Indian Institute of Technology Kanpur

National Programme on Technology Enhanced Learning (NPTEL)

Course Title
Applied Electromagnetics for Engineers

Module – 34
Electrostatics –I: Solving Laplace's equation in 1D

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Hello and welcome to NPTEL mook on applied electromagnetics for engineers. We were discussing in the previous module the Laplace's equations and its use to find out electric field, we setup the problem of parallel plate in a conducting system, we will solve that system in this module first to show you how one can calculate electric field E and corresponding in the quantity capacitance of these two parallel plates using, by solving the Laplace's equations.

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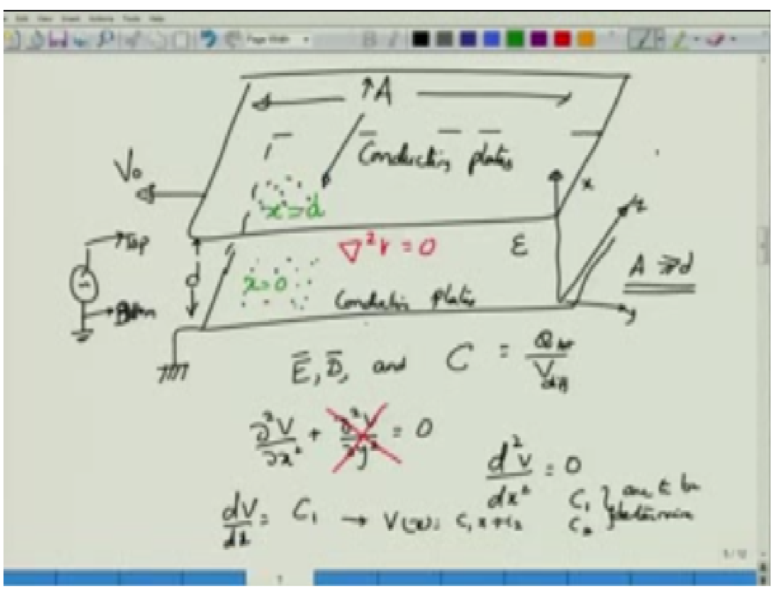

Now we see that Laplace's equation between the regions will have this form right, so $\delta^2 V=0$, why it should be equal to 0, because I am going to assume that the medium in between is filled with the perfect insulator having the permittivity of ε okay. So it is a complete insulator in between these two plates. As I said the plates themselves have an area A which is quite large compared to the thickness of the plate.

So A is much, much larger than D, there is a reason why I consider A to be much larger than D, because all the calculations that we are going to do are very nice and very excellent calculations as long as you are in the center of these plates okay. When you come to the edges, the electric fields, you know you will see that one, so that in other modules that these electric fields would not be normal to these conductors and they will actually bulge a little bit okay.

So this bulging phenomenon is what is called as bulging of electric fields is what is called as the fringing effect and the fields that you are going to generate will be what is called as fringing fields. So we are going to going to neglect this fringing field, because my goal is to try and show how to solve this Laplace’s equation and then maybe you obtain the capacitance value for this system assuming the condition very much larger than D.

Let us complicate our life in the first example itself. So I have $\delta^2 v=0$ in two dimension, this is nearly the case of a two dimension, I can write this as $\delta^2 v/ \delta x^2+ \delta^2 v/ \delta^y 2=0$ with the x-axis taken perpendicular to the plate, so this is how the x-axis is going, and then you can consider this to be the y-axis, and along this axis will be the length of the axis z which we do not need to worry about it at this particular point.

You do not even need to worry about this second term, this $\delta^2 v/ \delta y^2$ term, because the condition or if the symmetry of the problem is such that if you move around or if you move from the top plate to the bottom plate or from the bottom plate to the top plate you are actually seeing or you are moving at a certain distance right okay, and then you, so when you start with the bottom plate as you keep moving and you hit a distance of x=d which happens to be very quickly right, then you see that there is a second plate okay.

