Principles of Signals and Systems
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Lecture – 19
Laplace Transform – Convolution, Rational Function – Poles and Zeros, Properties of ROC

Hello welcome to another module in this massive open online course. So, we are looking at the properties of the Laplace transform let us continue our discussion.

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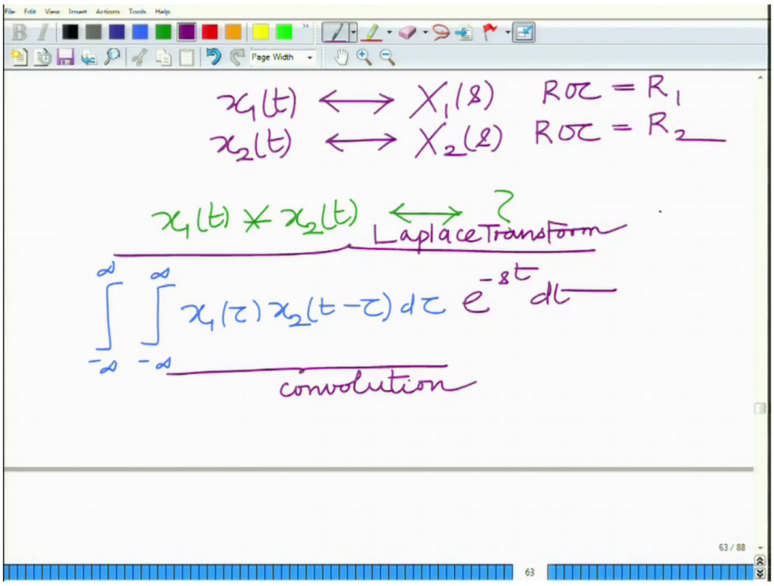

So, what we want to look at now is another interesting aspect of the Laplace transform is its relation to the convolution between 2 signals. So, we want to look at the convolution aspect or the convolution property of Laplace transform. Let us say I have 2 signals x 1 t this is X 1 s Laplace transform with ROC equals R 1 and x 2 t with Laplace transform X 2 s and ROC equals R 2. Then if you have the convolution x 1 t convolved with x 2 t we considered the convolution x 1 t convolved with x 2 t I want to find out what is the Laplace transform of x 1 t convolved with x 2 t. Well you can see x 1 t convolved with x 2 t can be expressed as integral minus infinity to infinity, x 1 t convolved with x 2 or x 1 tau times x 2 t minus tau d tau.

And we take the Laplace transform of. So, this is the convolution the inside integral and you take the Laplace transform of this e raise to minus s t so this is the convolution, this is the inner integral and basically outer integral is taking the Laplace transform of the convolution. And now this you can simplify this by modifying the order of integration as follows.

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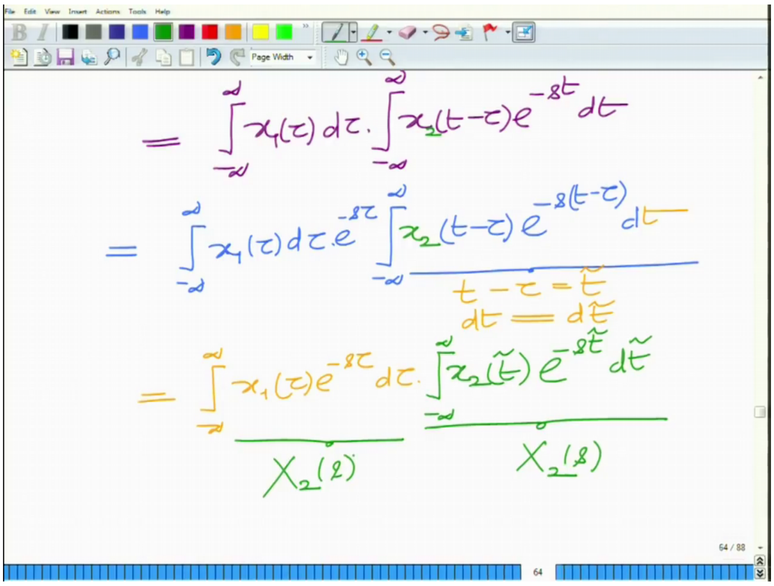

I can make this as integral minus infinity to infinity x 1 tau times d tau into integral minus infinity to infinity x 2 t minus tau e raise to minus s t d t. And now what I can do here is basically I can write this as integral minus infinity to infinity x 1 tau d tau integral minus infinity to infinity x 1 t minus tau e raise to minus s t minus tau d tau.

So, I can changing e raise to minus s t minus tau which means I will as it is a fact of e raise to minus s tau which I am writing outside so I am multiplying by e raise to minus s tau into e raise to plus s tau and now you can see if I look at this integral if I set t minus tau equals t tilde then I will have d t or minus d tau t minus tau equals by the way this integral is with respect to t so this is d t so I will have d t equals d tilde and I can further write this integral as e integral minus infinity to infinity x 1 tau e raise to minus s tau d tau into integral minus infinity to infinity x 1 t tilde e raise to minus s t tilde d t tilde.

And you can see this is basically your x 1 s so this you can see is basically your x 1 s and this which does not depend on t tilde is x 2 s so what you have is I am sorry this is basically your now let me just write it this is x 1 tau so this x 2, this is x 2 tau so this is x 2 t minus tau so again this is x 2 of t tilde and so this is basically your x 2 of s and this is in fact your x 1 of s.

