Indian Institute of Technology Kanpur
National Programme on Technology Enhanced Learning (NPTEL)

Course Title
Applied Electromagnetic for Engineers

Module – 27
Divergence, Curl, and Laplacian operations

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Hello and welcome to NPTEL mook on applied electromagnetics for engineers. In the previous module we discussed gradient which is one of the operations that we perform with a del operator that we introduced. In this module we will quickly look at divergence, curl and laplacian these are the additional operations that we perform more often than other operations. And there quite important on their own and they have a physical meaning.

Remember, that the physical meaning of gradient is that it allows you to determine in which way the function is changing at its maximum okay. That applied for the concept of a scalar field okay, you do not have a definition of gradient for a vector field, because at each point you do not have a scalar function, but you have a vector function. But there are certain operations that we can perform with the vector fields which group to be very important for our electromagnetic discussion.

And the first operation that I am going to discuss is what is called as divergence operation okay. To give you the physical meaning of divergence first of all, let us assume that I have some isolated point okay.

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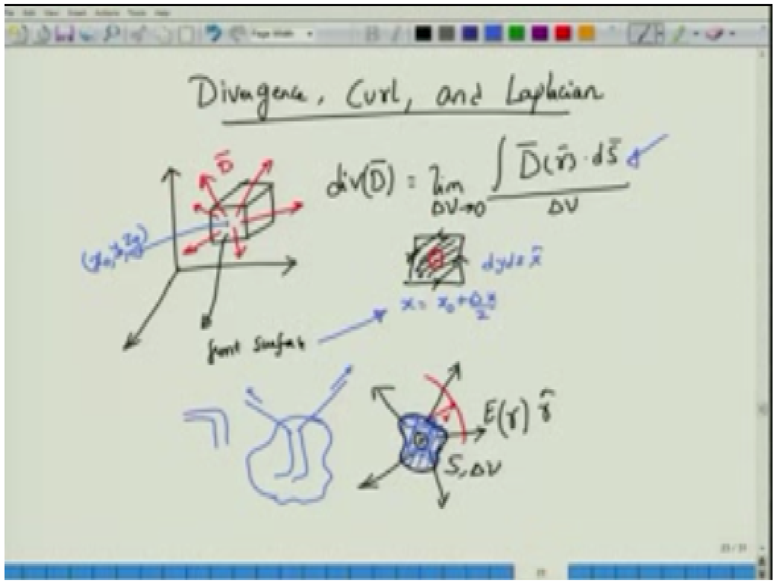

And then I place a certain charge here okay, do not look at the other parts of this slide here or the picture here, I will come back to that one, that is just a mathematical description of what we are doing. So you have a charge placed here and you know that if I were to place small test charge and you measure the electric field that is coming out of this particular charge, I know that this would be directed in this radial direction with the magnitude of decaying as you move away from the charge.

But nevertheless the electric field will be directed radially outward. So you might even write that the way in which this now the field can be written, can be written as some electric field E okay, as the function of the radial distance R, where the radial distance R will be this distance okay. So this is the radial distance R, and then the direction of this can be written as R^, R^ being the direction of the radially increasing vector.

So this is how you can actually write down the vector field and this is of course an example of a electric field vector around the positive charge right. Now imagine that I will take a small balloon or something, and then try to cover this balloon, or some kind of a box or this one that I will try to cover the charge okay. Assume that the thickness of this box is mimetically the box is empty; whatever so essentially it is not determined.

So what I am actually putting is just a virtual box kind of a thing right. And if you are observe just outside of the box, so let us say the inside of the box is not visible to you, if you were to just observe from the outside, then you will conclude that there must be a positive charge inside, because you are seeing these lines coming out right, you are measuring these lines and then you imagine that there is something, you know of a positive charge that must be kept.

Different way would be that if there is a tap okay, so the water is coming out or let us say the sprinkler that is present, so you have a sprinkler here, and if I cover the sprinkler with some unknown box or if I cover that one, so that I do not give you an axis. But let you see the outside of the sprinkler and water is coming out of this sprinkler right. So you will imagine or you will tell me that surrounding this box must be some source of a water which is why you are seeing some lines of water coming out, the sprinkler is sending you water as a turn in that one the sprinkler is now showing you I will flowing out water in this way.