However, if you stand in the middle and then move side way, so you stand here and then you move along the Y, so we assume that this length of this one and hence the total area of the plate is so wide, that you will have to move substantially in order to encounter the end and because of that reason we can neglect any variations of the potential V with respect to the Y-axis. As you keep moving along the side ways okay, either on to the left or to the right you encounter the fringing fields or you encounter the boundaries at a much, much larger distance compared to the fields that you are encounter.

So is the plates that you encounter when you move along the x-axis. So the fields are strongly dependent on X, and extremely weakly dependent on Y which allows me to completely neglect this $\delta^2 v / \delta y^2$ term. So the equation that I am trying to solve now becomes very simple, I have $d^2v/dx^2=0$ how do I solve this integrate once integrating once gives you dv / dx = some constant one integrating this one again gives you $V(x) = C_1 x + C_2$ this is something that you know from your differential equation, so integrating this one you get $C_1x + C_2$ where C_1 and C_2 are to be determined okay we do not know what these are and we need to determine do we have enough information to determine this constant C_1 and C_2.

Yes we do we know that the potential everywhere on the bottom plate happens to be equal to 0 and the bottom plate is actually kept at x = 0 and potential everywhere on the top plate happens to be equal to v0 and the top plate is kept at x = d okay.

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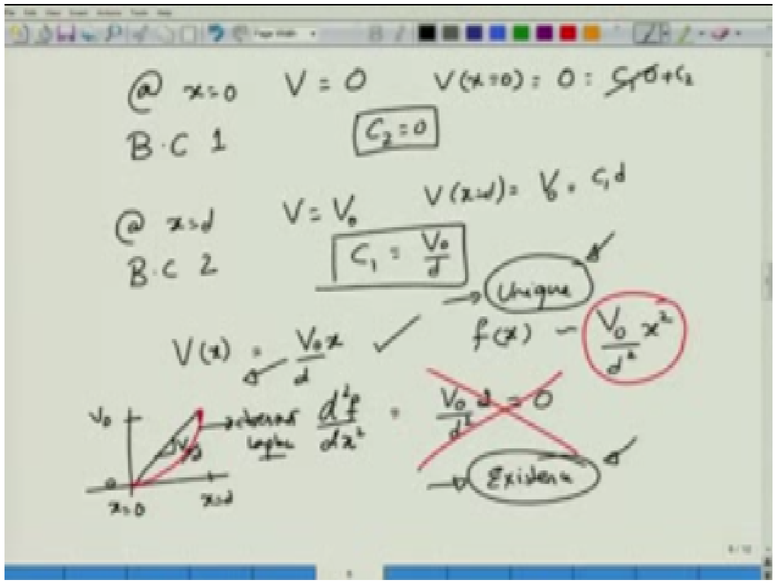

So I apply the condition that @ x = 0 the potential that I am finding or the potential that I have must be equal to 0 everywhere because this happens to be the bottom conductor which as grounded so go back to the equation V(x) and substitute x = 0 which will give you 0 okay that must be equal to C_1. 0 + C_2 right $C_1x + C_2x = 0$ clearly C_1. 0 will make this entire thing go to 0 and the condition that you are going to get for this one is that $C_2 = 0$.

And the condition that I have applied is actually the boundary of the problem right so the bottom plate defines the boundary and therefore this is called as boundary condition so I applied a boundary condition one saying that @ x= 0 potential v = 0 substituting into the expression that we have obtained allows me to find out $C_2 = 0$ @ x = d potentially sum applied potential V_0 so V @ x = d = V_0 the potential on the top conductor this must be equal to C1. d right x = d.

There is no C_2 here C_2 is already 0 so this allows me to write C_1 as V_0 /d so my general expression for the potential V(x) can now be rewritten as v0 / d so V_0 / d times x does it stratify the two boundary conditions the potential V(x) that we have written does it satisfy the potential are the boundary conditions yes, @ x = 0 this right hand side quantity V_0x/ d0 at @ x =d this right hand side quantity is equal to V0 which is what we have for the upper conductor or the top conductor okay.