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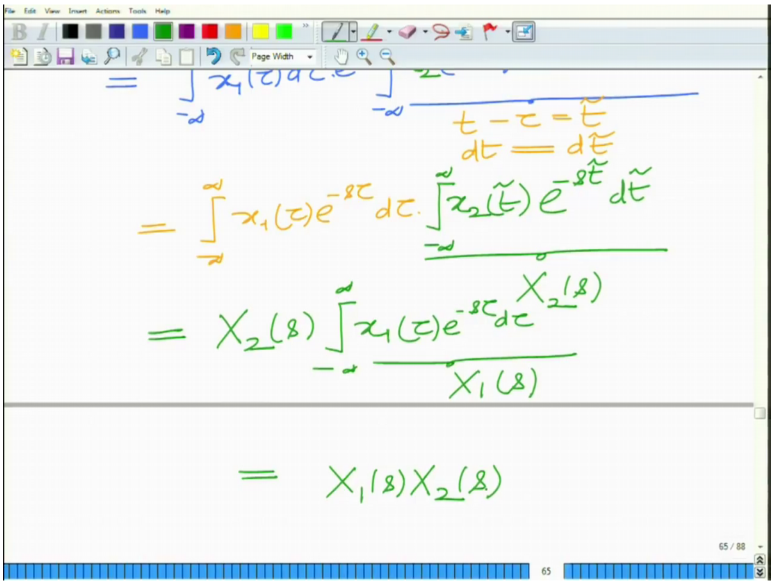

So just write one more step so this x 2 of s which comes out of a integral and what you are left with is x 2 of s integral minus infinity to infinity and this you can see is x 1 of s so what you have is x 1 of s times x 2 of s.

So, basically with the result that we have is that the convolution of 2 signals we convolved x 1 t with x 2 t the Laplace transform is given by the product of the Laplace transform so that is x 1 s times x 2 s.

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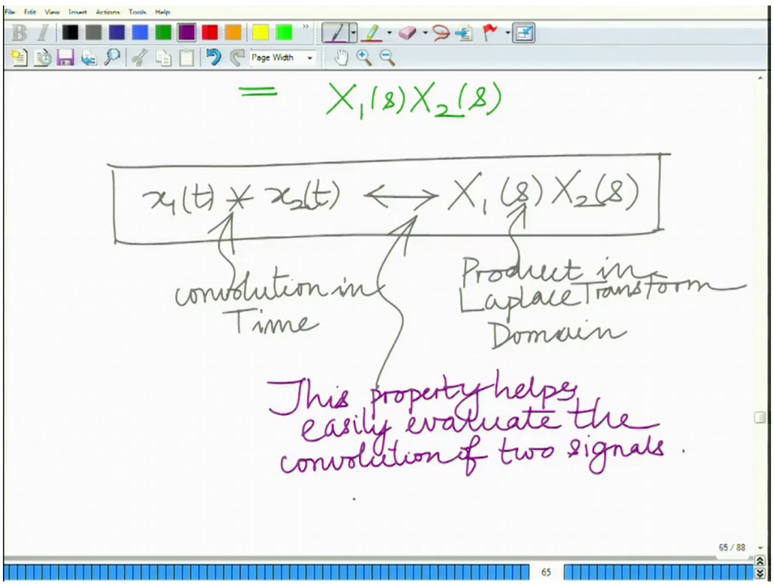

And this is a very interesting results so x 1 t convolved with x 2 t has Laplace transform x 1 s times x 2 s. So, convolution in time is equivalent to taken the product in the transform domain that is the Laplace transform of convolution in time the product in Laplace transform domain or basically the s domain. And this is an important product this is an important property what it does it helps easily evaluate so this property this helps easily evaluate the convolution between 2 signals because rather than evaluating convolution which is slightly difficult to evaluate typically.

Because it knows the integral if one has knowledge of the Laplace transforms of the 2 signals is then the Laplace transform of the convolution can be readily rather readily evaluated by the multiplication of the Laplace transform of these 2 signals that is in potential. So, this is a very convenient tool so transform representation gives the very convenient tool to evaluate the convolution between 2 signals. And hence as a result of that naturally it is easier to look at the output of an LTI system given in input signal because the output is nothing but the convolution between the input signal and the impulse response of the LTI system.

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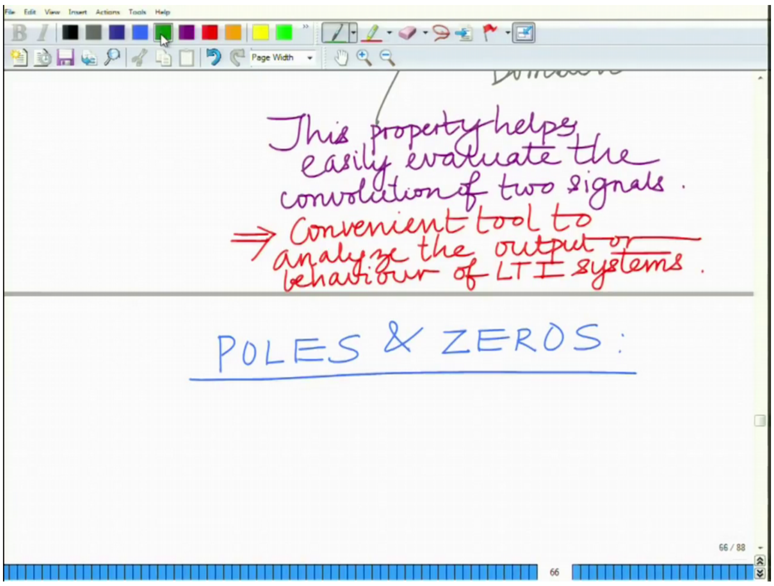

So, as a result, the properties and behavior which implies this that it is a convenient tool to analyze the output or behavior of LTI systems that is linear time invariant uses a convenient tool to analyze the output or behavior of LTI system that is for the Laplace transform is a very convenient tool about signal processing communication in several areas such as, control systems correct instrumentation etcetera where one needs to study the behavior or system analyze the behavior of system, examine the behavior or the output of a given system for different input signals and also analyze the impact of the system for given different signals characterize the interaction between various signals and systems alright.

So, let us now move on to another topic in the Laplace transform which is that of the poles and zeros.

