Indian Institute of Technology Kanpur

National Programme on Technology Enhanced Learning (NPTEL)

Module-51
Applications Matrix analysis of reflection From multiple boundaries

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Hello and welcome to NPTEL mook on applied electromagnetic for engineers in this module we will discuss a phenomenon known as van assent Papers which is again something that we don't discuss usually when we talk of total internal reflection. And then carry out a matrix analysis or formulate a matrix method of analyzing reflection from multiple layers of multiple boundaries okay.

So we go back to the total internal reflection and even assent waves first and then carry out the matrix analysis well this is the.

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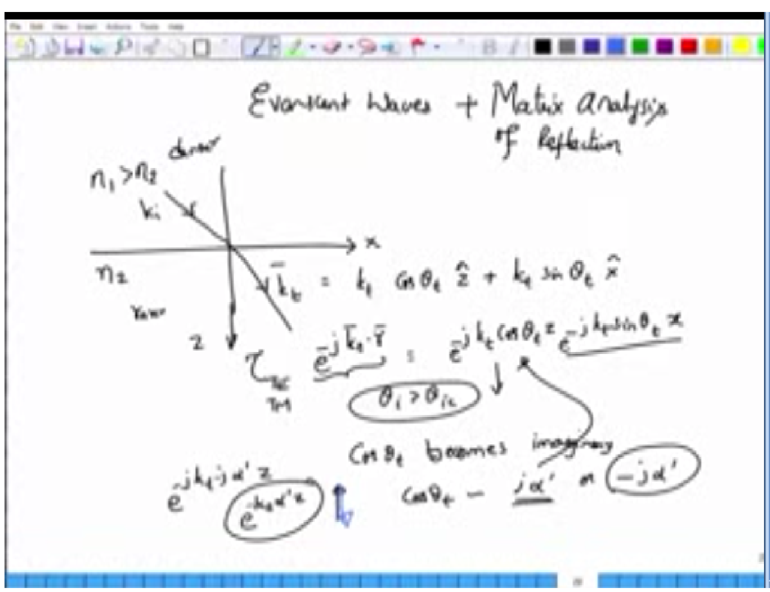

KI and KT vectors the incident and the transmitted wave vectors which we are considering we are of course considering only the total internal reflection phenomena therefore the first medium or the medium of incidence has a larger refractive index compared to the second medium which has a lower refractive index okay. So this we call as optically call denser medium this we call as optically rarer medium.

So denser and rarer medium now we have seen this expressions KT the wave vector is equal to the magnitude of the transmitted wave vector KT cos θ T Z had plus KT sin θ T X had this is obtained by simply decomposing the transmitted wave vector into two pieces right. We know that the transmitted fields will have some transmission coefficient whether that is TE or TM transmission coefficient.

But that x there is a phase factor e to the power minus J KT. is what I am very interested in substituting for KT you see that this phase factor can be neatly decomposed into a phase factor which is Z dependent and phase factor which is X dependent right. Now consider what happens when θ I is greater than θ IC we have already seen that in that case cos θ T becomes imaginary correct.

So this becomes imaginary which means we have to rewrite cos θ T as some J times let us call this as I do not know what we should call so let us call this as J α okay. Now I have two choices here should I consider it as plus J α or I consider minus J α right. But let us try and see what happens to the phase factor when we consider + J α and - J α and substitute that one into this expression.

So not α let us put out a different term for this so instead of +J α - J α let me call this as + G α Prime and - J α Prime okay. So when I substitute that back into the term I power - JK T cosθT into Z what happens to X is not really important because sinθ T is not getting imaginary so what happens to this one well you have E power minus J now KT is as it is cos θ T has become complex.

So I will put J α Prime into Z what is the product of - J into + J the product of - J into + J is plus 1 therefore this says E power KT α prime said now also prime is real KT is real Z is real and it is increasing as you go into the second medium right. Which means this wave keeps on increasing its amplitude as you go deeper and deeper okay? Clearly that cannot happen there is no amplification happening no amplifiers are connected.