I have now found out V(x) there are two questions I need to answer is this potential okay unique in other words is there another function of x which would satisfy the boundary conditions okay and stratify Laplace's equation unfortunately the answer is no okay so you might for example think of this I am just trying to pull a certain this one out from my side so let say V0 / d^2 into x_2 okay so this field clearly will satisfy the boundary conditions so at x = 0 this f of x = 0 and that x = d because d^2 in the numerator cancel to this one of denominator this seems to be satisfying the boundary condition okay now if I try to find out what is d^2 f / dx^2 right which is what I should be able to find.

When f of x is happens to another potential so new or another voltage then differentiate this one twice what you get twice differential of x_2 = 2 okay is this quantity = 0 now this quantity is= 0 so in this case you had a situation where the true solution, so if this is x = 0 and this x = d the potential at x = d happens to be V0 the potential at x = 0 is 0 the solution in between has to have a linear leave varying.

Because you see that this is the true solution right so this V0 / d into x the slope of this line is V0 / d and when you go as distance of d we actually reach the potential of V_0 the potential that we had here, was a quadratic one that is assumed f of x is a quadratic one which also starts a 0 here and ends but V0 but it does so in a quadratic manner unfortunately, this potential does not satisfy Laplace okay and hence cannot be a solution in fact the solution that you obtained if it is satisfies Laplace's equation.

If it is satisfies boundary condition that solution will always be unique, okay this is a very important thing to note the second question is can I always find the solution can I always Laplace's equation and then actually obtained a solution, I have already seen that if I obtain a solution subject to boundary conditions of the problem and the fact that solutions satisfies Laplace's equation that solution will be unique but when I always be able to find a solution turns out that except for a pathological cases.

Which we do not consider there is a nice theory which tells you that these Laplace's equations usually have a solution okay, so we do not really bother with the existence problem at this stage there only concerned with uniqueness and we have not proved this uniqueness in a very rigorous manner but we have given you the hint has to what can happen, so you might have a situation that once you obtained a solution if that solutions satisfies boundary condition and if a solution satisfies Laplace's equation.

Everywhere then that solution is unique no other method can give the different solution right because if the other solutions exist then either they must be some linear multiple of the existence solution which anyway you have taken care of or if other solutions exist then either they do not satisfy Laplace’s equation but satisfy boundary conditions or satisfies Laplace’s equation but do not satisfy boundary conditions, right. So in that manner you can actually see that there is only one unique solution for Laplace equation and that is what we are trying to find, alright. Now we have found a potential is it possible for me to find out the electric field why, yes.

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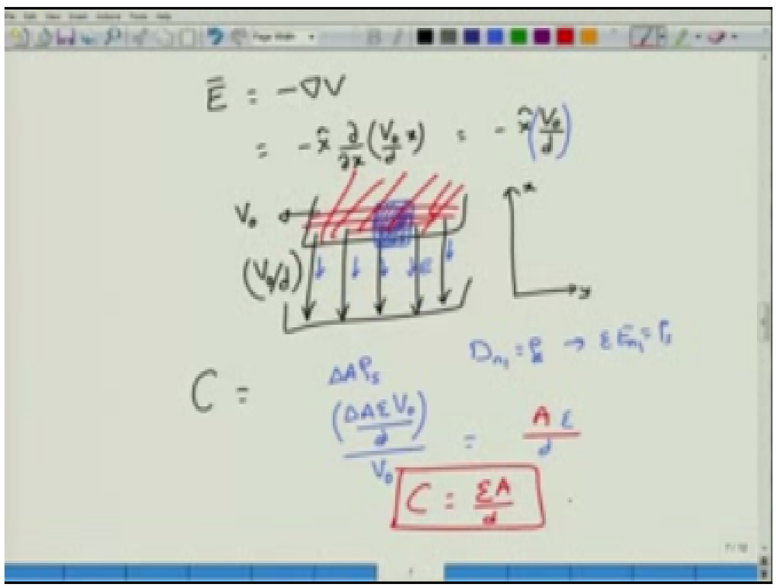

Electric field is given by – ΔV and in this case V is a function only of X so this becomes –x^ ∂/∂x or rather d/dx in this case since V is a function only of x of V0/ d x and what is this, differentiating this fellow what you get is – x ^ okay, V0/ d tells out that the potential you know is actually negatively directed which means that it is directed from the top plate that make sense because top rate is the one that was kept at a higher potential V0.