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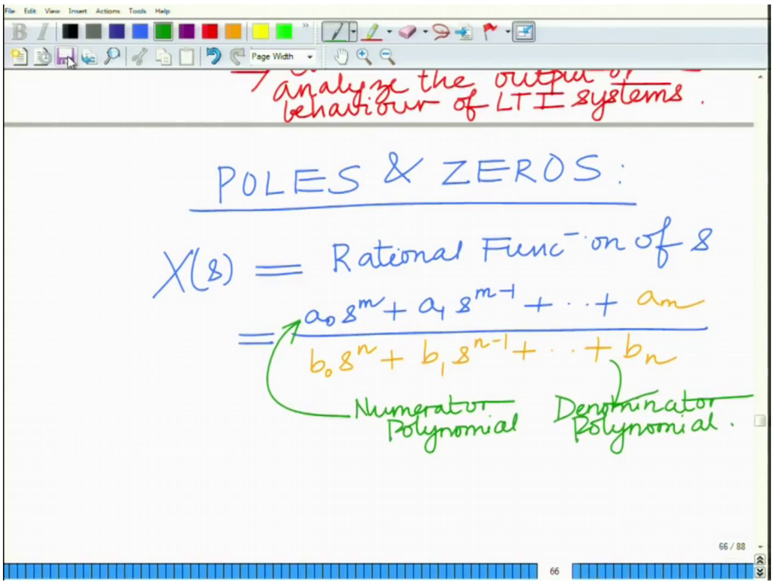

The poles and zeros now consider x s to be rational function of s. We considered the Laplace transform to be a rational which implies x s is the form of numerator polynomial divided by denominator polynomial so this is going to be on the form a naught s minus z 1 s minus z 2 s minus z m or let us write it this way this is of the form a naught s to the power of m plus, a 1 s to the power of m minus 1 plus, a m divided by b naught s to the power of n plus b 1 s to the power of n minus 1 plus, b n. So, this is of the form numerator polynomial divided by denominator polynomial.

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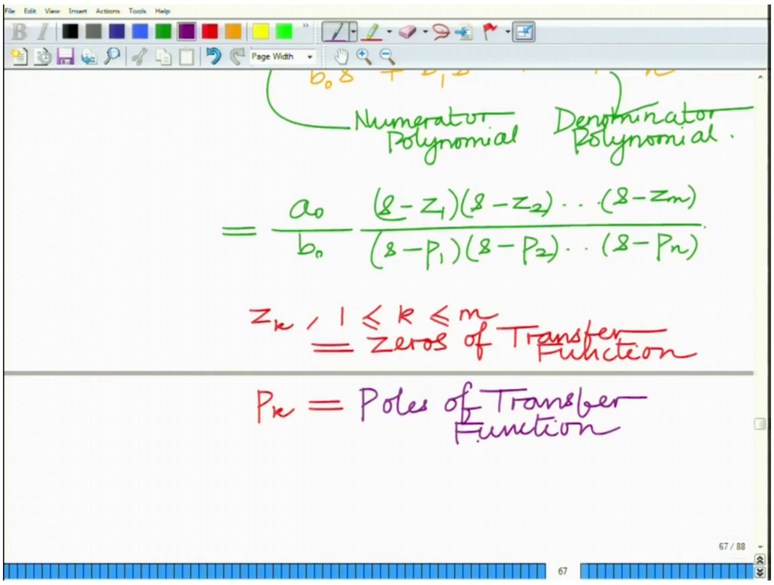

And therefore, this can be written as a naught over b naught into s minus z 1 s minus z 2 s minus z m divided by s minus p 1 s minus p 2 s minus p n where these roots of the numerator polynomial that is z k for 1 less than or equal to k less than or equal to m these are known as the zeros of the transform that is the roots of the numerator polynomial in s these are known as the zeros of the transfer function and the roots of the denominator polynomials are known as the poles of the transfer function z k are the zeros.

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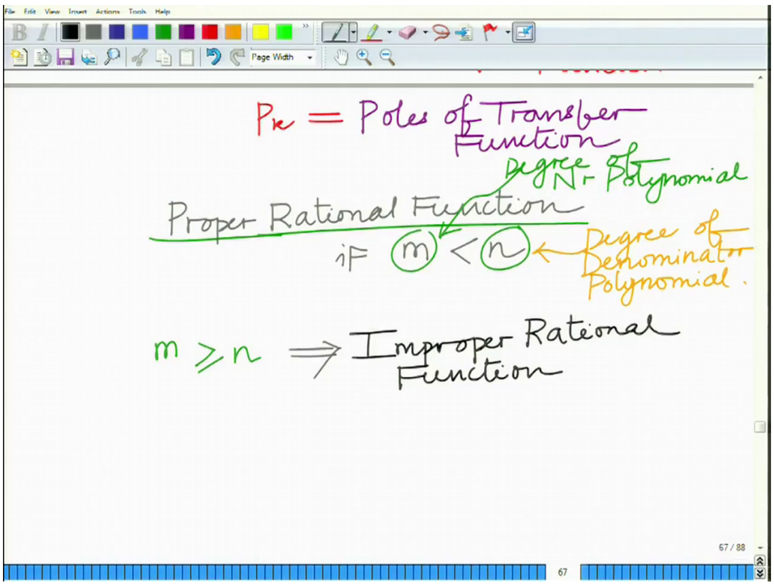

Now the transfer function P k are the poles of the transfer function or the poles of the transfer function and this is known as a proper rational function.

Proper rational function if m is strictly less than n then this is known as a proper rational function if m is greater than or equal to n, this implies that is numerator degree of the numerator polynomial. So m remember m equals degree of numerator polynomial and n is basically degree of denominator polynomial degree of numerator polynomial and if m is greater than or equal to n, this implies an improper rational function.

So, if the numerator polynomial degree is less than the degree of the denominator polynomial it is a proper rational function. On the other hand otherwise that is degree of numerator polynomial is greater than or equal to that of the denominator polynomial it is a improper rational function alright.