So this choice of + J α is wrong okay if we choose - J α what will happen instead of a + J here I should replace this + J α prime by -J α Prime in which case you have -J times - J minus J times minus J is -1 therefore there will be a e power -KT α Prime into Z okay. This will actually decay as you go deeper into the second middle now that is more like okay. I mean you know that the waves can decay but the waves cannot amplify and go off into infinity.

As said keeps on going to infinity so this - KT α prime because KT is real α prime is real can be rewritten or redefined as a single α this α can be I mean this KT into α prime can be redefined as α itself. So you have KT α Prime and therefore what we see is that along Z the waves area it is a very interesting phenomenon in which the wave actually starts to transmit along.

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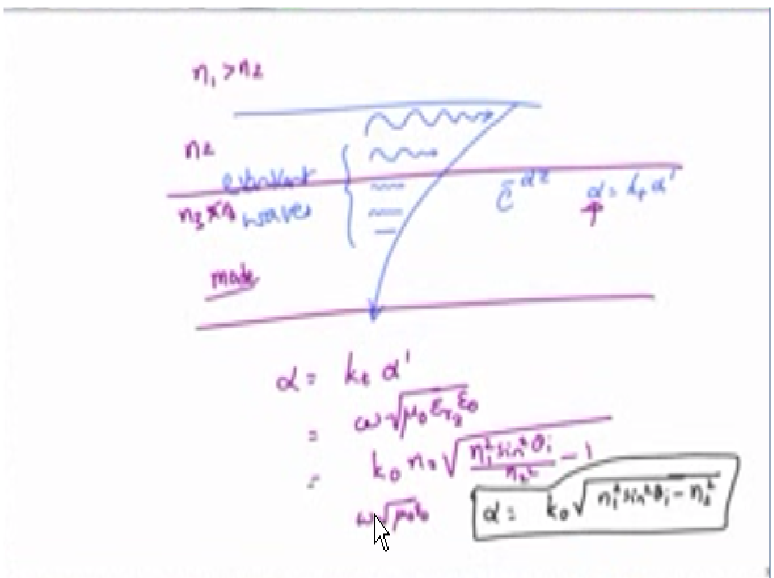

The interface but its strength reduces as it propagates and goes down into the second medium. So this error in the along the normal to the interface if you keep going the amplitude of as E power - α Z where α is equal to KT times α prime whereas along X it continues to oscillate. These waves which carry no energy but they can excite other modes or other structures are called as evanescent waves okay.

The evanescent waves are very important because if I choose one more structure so I have already chosen this one to be such that and one is greater than N 2and this is N 2 if I now choose a structure let's say N 3 and bring the structure close up so let us say I choose the structure over here and 3 not very far away okay. If I choose it very close by to this medium and let us say N 3 is greater than or N 3 is less than N 2 then the waves that are present here can actually excite the propagation of additional light that is formed by this structure of two parallel lines or this is a film of some refractive index N 3 that you can think of.

And it can actually excite waves or wave propagation or as we would actually call it the mode propagation or they excite the modes of this particular film which is formed by this purple color lines okay. Or between those two purple color lines. So this is the way of what is called as a variant wave coupling into optical waveguides which was very important in the 1970s and 80s when optical integrated circuits were first developed.

Even today this is sometimes used instead of having any generic boundary you actually put in a prism okay. Such that you constitute so you send light into the prism but you have the prism slightly above the optical film. So if this is the optical waveguide you have a prism over here you choose the gap such that you actually there are a few considerations in choosing the gap. But the gap has to be chosen such that it is not very far or not very close but an optimum gap if you choose and then have light incident into the prism okay.

And arrange the prism to the gap boundary to be a total internal reflecting boundary. The light will be reflected back into the prism okay. But there will be evanescent waves generated from prism in the gap and because of this film that is now kept right. It would generate modes inside the film okay. And you could actually couple light and excite light into the second medium so this gap coupling by innocent waves is a very common technique that was used and in some is sometimes used.

Even now in coupling energy or a coupling light into the second medium okay but we have not really found out what is the value of α well in one sense we have α is given by KT right KT can be KT times α prime KT is nothing but the frequency ω square root of μ0 ∞ R now which medium should be considered this is the second medium that we need to consider right. So this can be rewritten so times epsilon zero so this is nothing but K zero which is the free space wavelength or free space propagation vector as we would call it.