And from the top plate the electric field happens to have a constant value of the electric field, okay. So this – sign is telling you that the assumed x direction was from bottom to top so this was my x axis right this was my y – axis the electric field is directed from top to bottom therefore there is you know lines coming from the top conductor to the bottom conductor out there okay and the magnitude of these is given by V0/D.

So we see here that the electric field is you know coming from the top conductor to the bottom conductor and it has a value of V0/d so it is a constant conductor, now how do I obtain the capacitance well to obtain the capacitance I need to know what is the total charge told either on the top conductor or the bottom conductor now I know from boundary condition that if I consider you know a small region here, right.

Which might be the region that I consider over here than in this region which just extends below and I know that there is nothing in the below out there so if this small region which has an area of say ΔA right the corresponding charge enclosed must be equal to ρs ΔA, ρs being the surface charge density but boundary condition for D tells me that if I have this is a electric field the corresponding D field will also be like this, right. So D field happens to be normal to this top conductor and the normal D field because there is no second magnetic there is no second D field so there is only one normal D field that must be equal to ρs, okay. And D and 1 is nothing but ε times E and 1 that is equal to ρs.

So I can substitute for ρs as ΔA ε E and 1 but what is the electric field the normal electric field we have already you know obtained and that is given by V0/d okay so this is the charge that I have now this divided by the potential to which we have you know applied the potential difference between the two plates is actually given by V0 and this ratio will give you the capacitance this is the slightly more rigorous way of solving this one I am skipping that one in the interest that I can get to another interesting problem, so this is the basic idea which is still okay, so you see that the potential difference is v_0 that we have already specified the total charge is given by ΔA over this small patch ΔAεv0/d clearly v_0 cancels on both sides and what I obtain is ΔAε/d, okay.

Instead of considering this small patch of area I can consider the entire area itself so when I take the entire area I am trying to you know sum up many, many ΔA's such that they will all sum to one single A and then I get εA/d or Aε/d so I usually remember this in the form of εA/d and this happens to be the capacitance of a parallel plate capacitor. So you might have actually used these relationships in your earlier studies without really you know see where this relationship is coming from.

The moment you have kept to you know conductors are two different potentials or a potential difference between them there will be electric fields generated, because there will be a different potential there will be electric field and this electric field is usually directed from the higher potential to the lower potential and this electric field you know if it leaves a particular conductor it will always leave at a normal case, so as soon as this is uniform it will leave at a normal case.

If is not then it will bulge and show all the fringing effects which we will not deal right now. So our goal after looking at this parallel plate capacitor which might have seen to be too excessive is to consider another interesting you know example of a transmission line and remember parallel plates can also be used as a transmission line or their cousins parallel wires can be use as a transmission lines. Now we are considering another transmission line.

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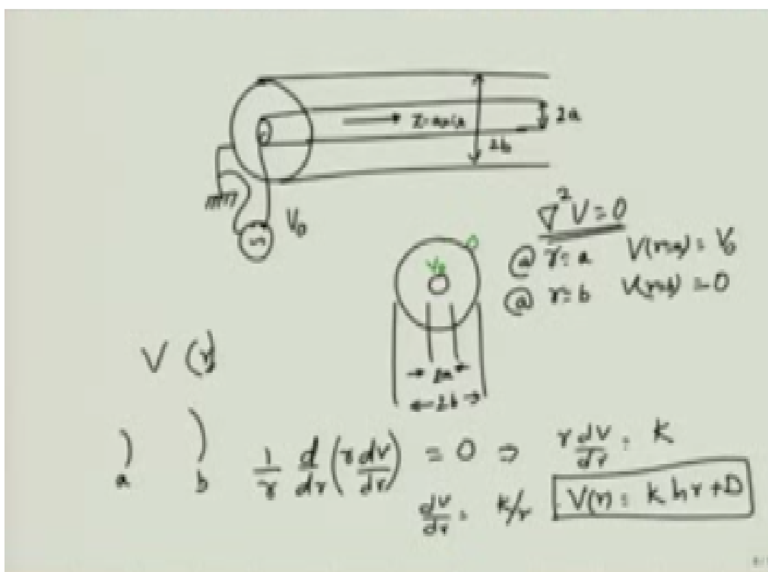