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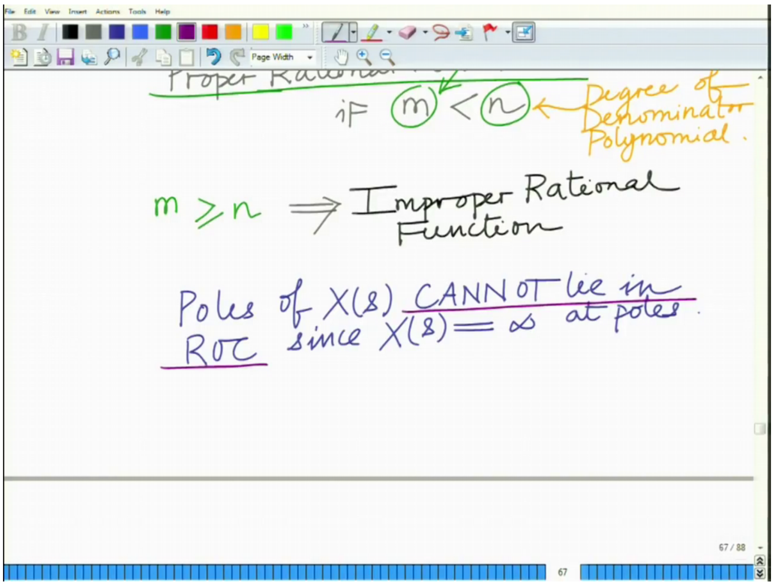

And now one thing to keep in mind is that poles cannot lie the poles of x of s poles of the transfer function cannot lie in ROC, since the x s is undefined at the poles. So since x s evaluates to infinity at the poles, since x s equals infinity at poles so poles cannot lie in the ROC poles of x s. So ROC the region of convergence cannot include poles of the transfer function x s. Let us take an example to understand this better ok.

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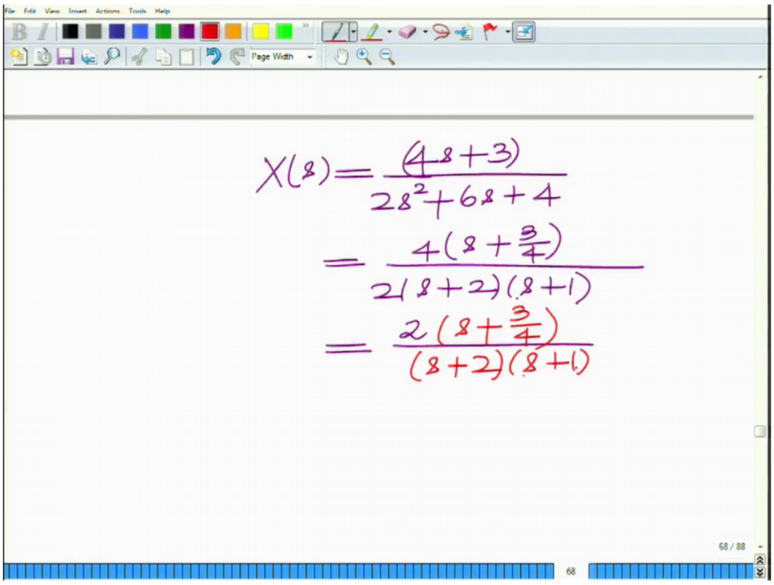

So, I have x s equals 4 s plus 3 divided by 2 s square plus 6 s plus 4 equals 4 into s plus 3 by 4 divided by 2 into s plus 2 into s plus one. So, this is equal to basically they can write this as twice of s plus 3 over 4 divided by s plus 2 into s plus 1.

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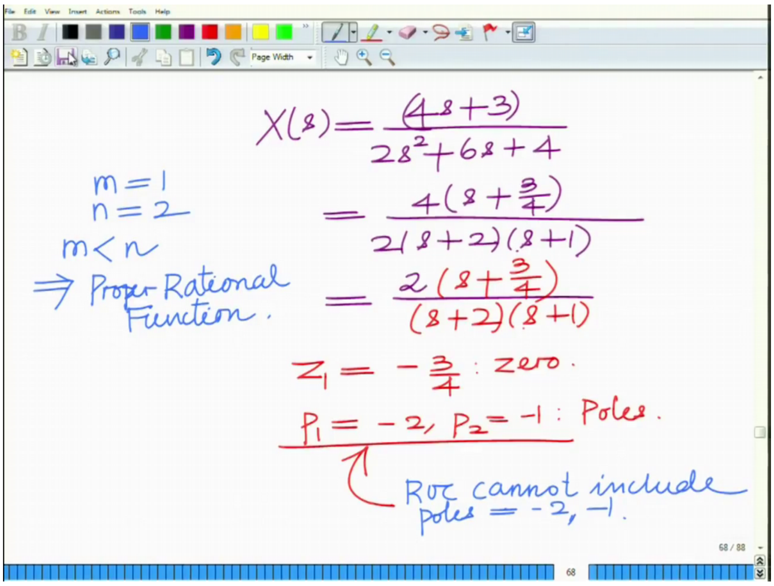

And now you can see here that the zeros z 1 equals minus 3 by 4, this is a 0 and p 1 equals minus 2, p 2 equals minus 1, these are the these are the poles and ROC cannot include these poles, the ROC cannot include minus 2 or minus 1. Roc cannot include poles that are equal to minus 1, so the poles are minus 1 and minus 2 and minus 1, the

ROC cannot include poles and also you can see this is a now m here degree of numerator polynomial m equals 1 n equals 2, you have m less than n implies this is a proper rational function. So, what we have just solved or just considered as an example is a proper rational function and the Laplace transform is a proper rational function alright.

Let us look at the some of the properties of the ROC.