Which is given by ω square root μ.∞ times N 2 N 2 is the refractive index of the second medium what is α prime α prime was this cos θ T right when θ is greater than θ I θ IC so this was actually square root of N 1 square sin square θ I divided by N 2square - 1 okay. So you can rewrite the value for α as K0 square root of N 1 square sin square θ I minus so this M 2 square can be multiplied and when you pull into that will cancel with the numerator N 2.

So this is the value of α that we were looking for which corresponds to how much attenuation or how quickly the fields are falling off in the evanescent region. So this completes our analysis of evanescent waves and the total internal reflection now what I want to consider is a very interesting topic.

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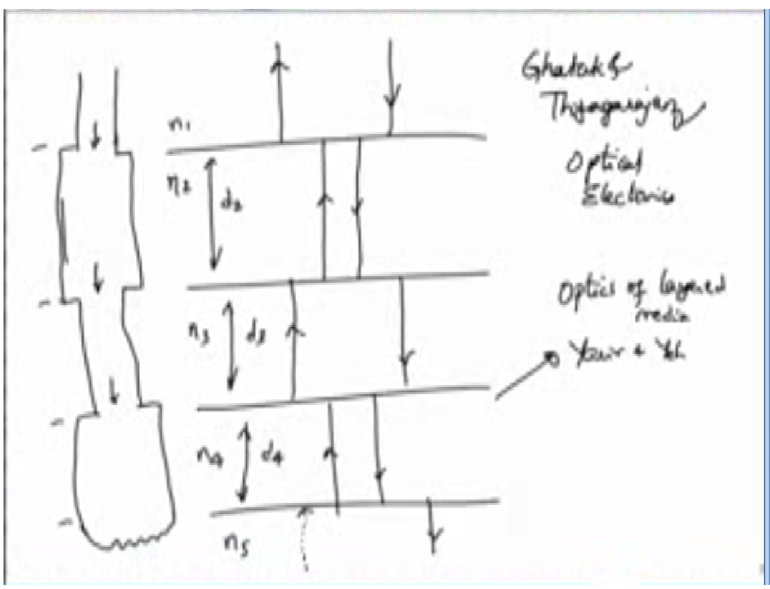

If you want more details on this topic I ask you to look at this book by Ghatok and Thyagaraj it's a very interesting book called optical electronics if you want even more details you can look at book called optics of layered media I believe that this is the correct title and this is by Yareve and Yew okay so you can look at this book or you can look at this book for the simplified analysis the idea I mean the development what we are trying to do is to put in some kind of a matrix method to analyze multiple layers okay.

The goal is to analyze a scenario where you have different layers or different material media this will have a refractive index N 1 this will have a refractive index N 2 and this will be at a certain length or thickness of D 2 this has N 3 and a thickness of D 3 and 4 thickness of D 4 and you have N 5. So this is a scenario where you have 1 2 3& 4 layers okay and 5 different media out there well.

We have already seen one way of understanding the propagation here so if suppose I send light in the normal incidence here I know that there will be a reflection back into the first surface there will be a partial transmission into the second surface right so the second media and because of the interference because of the interface between second and third media there will be partial reflection and there will be transmission there will be reflection there will be further transmission for the reflection.

And eventually you will see that light a partial amount of light has come out okay but for mathematical sense we also put in a back reflected or the excited kind of a source although there is nothing here so that is why I put this in the dotted lines right so it is one way to analyze you know one you know one way to analyze this particular structure is to actually it rewrite this one as a transmission lines.

So you actually have a transmission lines here right and then connect it to a different transmission line so let us say connected to a different transmission line then you connect it back onto another transmission line then one more transmission line which finally is the last medium is considered to be a load okay. So this is another transmission and this is the load okay so in between you have to write down what is the thickness or the length of each of the transmission lines and you can actually analyze the problem in this way.

There is no you have to successively first find the impedance here then transform to the impedance here then transform to the impedance here in order to obtain the overall reflection and in fact we have asked you to do that in one of the programming exercises given in the first week modules or something okay so you can analyze this way but this is well this is one way but there is another more popular way okay of analyzing this multi-layer films as we would call them and that involves the use of matrices okay.