If on the other hand I consider a black box and I will see that there are lines which are going in right. Then I would immediately conclude that there must be negative charge inside that black box, otherwise the lines would not have gone in. At the same time, if I consider box and then there is no line something like this, so this is a surface imagine that this is a closed surface, and you see that as many lines go in and as many lines come out, then you know immediate assumption about the scenario and the guess about the scenario is that, there is no source of charges inside, whether it is positive or negative.

Because if there was a source inside the positive charge, then there would have been some lines which are coming out okay so to quantify this, so if we are familiar with this phenomenon in which the lines are coming out and the spirit of the water is coming out the physical picture of this is that of something that is diverging. You can even go back to physics classes or optics classes and imagine that you know a small convex lens, and you should sending light here.

Then the light that comes out, the rays that come out of that one will be all diverging from the lens. Similarly, you have something called as a concave lens in which the lines actually start to converge into a particular point that point is called as the focus point. So you are familiar with this concept of emergence, this could be a divergence or this could be a convergence mathematically you can associate a positive quantity to divergence and a negative quantity to convergence.

And what you would have obtained is a very simple concept called as a divergence. Physically divergence is that is all that is about. However, mathematically divergence is not defined as so simple as I have written down. Mathematically, the definition of divergence supposes D is a vector field which will be changing at different points along the space. So mathematically the definition of divergence is given by integral of this D over the closed surface okay, divide this one by the volume of the closed surface okay.

So you have imagined the close surface in the case of the charge, that was this close surface and there is a certain volume, so there is close surface s and the certain volume for this close surface which we labeled as Δv and what happens if you start you know shrinking this volume, that is you take the volume Δv and then take the limit of this volume Δv to 0 right. If you take the volume and start to shrinking this volume more and more towards 0 and if you still see this quantity to non 0 rights.

So whatever the d(r) that you are going to obtain, ds if that quantity is non 0 then you say that this vector field d as a non 0 diva gents okay and then you represent that one by writing the diversions of D and defiling the mathematics in this way okay. So immediately you will be facing a couple of questions. First of all what is this Integral? That we have taken and how we evaluate the integral turns out that this integral.

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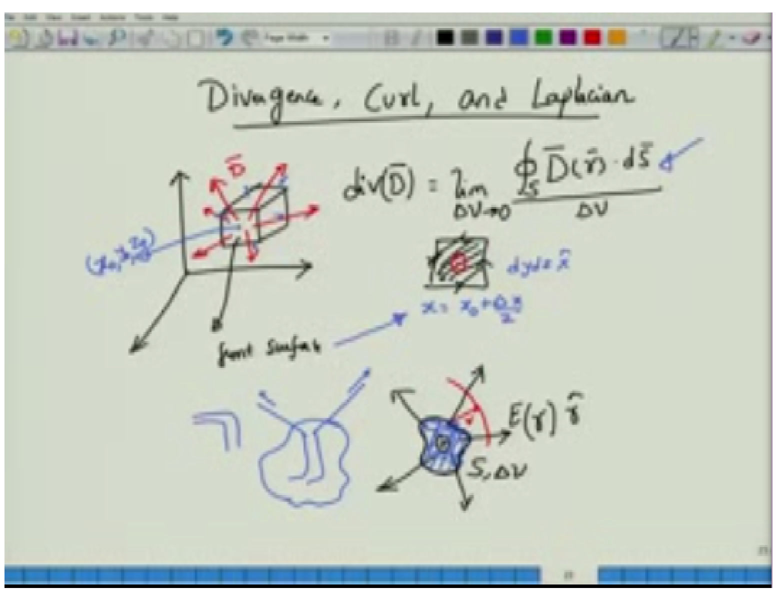

That we had in the previous example it was suppose to be a closed integral, so let me go back and re write this integral. This has to be closed surface integral okay. So what should be this closed surface integral? And how to evaluate it okay, so that is the objective that I would just very quickly give you an idea of okay. So I know that this is the vector field D that I have here okay and this is the surface that I have picked. This surface is that of cube and as I know that the closed surface of a closed cube, will have 6 surfaces right.

So that it will be made out of 6 surfaces that will be one front surface and there will be one back surface and then there will be one side surface to the right and there will be side surface to the left, then there will be a surface on the top and the surface on the bottom. What I have to do is to find out, on this surface how does the vector field D behave and then I have to take the dot product of this D at different points of the surface. And then multiply that one by the ∝ surface area okay. To make my concept or what I am saying little more clear.