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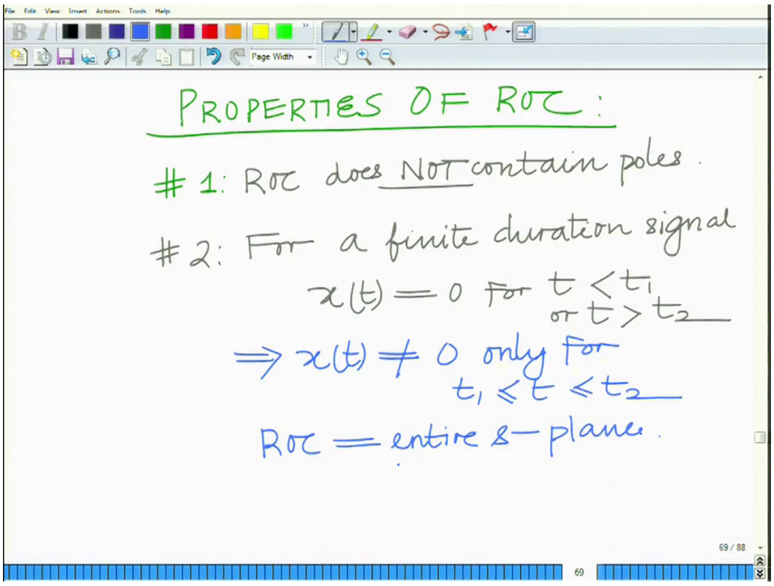

Now the properties of ROC, the first property as we already seen ROC does not contain poles. For a finite duration signal that is x t equal to 0 for t greater than or t less than t 1 or t greater than t 2 implies that signal is non-zero only for t 1 less than or equal to that is non-zero only in interval t 1 to t 2 it is a finite duration signal for such signal the ROC is typically the entire s plane. So this is a finite duration signal implies x t equals not equal to 0 only for t 1 less than or equal to t less than or equal to 2, then ROC equals the entire s-plane.

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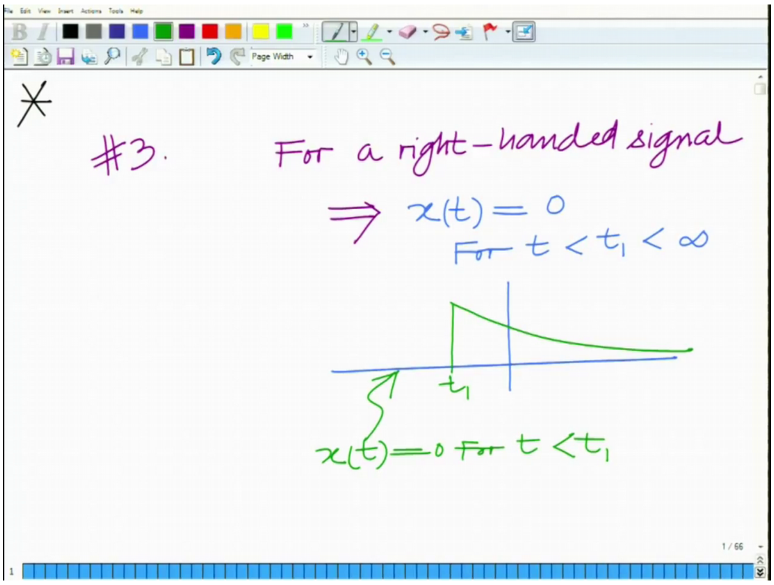

Now, for a right handed signal this implies that x t equals to 0 for all t less than some t 1 which is less than infinity that is this right handed signal is 0 it is some signal that looks like this.

So, at t 1 for all t less than t 1 this is 0 and non-zero only for t greater than t 1 that is it exists only on the right hand side it is non-zero only on the right hand side.

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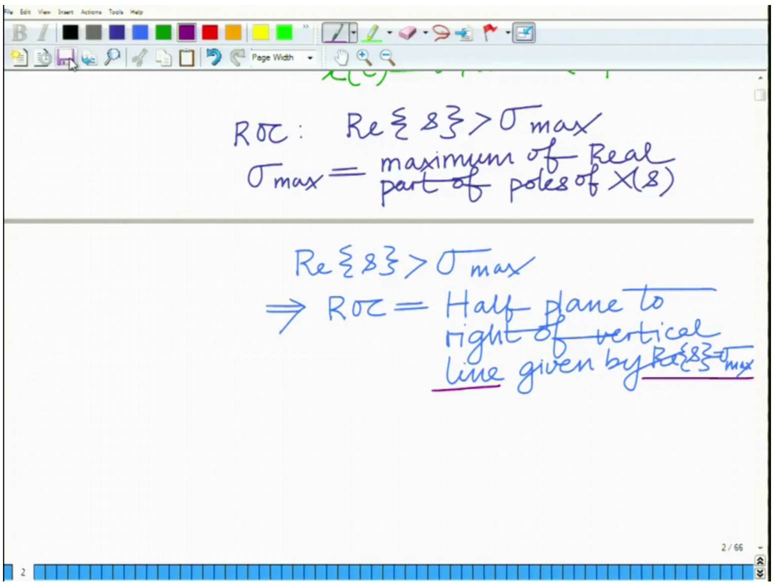

For this kind of a signal for a right handed signal the ROC is of the form the real part of s is greater than sigma max where sigma max equals the maximum of the real part of the poles of x s of the Laplace transform maximum of real part of poles of x s and this ROC remember the ROC is real part of x s greater than sigma max. This implies that the ROC is to the right half ROC equals the half plane to right of the vertical line given by real part of s equals sigma max. So, this is to the right of the line given by real part of s equals sigma max.

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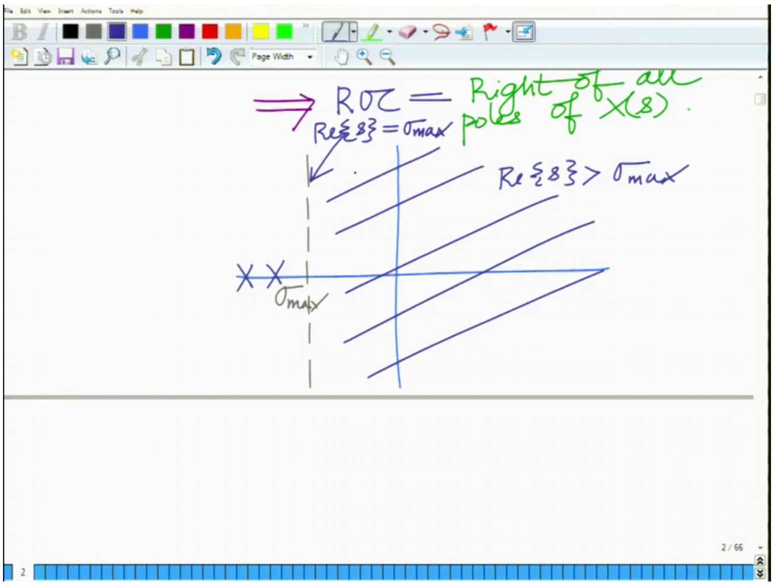

Implies this is to the right of all the since this is greater than the real parts of all the poles that sigma max is the maximum of the real part all the poles implies that the ROC is to the right of all poles of x s and the ROC will look something like this that is you have your s plane and you have your sigma max alright. So, this is let us say your sigma max then the ROC will be sigma max ROC will be a real part of s greater than sigma max and remember all the other poles can only be on the remember the ROC cannot contain any poles.

