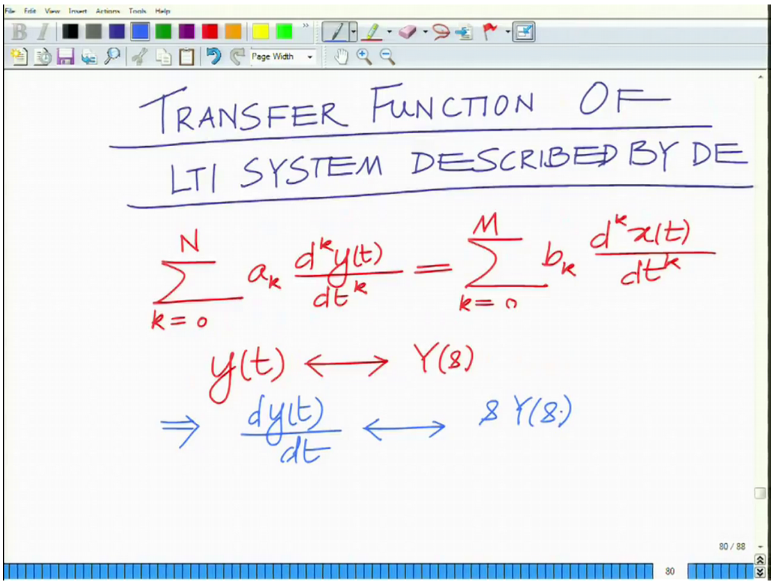

We have the transfer function of the LTI system described by the referential equation, described by a differential equation; that is I have sum k equals 0 to N a k d raise to k y t dt k; that is equal to sum k raise to 0 to M bk d raised dk xt dt d to the, and d k d to the k xt over dt k.

Now, we know from the properties of the Laplace transform that yt has Laplace transform Y s implies. So, yt implies d raise to d t. Remember d yt over dt has Laplace transform s Ys. In fact, this implies progressing this way.

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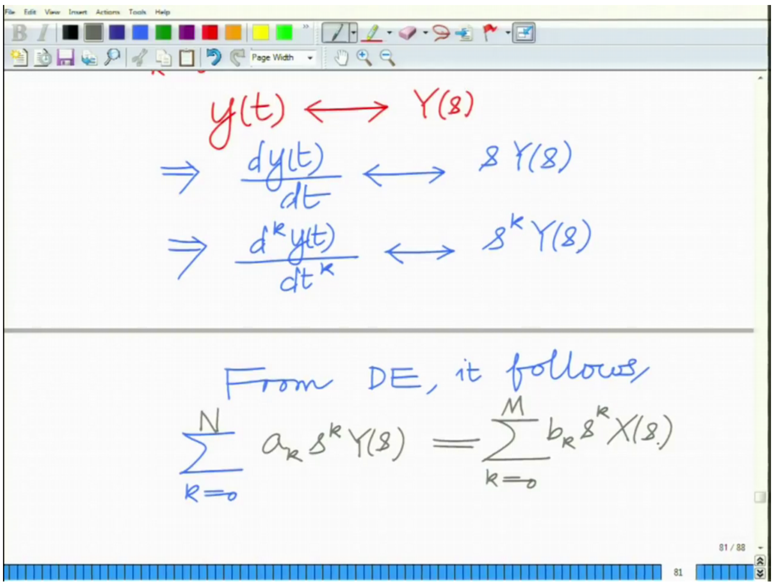

We can show, one can show the d raise to k y t over dt k or for that matter d raise to k yt or dt k has Laplace transform s to the power s raise to k Ys.

Similarly, d raise to k d raise to k xt dt over dt k all right. The k th order derivative of the signal xt with respect to time, has a Laplace transform s to the power of k X s. So, substituting this we have. So, from the differential equation it follows, it follows that we have summation k equal to 0 to N a k sk s raise to k Ys equals summation k equal to 0 to M bk s raise to k X s.

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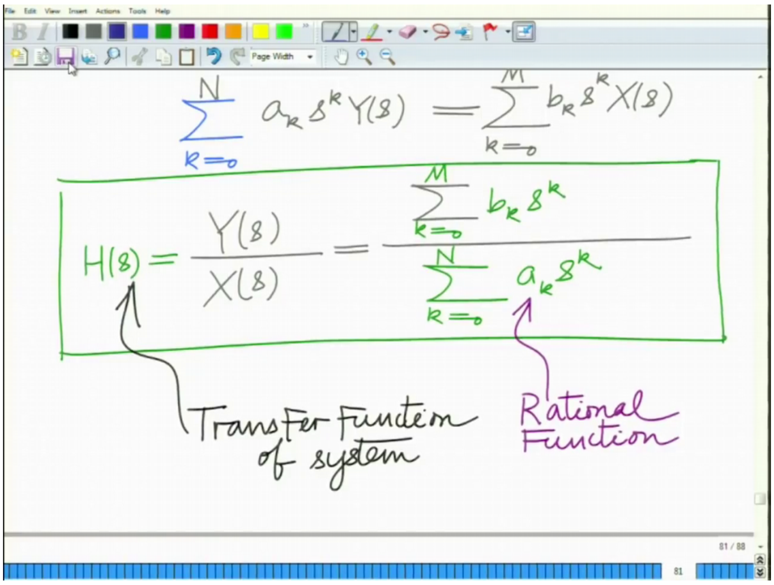

So, implies taking Ys common on the left Xs common on the right.

So, Ys by X s, if you look at this. This is equal to summation k equals 0 to M b raise to k x k s k divided by k equal to 0 to N a raise to k a k s raise to k and this is a transfer function. In fact, let me, we remember we said H s is defined as the impulse response that is the Laplace transform of the output divided by the Laplace transform the input. So, this is the transfer function of the system. This is the transfer function of the system, and remember this is a rational function. You can clearly see this is a rational, then you can clearly see that this is a rational function ok. So, all right. So, basically that completes our description of the definition, and the properties of the Laplace transform.

So, for, with this what we have done is, we have completed our discussion of the definition, definition of the ROC properties of the ROC, some of the properties of the

Laplace transform and the description of the properties of the Laplace transform with LTI, or with respect to LTI systems and several other aspects. So, this completes our discussion of the theory, behind the theory and applications theory, applications and the properties of the Laplace transform. And in the subsequent modules, we look at some of the examples related to Laplace transform solving problems, involving Laplace transform as it applies to the principles of signals and systems all right. So, we will stop here.

Thank you very much.