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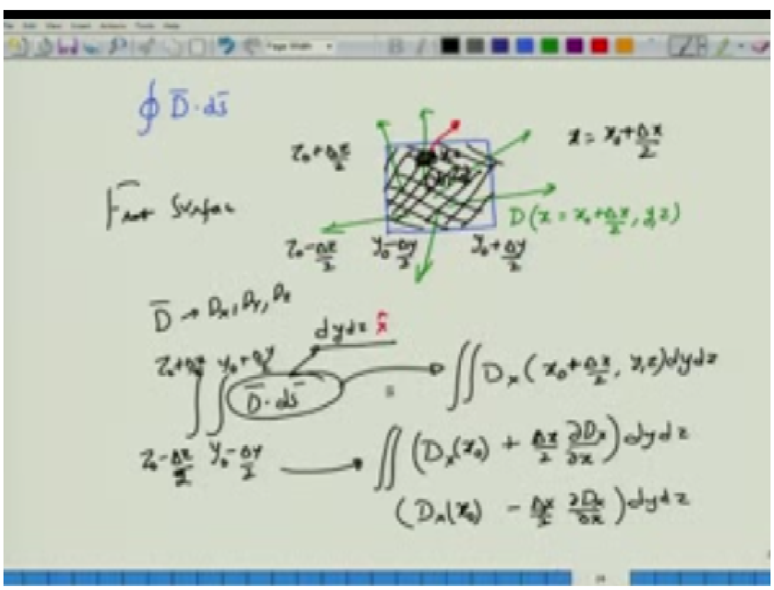

Let me redraw that situation only with the front surface over here, let say d vector is always going in this direction okay. So the d vector is in this particular direction and then I have this surface over this which I am trying to evaluate. Let us also consider this surface to be center point to be made out of y_0 and z_0 okay this is at the constant x= x0 Δx/2 okay the reason for the Δx/2 is very clear.

If you go back to this unit cube picture the center of the cube will be having the point x0, y0 and z0 and the cube itself has the volume of Δx, Δy, and Δz. Therefore the front surface is actually given at x0 Δx/2 the back surface will be given x0 - Δx/2 okay and side surface will y= y0 + Δy/2 and the left surface will be given at y = y0 - Δy/2 and so on okay. Focusing our attention on only on the front surface, so I have to find out what is the value of this d vector at by fixing x = x0 Δx/2 and allowing it to vary with respect to y and z.

In the limit of y0 - Δy/2, y0+ Δy/2 and z0 - Δz/2 to z0+ Δz/2, so I have to find out let say I pick this particular point. So I pick my no ∝ area, the area will be given by d/dz you know at that particular time and then I have to find out what is the value of the vector field d you know evaluate at x = x0 + Δx/2 and the corresponding value of y. y goes over this limit and z goes over this limit, but over this small ∝ value.

I have to find the exact value of y and z and then multiplied that value of or find out the value of d at56 that point multiplied by dy dz which gives me the infinity symbol area now here is a problem this is a surface integral okay as we call it but then there is a dot product sitting here what is the dot product imply okay let me raise this is not the definition right, so what is the dot product do to us the dot product means that I have to consider okay not any component of d but only that component of d which is parallel to the infinity symbol surface area.

For the surface area that we have considered the corresponding vector element will be given by dy / dz along direction so even though my d itself, might have component as Dx Dy and Dz not all 3 of them will be giving you a non zero value if I find out the value of d.ds over here the corresponding component that I am interested will be only the dx component because if I take the dot product of d with respect this surface element dy dx x at only the dx component will come out.

So what I have for the surface integral of over ds over the front surface in which y will be equal to y0 – Δy/2 to y 0 + Δy / 2 z0 – Δz/ 2 to z0 + Δz / 2 the infinity symbol area this integrant will actually be equal to Dx evaluated at x 0 + Δ x/2 y and z multiplied by dy /dz because the vector area here ds is given by dy /dz $x^\wedge$ and the component of d that will give you non xer4o value with respective this integrant will be only the x component and this x component is being evaluated at x0 + Δx / 2.

I also know how to do this one right I mean if consider what is dx as a Taylor series expansion then I right this as dx at x0 + sum Δx / 2 which that small displacement from x0 that you are looking at and then there is a partial derivative Δdx / dx right this is papering Taylor series only to the first order and then integrating this one over dy /dz with the appropriate limits I am not writing the limits again but this is the way you would have to write down the appropriate limit okay.