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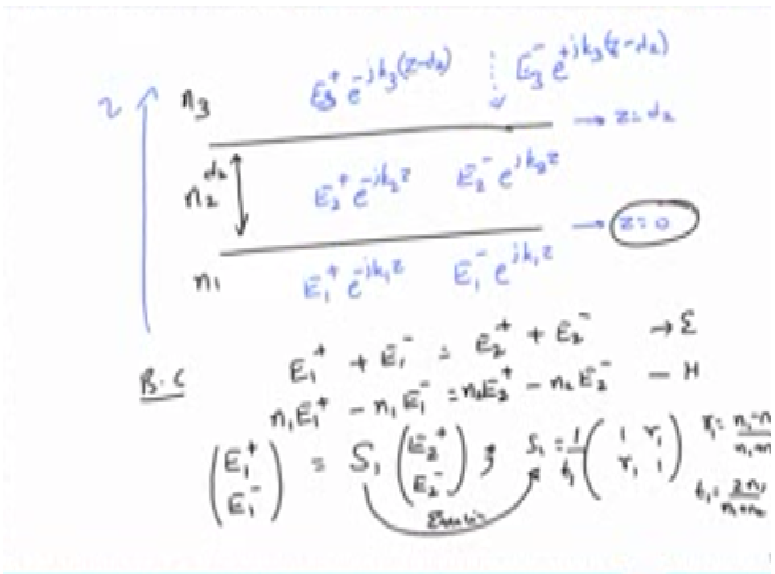

Let us develop this matrix analysis just for a single film or two boundary scenario extension to multiple layers is extremely simple we will talk about that one the scenario that we are sure we want to analyze is given by this one so you have a second medium of thickness D 2 okay having a refractive index of N 2 while in the third medium I have refractive index of N 3 the first medium or the incident medium has a refractive index of N 1.

For simplicity I consider only the normally incident waves okay so you could of course extend this analysis to the oblique incidence case as well but in the oblique incidence case you have to keep in mind whether are you analyzing T modes or TM modes okay and you have to use the corresponding formulas for γ T and transmission of TE appropriately or γ TM or transmission coefficient for TM and keep calculating θ T at every interface.

Now a little bit of complications are there so we want to avoid that complications to get you the essence of the method and this is the simplest case that we can think of is that offer normal incidence where there is no calculation of θ T θ T will be equal to 0 θ I will also be equal to 0 okay for we can begin the matrix analysis just as a reminder for ourselves what are the electric fields in the incident region.

There will be an incident field which would be propagating let us say this is the Z direction that I have chosen so this would be even + E power – JK1 ZK1 corresponds to the incident wave vector okay and there will be a reflection wave also because of the first medium interface medium interface between the first and the second medium that will have an amplitude of even - 1 meaning this is the first medium minus indicates it is the reflected light .

So it would be traveling positively in this way inside here you have let us say e 2 + e power - JK 2 Z and the reflected light would be e 2 - e power j k2 z this node is equal to 0 since this is Z equal to D 2 we write this as e 3 + and instead of writing this as K 2 Z it would be better if we write this as K 3Z - D 2 okay otherwise you will have to keep track of this e power minus JK z or KD 2 kind of a phase factor.

I do not want to carry this phase factor therefore I choose my coordinate system itself such that Z equal to D 2 stands as the Z equal to 0 kind of a plane for the third waves in the third medium as I said there is nothing coming off in this direction I am not kept my source here yet if suppose I had that one at least mathematically I can have I can write this as e 3 - e power - J + J k3z - D 2 okay.

So these are the waves that I have and these waves are actually trans you know going around in this particular way what about the magnetic fields well the magnetic fields will have for the positive z travelling waves right or the waves which are propagating along the forward direction they would all have the amplitudes reduced by the appropriate medium refractive index not medium refractive index by the impedance.

Since impedance is inversely proportional to refractive index the amplitudes of the H field will be multiplied by N okay. So e so rather H is e by Φ since Φ is inversely proportional to refractive index H will be N times e okay and for forward waves it would be positive and for the waves which are propagating along - center is reflected fields there will be minus n times E –right.