So, all other poles are to the left so the ROC is basically to the right of all the poles so ROC is on the left of this line given by real part of s so this is the line which is real part of s equals sigma max and of course this line itself is not included in the ROC.

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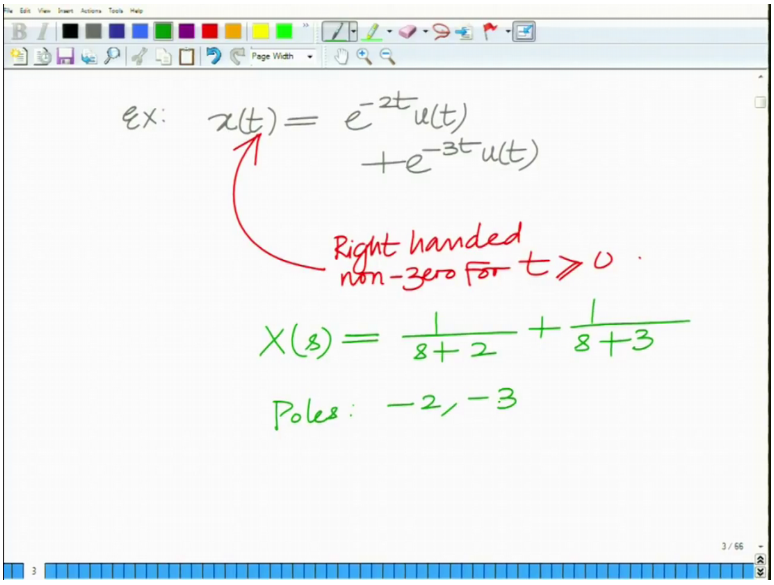

So, let us take a example, let us look at a simple example to understand this. Example consider x t equals e raise to minus 2 t; u t plus e raise to minus 3 t u t this is a right handed signal you can clearly see that correct it exists only for t greater than 0; non-zero only for t greater than or equal to 0 correct it is a right handed signal. And if you look at the Laplace transform of this that will be x s equals well that will be x s equals 1 over s plus 2 we have seen that plus 1 over s plus 3 the poles are minus 2 s equal to minus 2 comma s equals minus 3 and the ROC.

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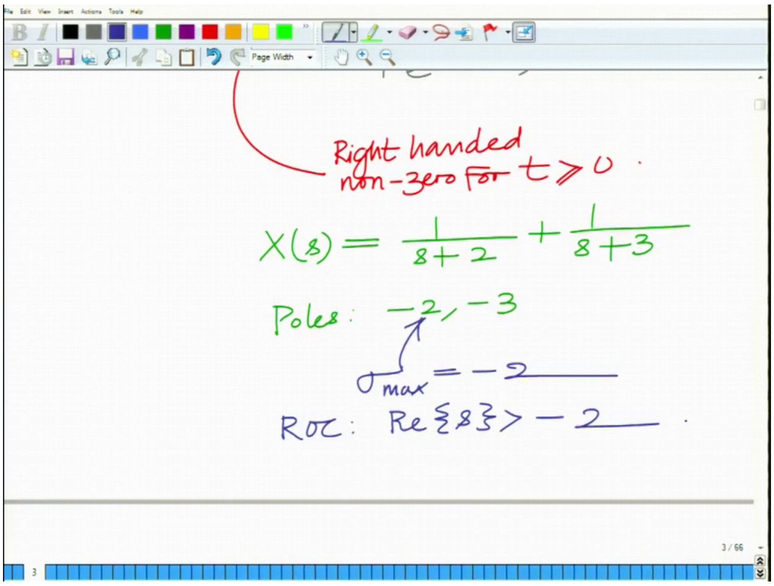

Now the maximum remember look at this minus 2 minus 3 this is sigma max is minus 2, maximum the real part sigma max is minus 2 and therefore, the ROC is real part of s greater than minus 2 and so this is the ROC and the ROC looks like this.

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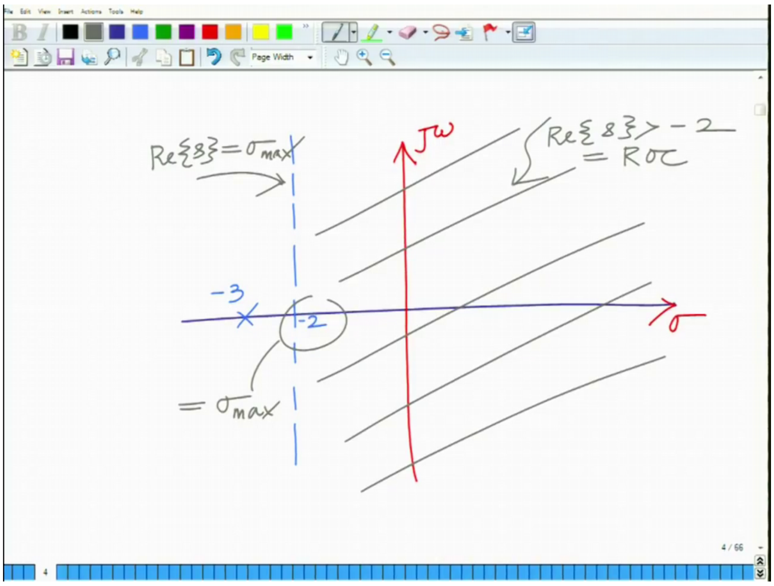

If you plot this, this is your sigma, this is your s plane, this is your sigma, this is j omega, x axis, y axis is the j omega axis. This is the line sigma max s equals this is a line a real part of s equals sigma max that is minus 2 and remember somewhere here you have minus 3 and the ROC is this part that is real part of s greater than minus 2, so all the other poles lie on the left of this line real part of s equal to sigma max, ROC is to the right of all poles.