You can do this is what we have done is only for the front surface you have to have cover first of all this entire area by moving your Δx, Δy infinity symbol area around this limit okay and then once you have found this one out you then have to put all these quantities together for the front surface that you have optioned you find that this dx x0 dy , dz okay and for the back surface because the back surface is evaluated x0 – Δx what you get is dx x0 – Δx/ 2 Δ dx by Δ x okay.

Multiplied still by dy dz and then you know you can remove some of them because sum of these quantities will act to actually cenacle with respect to each other so when you put all these surfaces together the front surface the back surface and all them and combine them in order to evaluate the integral you will obtain you know you will have Δx, Δy, Δz that gets canalled by the Δx, Δy, Δz from the volume integral in the denominator/.

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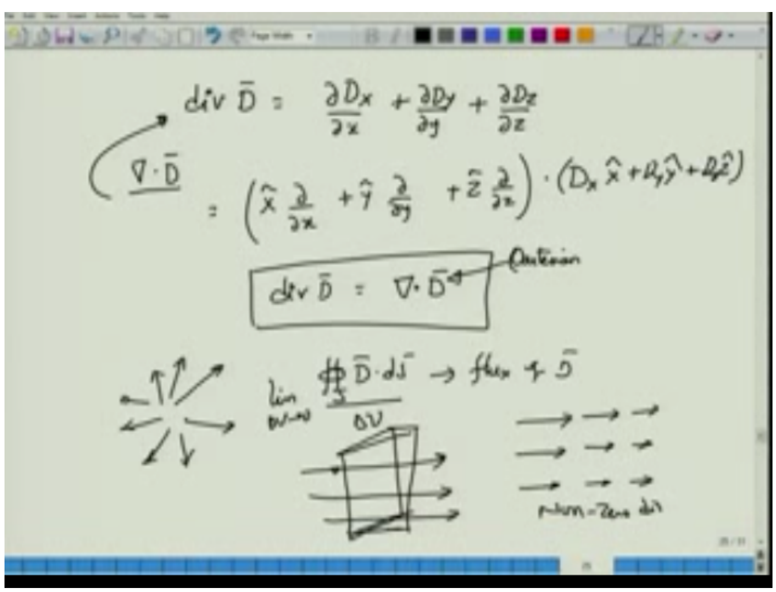

And eventually what you obtained is that the divergence of this vector field quantity to be given by $\partial Dx/\partial x + \partial Dy/\partial y + \partial Dz/\partial z$ okay. Again I can use my ∂operator okay in order to write this equation slightly better in terms of a simplified notation, consider what is this Δ.D okay. Remember that D is a vector field which means that D is a function of x, y and z and at every point you have three components for D that would be Dx, Dy and Dz.

And if I consider this operation Δ.D I know what is Δ right that is the operator x^ ∂/∂x + y^ ∂/∂y + z^ ∂/∂z so if I now take the dot product of this with respect to dx x^ + dy y^ + dz z^ I clearly obtain the above expression so divergence of D which I have calculated or rather I have just described how to go above finding that one out in the Cartesian coordinate system is actually given by Δ.D.

Therefore when you write Δ.D you have actually obtained the divergence of D okay, so this is the expression unfortunately this e4xpresion of Δ.D ho0lds only in the Cartesian coordinate system for the other coordinate systems the expressions at the in full to obtain okay but nevertheless you can obtain them you can use the tables in order to find out what would be the corresponding expression for divergence in the cylindrical and in the spherical coordinate system, okay.

So this was all about divergence the physical significance of divergence is that quantities which exhibit or the fields which exhibit a phenomenon in which the field lines or the flux would becoming out by the way the flux is simply the closed value of the vector field quantity over the surface that you are considering so if you consider this one this is actually the flux of Das we would call it okay.

So if you find that this quantity divided by Δv in the limit of ΔV going to 0 happens to be a positive quantity then you say that this is a divergent field if it is a negative then you say that the field has a negative divergence and if the stops quantity will be equal to 0 then you say that this quantity has no divergence for example consider the field quantity that would be uniform along some axis right, so you can imagine that there are this arrows which are all uniform along the entire space and then imagine.

That there is a closed surface that we have made so you can actually image a close surface okay, in the form of a same thing that we had imagine the cube that instead of cube we can imagine this any closed surface and we would see that there will be as a lines as entering as many lines as that would be leading, indicating that this quantity when you calculate the divergence of this one will exhibit more divergence at all so this is a 0 divergence field but at the same time this quantity you know whose amplitude.