Because these are propagating along minus Z direction okay so the H fields have to go along - so that if the fields are all if the electric fields are along X then X cross minus y it should give you a long - Z so we this is the ampere but let us not put that one down right here we will do that when once we write the boundary condition for the first medium well boundary condition for the first medium what is the boundary condition here I know Z equal to 0 so even +even - must be equal to e 2 +e2 - in terms of the electric field.

What about the magnetic field well this would be even + N 1 - N 1 even - to be equal to e 2 +which is n 2 e 2 +- n 2 e 2 - okay so this for the electric fields these are for the magnetic fields now this gives you two sets of equations I can rewrite the equations in terms of the matrix form by writing this as the left hand side as even + and even - ok this can be written equal to as a matrix s 1 times e 2 +and e 2 - okay where I will leave this matrix to find the matrix s 1 as an exercise for you just have to add and subtract the equations after multiplying appropriately.

So you can take this N 1 on to the right hand side add the 2 then you will express even +in terms of e 2 + and e 2 - similarly if you subtract you will press even - in terms of e 2 + NT 2 – so if you do that one you will realize that this can be written as 1 by t1 1 R 1 R 11 e 2 sorry so that is the S matrix okay so the s 1 matrix which is applicable at the first boundary set equal to 0 seems to have a structure in which the it is 2 by 2 matrix okay because we are not considering the polarization this is a 2 by 2 matrix.

Otherwise you will have to consider a 4 by 4 matrix so T 1 is a transmission coefficient from the first medium to the second medium and R 1 is a reflection coefficient from the first interface so clearly R 1 is equal to N 1- n 2 by N 1 + n 2 and transmission will be equal to 2 N 1 by N1 + n 2 okay so these are something that you already know of so let this exercise you know you calculate that exercise I mean calculate that one and you know show that that is the correct expression.

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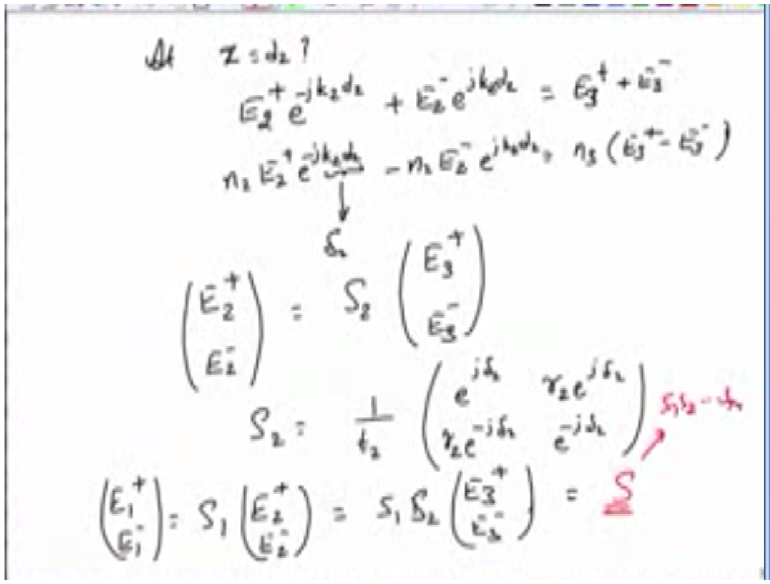

Now what is the boundary condition at Z equal to D 2 well this is the second medium interface right so again if I calculate that there will be a phase factor for e 2 + so that would be e 2 + e to the power – J k2 d2 correct + e 2 - e power jk2 d2 this should be equal to e 3+ e 3 - okay similarly for the magnetic field if you apply now that would be N 2 right this is N 2 e 2 +E power -JK 2 d 2 - n 2 e 2 - this is the reflected magnetic field this must be equal to n 3 into e 3 +- v 3 - okay so i hope you do not have any difficulty in writing these equations.

Now you can substitute K 2 Δ D 2 as you know some number Δ - okay and rearrange the equations again in the matrix form with the goal that you express Z 2 + + e 2 - which are the fields in the second way in terms of the matrix s 2 times the fields in the third medium III +and III – okay so when you write it in this way and calculate what is s 2 you will see that this would be equal to 1 by T 2 e to the power J Δ - R 2 E power J Δ - R- E power - J Delta 2 and E power - J Δ – okay.