So, this is real part of s greater than minus 2 and this is your ROC for the right handed signal. So, this is the ROC for the right handed signal and this quantity here this is equal to sigma max and this is the line real part of s equals sigma max and ROC is the regent of the right of this line real part of s equals sigma max and in fact it does not include this line real part of s equal to sigma max that is something important to remember ok.

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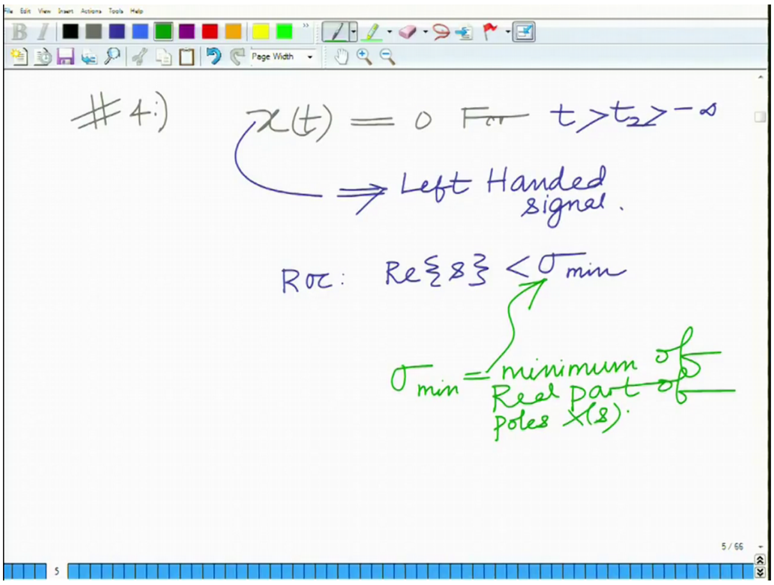

Now, let us look at what about a left-handed signal, for the left handed signal everything becomes the opposite of what we said so far for a right handed signal. So for x t equal to 0 for well t greater than t 2 a greater than minus infinity that is t 2 cannot be minus infinity and for this that is this is termed as a left-handed signal. This is termed as a left-handed signal and ROC of this is of the from real part of s less than sigma min where sigma min is the minimum of the real part of the poles of x s that is what we said it is opposite of the right handed signal sigma min equals minimum of real part of poles of x of s and this is a minimum of the real part of poles of the x of s ok.

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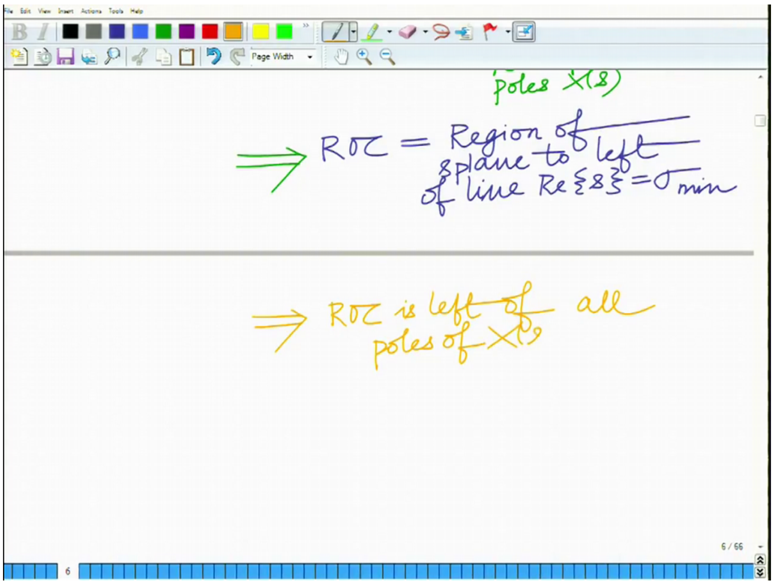

And implies the ROC equals region of s plane to left of line real part of s equals sigma min. So, ROC is the region to the left of this side in fact not including this line real part of s equal to sigma min and this also implies that is it is to the left of all poles of x of s implies ROC is left of all poles of x of s.

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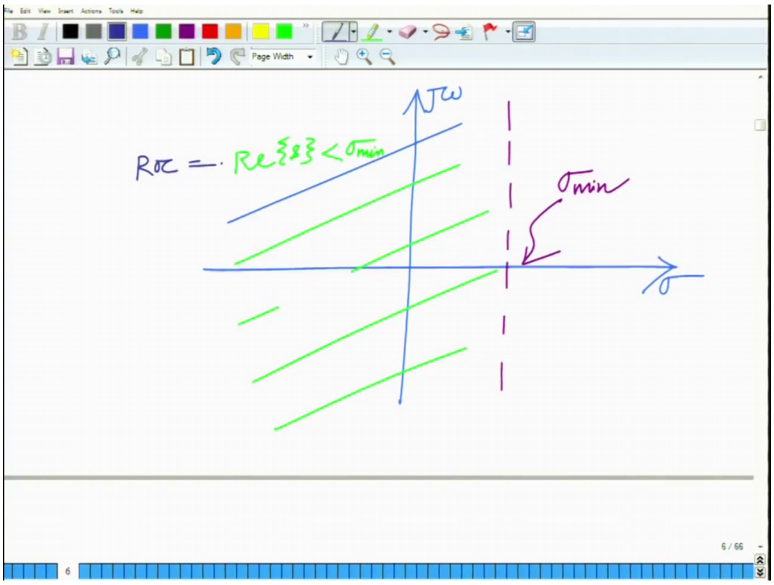

And the simple way to look at it again once again simple illustration s plane sigma j omega let us just do make it similar let us say this point is your sigma min, this line is real part of s equal to sigma min then the ROC will be less real part of s less than the ROC will be real part of s less than sigma min and this is your ROC, ROC equals real part of s equal less than sigma min.