Is decreasing as you can see as you know towards the right okay this actually has a non zero divergence this is slightly non intuitive because divergence we normally assume it to be something like race coming out or race that is going in that you can also have a non zero divergence for the case that we have considered here okay this is the all about divergence in the Cartesian coordinate system, we will immediately see when you talk of Maxwell's equation to divergence is very important we move on to the next calculus bit that we are going to do with.

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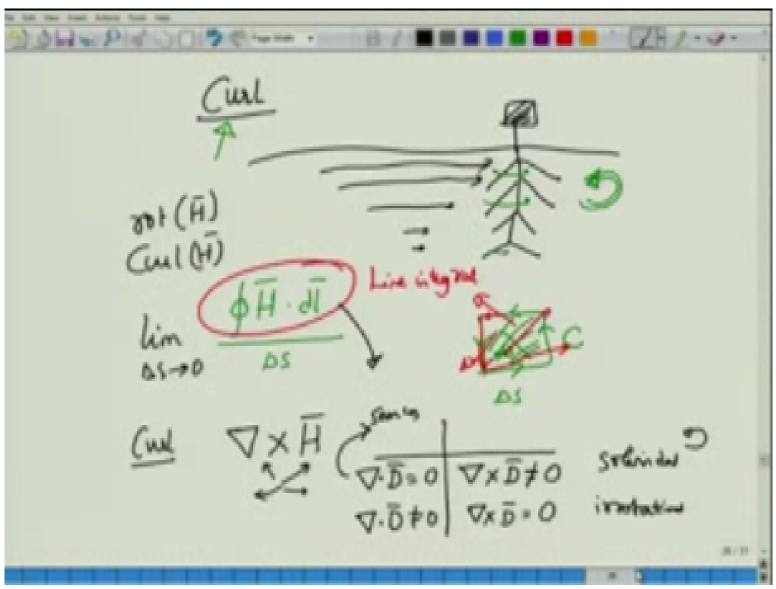

Vector filed and that quantity is called as a current okay instead of developing the complete mathematical picture of that one, I will first give you the physical insight of a current imagine that I have surface imagine I mean in front of river okay so this is a river bed and there is water current will be strong that the surface and it would be weak as you keep going towards the center of the water surface right, so you imagine that we are moving in inside a submarine or something and then measuring the water current.

Water will be very fast in the surface and this go down the water current will be reasonably small and then you imagine that there is a paddle be lower here so I have a paddle wheel here and what is the paddle wheel do now I mean if I imagine the paddle wheel out here so I connect this one this is key with the paddle out there you feel it this current is stronger here right at the top and the current the water current is weak at the bottom so the weak that are associated the bottom will be pushed with larger force okay, compared to the wings that are at the bottom so the effect of this one is that the paddle wheel.

Will start to rotate you can actually do this simple electro size by sitting up a swirling whirlpool in your bath tub and actually placing a small paddle wheel out there the paddle will actually starts rotating as the water speed is you know different a different points on the paddle, so because it is rotating in the inn of the directions if the water current is going in other way then paddle will reveal rotates so you can actually associate a clockwise with some plus quantity and then anticlockwise which is minus quantity but never less this type of vector filed right.

Where is the quantity if causing is passing a paddle wheel to rotate is called as a curl or this you know this is called as a rotation field, and the quantity by which you rotate is called as the curl of that particular field okay. And mathematically the definition of curl is that I will take the vector field so let say that vector field happens to be H and then evaluate this H over the closed line integral okay and after evaluating that one over the close line integral if I divide this one by the surface that would be the open surface that I have.

So imagine that this is the closed line integral or the contour that I am imaging okay, and this is total area the open surface will be the one which have the ∞ area of Δf okay and if I have a vector filed that would be changing over you know over this particular loop in an arbitrary manner then what I am trying to find out is the dot product of this vector field over the contour right, so this numerator which what call as a line integral so this a example of a line integral you will actually feel these integral and other quantities in textbook or in the problems that we are going to discuss okay.

And if you take the limit of this line integral the close line integral that you have consider over the contour that you have imagine and then divide that one though the surface that the open surface that the contour extend over and then you take the limit of this ΔS is going to 0 that is we start shrinking and shrinking and the result that you obtained is what is called as the curl of H or sometimes denoted by rotation of H so for this quantity to the non zero the line integral of this kilo should be non zero.