Which we call as a quaxial transmission line, so we have already discussed this quaxial transmission in the quaxial line what you have is a inner conductor of diameter 2a and an outer conductor of diameter 2b. Usually you connect the outer conductor to ground and then the inner conductor you connected to a voltage source okay, so you connect a certain voltage source to this one so that inner conductor is usually at a higher potential and a constant potential we will again consider that the inner conductor is at a potential of v0 okay. But then we have a small problem here, well the structure is actually cylindrical you know system right, it has a cross section which is circular but then it also has a axis along z axis okay.

As long as we consider a uniform cross section and the cross section appears to be something like this okay, the inner circle has a diameter of 2a the outer circle has a diameter of 2b as we have discussed okay, so in this case and I have also kept the inner conductor to be at a higher potential which is at v_0 and I have consider the outer conductor to be at a potential 0 volt okay. So if this cross section is maintained as you keep going along the z axis then it is possible for us to obtain electric field and calculate the capacitance of this structure, okay without actually solving Laplace's equation in the three dimensions.

So we are going to solve Laplace's equation in the two dimensions so you have $\nabla^2 v=0$ and now we need to first specify the boundary conditions as well, well we have already specified the boundary conditions by keeping the inner conductor at v_0, thus at r=a right, this is the first boundary condition the potential at a will be equal to v_0 and the second boundary condition is that at r=b where r is the radial distance, okay.

The potential v at r=b will actually be equal to 0 okay. Now v can depend on r, ϕ and z in general since we are considering only two dimensional Laplace's equation so we have no z dependence but now can r potential depend of φ, again do the test suppose this is my coaxial cable okay now if I keep moving around this coaxial cable does this cable look any different right so imagine this and then keep moving around like this does looks any different in this case the pen looks different because there is some mark out there.

But if you just imagine this surface and you keep moving around there is absolutely no difference so the potential does not have a φ component that is as you move around there is no difference out there, however if you keep moving away from this one right and you reach the potential or you reach the circle at r = b then the potential have to drop to 0 there. So the potential must start here at V0 and as you move away from it and reach the second conductor you must have to I mean it has to drop to 0 there.

So this potential is actually function of only r okay, so likely for us our problems are not so difficult we have the potential only as function of r. only small difficult is that we need to know what is this $\nabla^2 V = 0$ in the cylindrical coordinate system and I would not you know I bore you with the details of derivation of that one but I will just give you quickly what is the corresponding expression, since V is function only of r I can replace the partial derivatives with the full derivative.

So I have 1/r d/dr or dv/ dr this being the ∇2 v component for which this particular problems is concerns so for this problem ∇2v = 0 actually reduces to this equation. Vr of course not considering the case when r = 0 our boundary start at a and then end at b therefore this equation equal to 0 also implies that r Dv/ dr must be some constant k and therefore dv/dr = constant k /t which allows me to you know integrate once more and then say v® = k log R +d.

So this is the general expression that I have I need to apply the boundary conditions and then find out the actual values of k and d.