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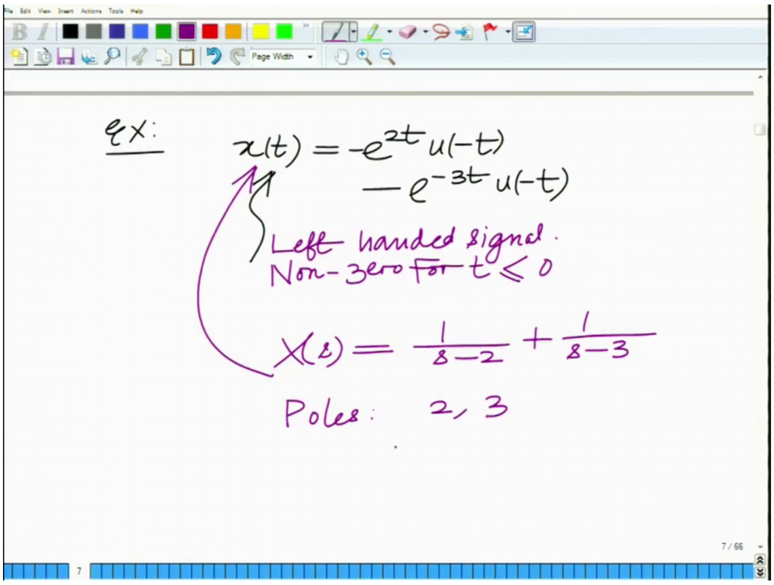

For instance, let us take a look at a simple example again. Consider x t equals e raise to minus e raise to minus 2 t u minus t minus e raise to minus 3 t u minus t. Now this is clearly have a left-handed signal exists or define non-zero only for t less than 0, non-zero for t less than or equal to 0.

Now if you look at the Laplace transform that will be again remember minus e raise to minus so this will be minus e raise to 2 t u minus t this is Laplace transform 1 over s minus 2 plus 1 over s minus 3 you can check that which implies the poles are 2 comma 3 sigma min.

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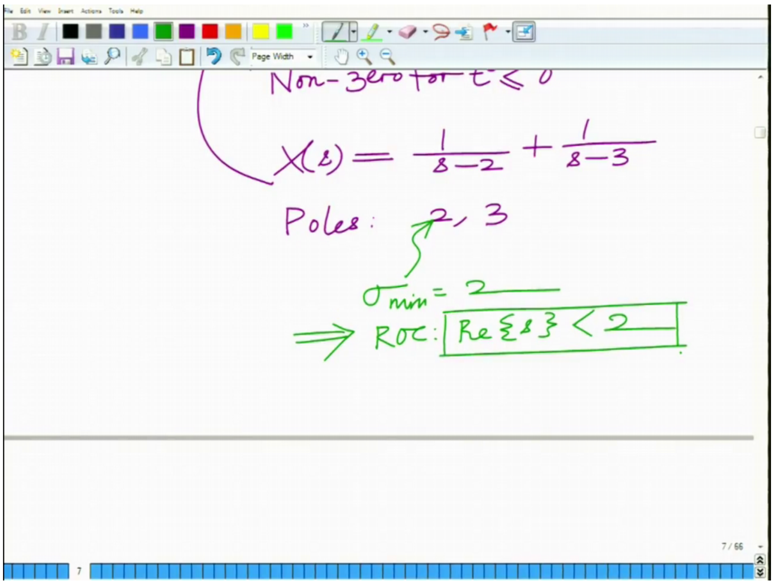

The minimum of the real part of the poles is 2 sigma min equals 2 implies ROC is real part of s less than that is your ROC.

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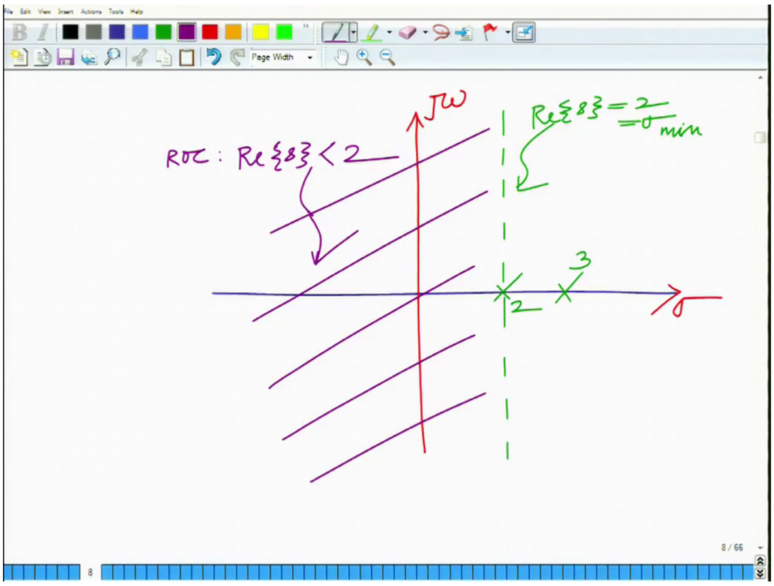

And the ROC plot again is given similarly ROC is plot is given; similarly this is sigma this is j omega axis and this is your pole which is 2 and of course, this is the pole which is 3, this is the line real part of s equals 2 that is equals sigma min and the ROC is to the left of this that is real part of s less than 2 and you can see all poles have to lie only to the right that is ROC has to lie to the left of all poles, ROC has to lie to the left of all the poles of x s alright. So, we will stop this module here and continue with other aspects on the sub sequent modules.

Thank you very much.