I would not give you the problems for line integral I will leave that exercise for you or in the tutorial problems that I will give you, you can find how to solve this line integral but please remember this is a close line integral and this is Δf unfortunately for as the expression for this cur or not so simple and in fact you can show that the curl of the filed in terms of in the Cartesian coordinate systems can actually be given by this curl you know taking a ∇ operator and taking the cross product of the ∇ operator with respect to the magnet I mean which respect to the filed H okay.

So this is your curl operation this is a point operation as just the divergence operations are, you now have you know you can find out the expression for curl H in different coordinate systems as usual this is easily complicated in the spherical and the other cylindrical coordinate system, it is not and that is very easy in the Cartesian but you can you are able to find out the corresponding elements for this curl operation okay, so far we have actually see two operations one is called and other called as divergence.

Based on that we actually have two kinds of field okay, one field where the divergence is 0 okay, and whereas the curl of the field is non zero okay and then you have the case where the divergence of the field is non zero and the curl of the field is = 0 the field here in which the divergence is = 0 is represented of the charges you know or the electric field that are coming out and actually denote that there are sources of divergent out there and if ΔD the divergence is = 0 and the curl is non zero then the field is called as solenoid field okay because there is a rotation only and no divergence and then you have other scenario where ΔD is non zero but there we cross D=0.

Then this is called as irrotational field that is there is no rotation to this particular field the paddle wheel that you put in will not rotate okay, you will soon see that magnetic fields are usually the once which have no divergence that is ΔD=0 but they will have nonzero curl so that is they are the so Reynolds fields where as the D vector the electric flux density vector usual has divergence but not rotational of course you can also have a scenario where both divergence and curl of a vector field are 0, okay.

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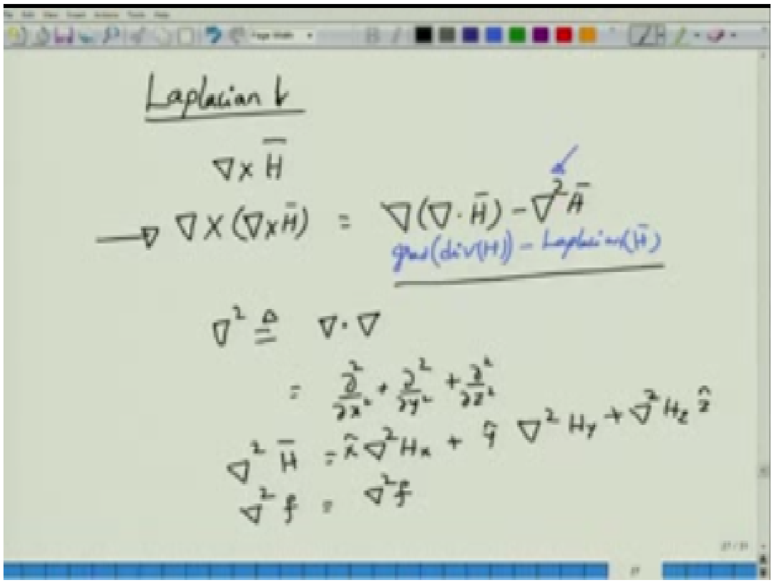

Finally I would like to talk about what is called as a laplacian okay, laplacian is something that I will be using to develop where the equation and therefore I will just very quickly talk to you about what is that, so I already you know what is curl of Hϕ, if I given what is the value of Hϕ then by looking at certain of the expressions I can find out what is this curl of Hϕ okay. Now if I take the curl of this quantity itself then there is a vector identity which you can prove okay, but it is actually quite TDS to prove.

But this would be the vector identity okay, this operation you already know so this is the divergence operation so this is divergence of Hϕ and this operation after taking divergence you are taking the gradient right, so you take the gradient of divergence of Hϕ minus we do not know what is this operation so we give a name to this as laplacian okay, so I have a name called laplacian and in terms of ∂ operation laplacian is actually given by the dot product of ∂ with respect to itself, okay.