So please verify where Δ 2 is of course K 2 into D 2 so please verify that this matrix what we have written is the correct matrix expression that we have okay now what is the advantage of writing all this well I have expressed e 2 + and E 2 – in terms of e 3 plus and E 3 - but I already know that e 2 + and E 2 - right times s 1 was equal to even + and even - right now this is nothing but s 1 since e 2 plus and E 2 - is related to e 3 + and D 3- why are the matrix s2 - .

So I can rewrite this as e 3+ e 3 – so essentially what I have is a single big s matrix okay which will be equal to s 1 s 2 all the way up to say s of n - 1provided you have n number of media that you need to consider so for each media you will have to find out what is the corresponding matrix s okay and that structure of the matrix s is very easily written down here for s 1 we have written s 2 we have written for s 3 you can very easily write this as you know 1 by T 3 E power J Δ3 and so on.

And so forth okay so this overall S matrix will be a 2 cross 2 matrix and therefore you can even write this as ABC and D okay thus relating.

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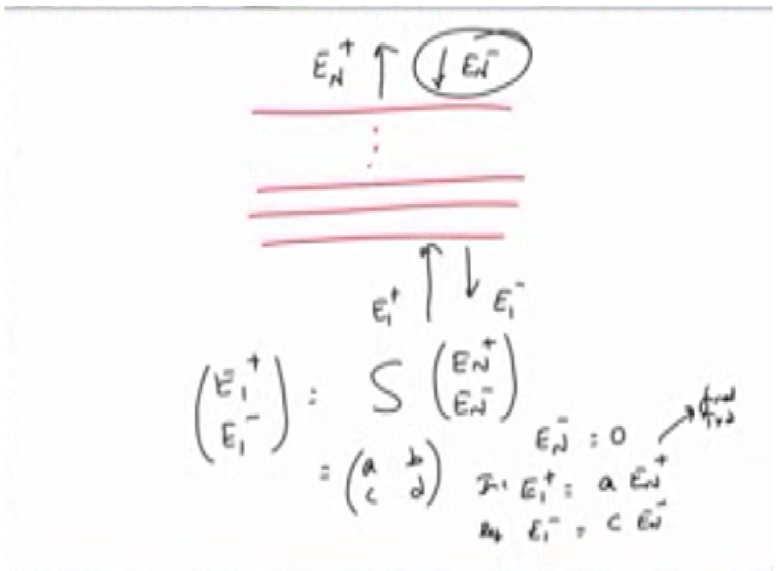

So you might have multiple layers ok so you might have multiple layers out there but if the incident fields are even + and even- and finally the transmitted fields are say e n+ + e n- although as I said this does not exist if you assume the source to be low-key in the first medium and for whatever the distances or whatever the thickness of each of the materials be and for whatever wavelength you can easily change the value of K and obtain the new values of Δ.

And for this case you can write down the overall relationship between even +and even - as a big matrix s times en + and en - okay so this overall matrix as I said will be a two by two matrix can be written as a b c and d now obviously I know that in this scenario I don't have any en – ok so after writing all the entire matrix I can substitute en - equal to zero when I do that I see that even + is simply given by a times en + okay.

This is the final transmitted electric field this is the incident electric field and then you have the reflected electric field even - related to C times en - right en + okay now if you want to obtain what is the ratio of say en + by even + .