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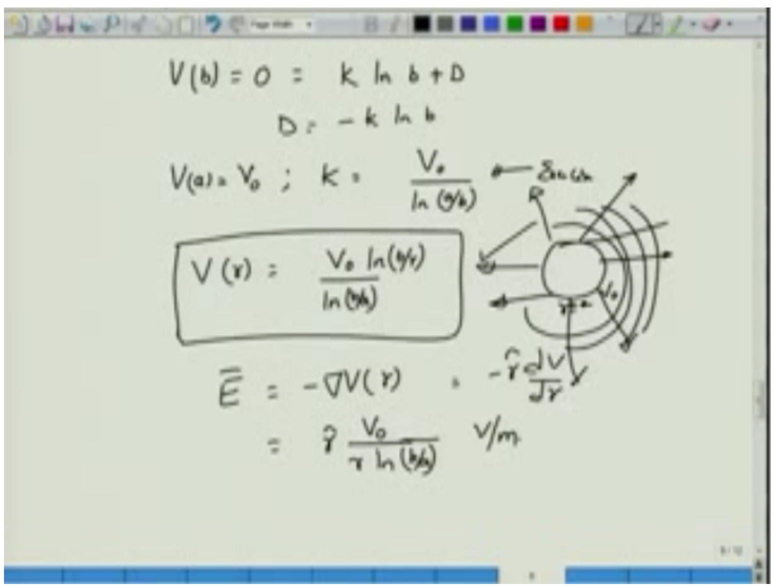

So I will do that one by first considering the boundary condition at b because I know v(b) the potential at that r = b will be = 0 and this is = k log (b) + D and D of course you know solving this one gives you –k log b, then v (a) = V0 that is a inner conductor is kept it a potential v0 so this is V(a) = V0 okay. And this one once you put ion you can then solve for k, k happens to be v0/log (a/b) I will just leave this as a small exercise to you to show that this is the this is indeed the correct solution, and therefore v® anywhere between the regions of inner and outer conductor this v® is given by v0 / log (b/a) b is large than a time slog of (b/r)

So this potential does it satisfy Laplace's equation I hope you can show that it actually satisfies Laplace's equation and what happens when r = b the potential drops term to 0 because this wound be log of 1 so it is gone what happens when r = a this would be log of b/ a in the numerator divided by log which is the natural log I am talking about log of b / a, so that will actually b = v0.

So this is the unique solution for v how about electric field e? E is minus gradient of v again you need to find out what is the expression for the gradient in cylindrical coordinate system and then you can you know show that this is simply equal to –rˆ dv/dr I just now obtain this one from the math hand book and then if you put Interviewer: his equation and then solve it you obtain the electric field as radically directed r0 v0/rlog b/c so there is some constant out there but then the electric field actually along radial but as if keep going away the electric fields is actually dropping down oaky and then the potential of course starts off with V_0 at r=a and then in logarithm b/a goes down to 0.

So the constant potential so this is at r=a the potential is V0 the potential actually goes as a log all the way up to 0 so you will have constant potentials okay so these are the constant potential contours and then the electric field will be perpendicular to this and the electric field in this case will also be going down in terms of the radial direction and it will be going as 1/r do not be the constant electric field out there.

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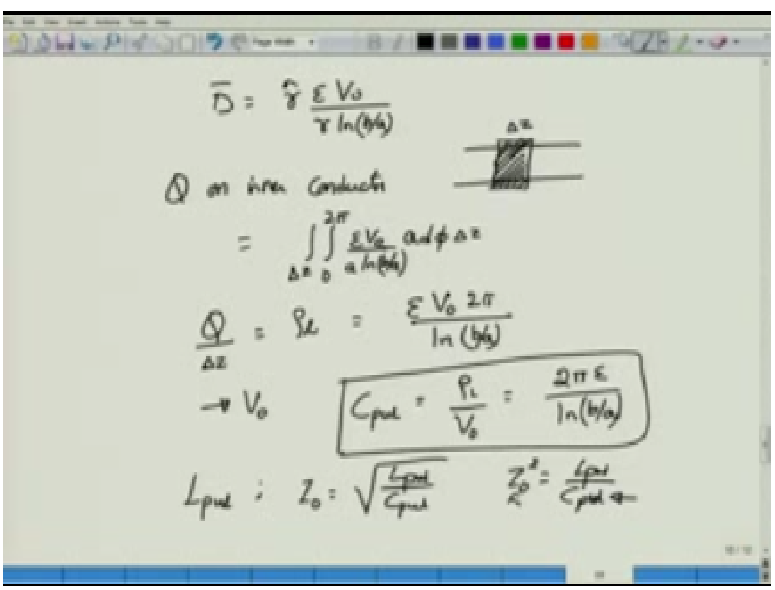

And this is the electric field at you are going to obtain now that you have obtained you know the electric fields then what will lead to do is to kind of find out what is the overall charge so I know D is related to electric field in terms of ε times E so D will be given by r^ εV_0/r log b/a now what I am interested is to actually find out what is the charge on the inner conductor again I have to enforces that I am not going to consider the Z axis okay.