For the partition coordinate system this is very simple this becomes $\partial^2/\partial x^2+\partial^2/\partial y^2+\partial^2/\partial z^2$ okay, so you can apply this ∂^2 on to a vector or you can apply this ∂^2 on to a scalar, the meaning of application of this ∂^2 or the laplacian on vector is that you are applying this on to each of the components and the result will be a vector okay, so observe that ∂^2 goes on to Hx and xϕ, ∂^2 goes on to Hy and yϕ+∂^2 goes on to Hz and there is a zϕ of there.

Whereas when you apply this one to a scalar quantity or only applying it to the function of f itself okay, this laplacian will be very important and this equation that we have written curl of, curl of H being given by gradient of divergence minus laplacian will be the starting point for us to derive the expressions for wave equation. There couple of notes that I would like to mention to you so that you are careful even you apply curl operation, divergence or other things.

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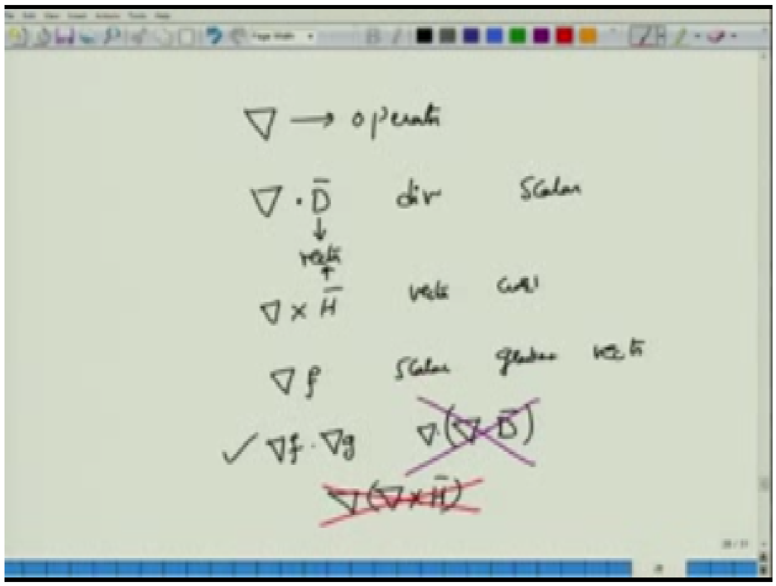

∂ is an operator okay, so ∂ is a operator this ∂ can operate on a vector field okay, to give you a divergence the result is a scalar so operates on a vector field but then the result is a scalar okay, and then you have curl operating on Hϕ the result is a vector field okay, and what you obtained is a curl okay, so you operate on a vector and then you obtain a curl okay. Gradient is something that you will have to operate only on the scalar okay, but the result of this scalar will be a vector okay.

So this is a legitimate operation ∂f. ∂g is a legitimate operation but ∂.Dϕ okay, ∂.(∂.Dϕ) is not allowed why is it not allowed, because the result of ∂.$\bar{D}$ will be a scalar and I cannot take a dot product of a scalar with a vector, so this is not a illegal operation whereas this is a legal operation. Similarly, this is also not a legal operation, because curl of $\bar{H}$ will give you a vector and you do not take the you know the gradient of a vector.

So these are the things that you should be little be careful about, when you operate on this and there is a list of identifies the vector identities that are involved with respect to the ∂ operator you do not have to you know remember everything but you will have to be able to recall some of the important ones, especially the one that we talked about in the context of laplacian okay, you have to you know remember of some of those and for all the other expressions you might look at the text book at the back of our text book Jaundy cross electromagnetics we have expressions for divergence, curl you know laplacian and not only the Cartesian coordinate system but also in all the other two coordinate systems the spherical and the cylindrical coordinate system.

We close this module by reminding you that we have actually looked at several accepts here all though I have not developed them completely in detail, I will leave that one to you to understand that because the goal is to move to Maxwell's equation as fast as possible the idea is that we have looked at a line integral, surface integral we have not looked at but it s very easy to understand a volume integral okay, associated with the surface integral close surface and dividing it by the volume gave as the divergence which is the point operation at a particular point you can find out what is the divergence of a vector field.

And similarly you had a line integral divided by the surface open surface of that one and then taking the limit of the surface going to 0 give you the curl operation again which was the point operation, okay. So this and gradient of course we have already talked about in the previous model, so this operations will be freely used when we talk of Maxwell's equations in fact we will express Maxwell's equations in these operator calculus terms and therefore you should study them before moving on to the next module thoroughly so that you understand what divergence, curl gradient, laplacian all these mean, so thank you very much.

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