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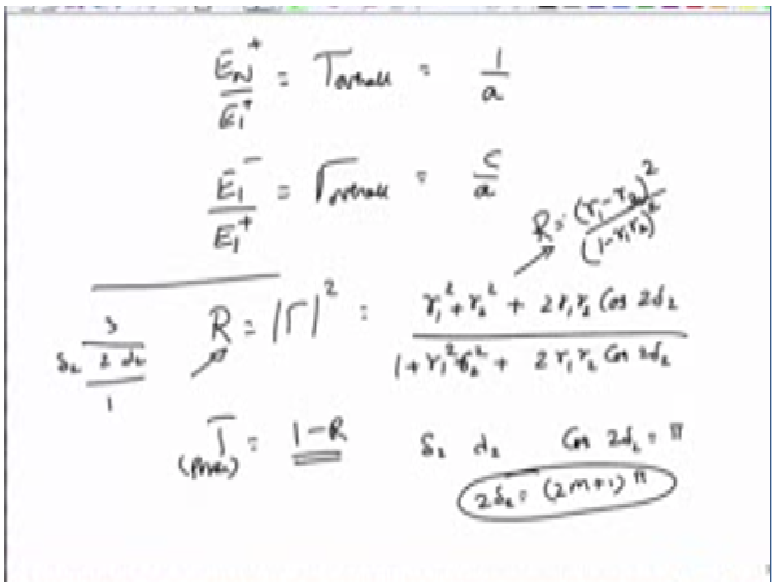

Which should tell you the overall transmission coefficient okay this is just the amplitude ratios these are not the power ratios these are amplitude ratios this would be equal to this would be equal to what 1 by a because e 1 + is equal to a times en + similarly if you now want to find out what is even - by even+ which is the overall reflection coefficient this would be equal to C by a right.

So that is all that is there this one now let's specialize or let us carry out the analysis for the case of that single film and 2 layer condition that we considered okay in the previous development and for that case that is for this particular case you have the medium one medium 2 and medium 3 with an appropriate length of D 2 or the phase factor Δ2 to the transmission coefficient sorry the reflection coefficient R which is related to the magnitude square or squared magnitude of the reflection coefficient.

This is the overall reflection that we are talking about a single film and this would be equal to R 1 square + R 2 square you can show this I will leave this as an exercise for you r 1 r 2 of 2 Δ2 divided by 1 + R 1 square R 2 square + 2 R 1 R 2 cos of 2 Δ 2of course you can find what is the transmission so this is the power transmission coefficient as 1 – R because these are powers are actually getting conserved power or energy is getting concerned okay.

Now let us see this power reflection coefficient or the reflected power as we start varying Δ2 either by varying the thickness of the film D 2 there comes a point when cos 2 Δ2 will be equal to π this will happen whenever 2 Δ 2 is some odd multiple of π correct so whenever there is some odd multiple of π when this happens is that correct yeah cost 2 Δ 2 is correct so Δ 2 can be written as 2 M +1 into π by 2.

But that is okay so whenever the argument becomes π or in one of the odd multiples of π cos 2 Δ 2 becomes -1 so the numerator for this can be written as R 1 - R 2 square okay and the denominator can be written as 1 - R 1 R 2 whole square so this would be the value for power reflected which is R 1 - R 2 whole square by 1 -R 1 R 2 whole square now we are actually at a very interesting point.

Suppose I consider a film and I take the same medium as this you know the third refractive index medium the third medium to be having the same refractive index as the first medium so clearly R 2 and R1 will be let us say this is the refractive index 1 the next one is about say 3 and next 1 is also 1 so then what is R 1 R 1 is 1 - 3 by 1 + 3 which is - something so - 2 or something 4 - 2 / 4 or something right.

Similarly what will happen to R 2 is given by n 2 - N 1 which will now be equal to - r1 right so when that happens then r1 r2 will be equal to - r1 so I can substitute for this in the in the expression that we have derived and I obtain reflection R to be some - R square divided by 1+R square okay there is nothing much interesting happening out there okay.

Suppose I adjust this in such a way that r2 equal to r1 then capital R can be made equal to 0 right so I can actually have no reflection coming back and the entire light could be transmitted into the second or transmitted into the medium out there okay we will see what kind of a solution that needs to be you know employed in order to solve this problem in order to make R equal to 0 in the next class where we discussed two important applications one is the multi-layer film that is whatever we have discussed we will extend the analysis from one layer to the multi-layer case okay.

And make a few of the remarks about that and then we discuss one film scenario where this kind of a situation is employed in order to make it act like a resonator called as a fabry-perot cavity or fabry-perot cavity resonator we will see these two applications but before that you should do these exercises and actually verify convince yourself that the expressions that we have obtained for the overall transmission coefficient reflection coefficient and for a single film the expression that we have obtained is actually correct so until then thank you very much.

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