So or I will consider inner conductor means actually want to consider very small amount of Δz okay and then put a surface around it so this is the inner conductor over which I am trying to find out so once I find out what is the charge here then I take the charge divide by Δz so I have to take away all the z dependence okay.

So you can show that this is nothing but integration over Δz 0 to 2π right because this one is happening at r=a we have to evaluate D at a okay and ds is given by r or adϕ right so this adϕ along r direction and Δs so I am going to integrate this one over Δz as I told you and you evaluate D at A so r=A will be the condition so you have εV0/a log of b/a right.

And then charge divided by Δz is what I will call as line charge density ρL and this line charge density ρL will be equal to after you integrate this equation and since this integral over Dϕ will simply pulled out it to 2π to you 2π at the outside of the integral so you get εV0 2π a will cancel in the numerator and denominator you get log of b/a okay.

Now what is applied potential the applied potential difference between the two conductors is V0 so now if you define what is called as the capacitor per unit length it get ρL/V0 which is a line charge density on the inner conductor I am only considering the inner conductor you can do the problem by considering the outer conductor is slightly more difficult to apply gausses law top the outer conductor.

Therefore we did not consider that but over the inner conductor you have to consider the closed surface and this surface has to have a small you have a small length Δz along with it okay then you divide all the quantities with respect to Δz in order to take away the z dependence so now you have a capacitors per unit length ρL/V0 which is given by 2πε/ log of b/a.

This is the very important relationships for quaxial cable you should remember this quantity or this particular expression and you might has how about the inductor per unit length when why am I interested in inductor per unit length because I know that for the quaxial cable the characteristic impedance Z^0 can be given by inductiveness per length divided by capacitor per length.

And there are ways of actually measuring what is Z^0 okay so since I know how to obtain Z_0 I can first get this I can Z^0=Lpul/Cpul and I know how to measure this left hand quantity Z^{02} I know how to calculate these capacitors, calculating inductions in this case is simple but it is not really necessary. So if you know what is the z_0^2 co-axial cable, you know what is the capacitor per unit length co-axial cable calculate from this expression, you can easily substitute for C_{pul} and then write down or find out what L_{pul} which is the length and this given by 1/ $z_0^2C_{pul}$. So this completes our couple of solutions for lap lasses equation.

You might have a lot of questions over here, so what we have covered is just merely scratching the surface. We will consider the two dimensional equation and solving in the next module but here is one very important question. We talked of parallel plate capacitors, we talked of parallel wires, we did not calculate or we did not solve the equation for the parallel wires but you can do as well. And then we can calculate the capacitors of a very important transmission line called as co-axial transmission line.

However you must have notice something co-axial lines, wires, plates they all are suppose to be used at a very high frequency. Which means that the potentials that you are applying are not going to be constant potentials right, so there will be change in respect to the time. So in those cases whatever the capacitors that we have calculated, is this formula valid. The formula that we have calculated comes from the electro static point of view. Where the fields are not varying with respect to time, so strictly speaking these formulas and the field that we have obtained is all valid only at Dc frequencies.

That is only at dc not when the frequency is different. For low frequencies one can neglect the changes in the electric filed and consider, pretty much to be constant, but to be really true, these equations must have, what is called as the correction term. And this correction term can be calculated using Maxwell equations or you can use what is called as quasi statics analysis and then calculate the correction term and keep adding these correction terms until your solution essential converts.

And this quasi statically analysis is a subject of some other module in this particular course. So in the next module we will talk of 2 dimensional Laplace equations and then solve the equation to round up our study of electro static, until then thank you very much.

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