Indian Institute of Technology Kanpur

National Programme on Technology Enhanced Learning (NPTEL)

Course Title
Applied Electromagnetics for Engineers

Module – 11
Graphical aid: Smith Chart Derivation

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Hello and welcome to the NPTEL mook on applied electromagnetic for engineers, in this module we will derive a graphical aid to help us solve all transmission line problems for most of the transmission line problems this graphical aid or graphical tool is called as smith chart it was invented by an electrical engineer from US his name was P. H. Smith and this was invented wave back in the 1930's infact around 1939 as when he published this chart.

At that time there were no computers and therefore calculating or solving the transmission line problems so actually very hard because it frequently required us to transform impedances and we have already seen that to transform impedances on a transmission line as you move along you will have to solve that particular complicated equation which relates the input impedance the load impedance the characteristic impedance and the length of the transmission line through that complicated formula.

And if you have to do it repeatedly that would be very tedious for someone to do in the absence of any computer today computers are faster they exist and first of all and then they are very fast, so you could do all these problems instead of using a smith chart on a computer but then the advantage of having smith chart or learning the smith chart technique is that it provides you a graphical understanding of how the impedances are transforming along the transmission line, okay.

On interconnected transmission line so in a complicated circuit especially in a microwave system where you will have multiple sub systems this subsystems will be connected by wired which will act like transmission line or interconnects and the impedances keep changing as you go from one point to the other point on this complicated microwave circuits and smith chart help us find this you know changes along the impedances in a graphical way.

It does not always give you a very accurate answer, okay for the accurate answers you should use the equation solve those equations on a computer may be you write some programs in one of the favorite programming languages that you know off but then the advantage of having smith chart is that although it does not give you very good accuracy okay it still gives you reasonable accurate values and most importantly the graphical understanding of how impedance is changing.

So let us do the following let us first understand the basics of smith chart and then in the next few modules we will look at the application of smith charts, so this module is trying to help you in understanding the basics of smith chart the origin of the smith chart and let us begin by actually noting down a particular relationship between the generalized reflection coefficient and the impedance of the transmission line along any point on the transmission line, okay.

What is this generalized reflection coefficient, I know that if I consider a transmission line this should be the, this is the transmission line and then I terminate this transmission line with the load impedance, okay. So this piece is the transmission line terminated by this load, in general the load would not be equal to in magnitude or in angle or in general it would not be equal to the characteristic impedance of the transmission line.

We will also assume that the line that we are considering is lossless so essentially what we are talking is transmission lines characteristic resistance and not really transmission line characteristic impedance, however the general trend is to simply use the word impedance even though we actually mean real value of Z_0 that is equal to R_0 although we know that it is real because it is a lossless transmission line.

We still continue to call this as a transmission line sorry characteristic impedance okay, so having said that and having met the point that whenever you terminate the transmission line with the load you are not going to be equal to the characteristic impedance all the time infact it cannot be in most cases because the load would be completely different and your transmission line may be not will not have the exact value.

Even if it does it would be there only at a particular frequency and not over the range of frequencies at which the load is expected to operate, so because of this there will be reflections and we know that and we also know how to calculate this quantity called as load reflection coefficient, load reflection coefficient tells you basically the mismatch that exist between the transmission line and the load at the point on the load itself.

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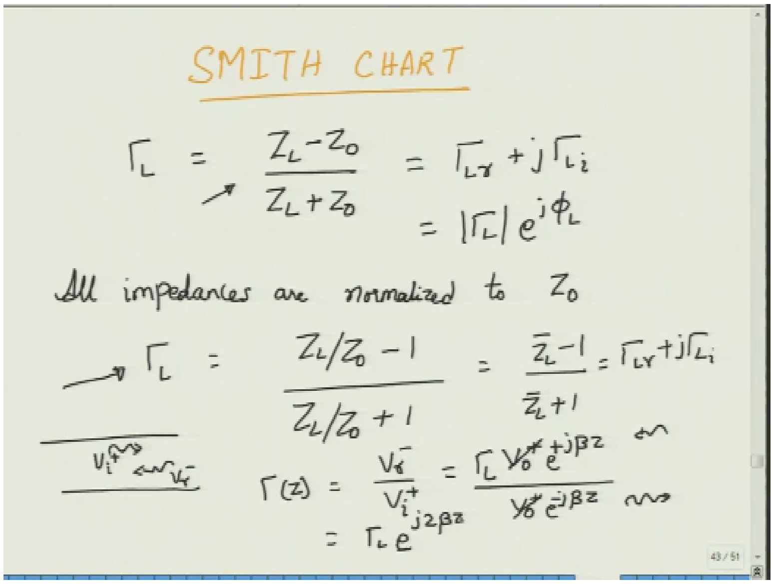

That is I denote the load reflection coefficient which is a complex quantity please remember that as Γ with a subscript L, L standing for load this is given by $Z_L – Z_0 / Z_L + Z_0$ in most case Z_L would be complex and even if it is not complex if it is real it would usually be real only at a particular frequency as you move away from that operating frequencies Z_L becomes complex and hence your Γ_L would also be complex.

We also know that a complex number can be represented in two equivalent ways one is called as the rectangular representation, okay I which you specify the real and imaginary parts of a complex number and the other would be to specify the magnitude and the angle the angle with respect to usually the x axis this second representation is called as the polar form of representing the complex number.

And since Γ_L happens to be a complex number we can equivalently represent this by the so called rectangular form or in the polar form where for the polar from I specify the magnitude of Γ_L and

also specify the angle $Ø_L$ now in smith chart all impedances whether it is load impedance or source impedance or the line impedance that you would consider along the transmission line.
All impedances are normalized, normalized to what?

They are normalized to the characteristic impedance of the transmission line why is this so, the answer is very simple if you where to actually put down the real values of Z_L or the impedance line impedance at any point you would then need a large number of smith charts because if you start changing the characteristic impedance of the transmission line by using or employing different transmission lines then you need different smith charts, okay.

So this problem is over come by normalizing all impedances on the smith charts so every impedance that you plot on the smith chart which we will see later on is actually normalized to the characteristic impedance of that particular transmission line characteristic impedance which is Z_0 okay, if you change the transmission line no problem because you again take the impedances normalize it with the new value of the characteristic impedance of a different transmission line.

So this is just a way of ways to use a single chart a universal chart for any type of transmission line whose impedance is Z_0 and if the impedance is different you simply normalized every impedance by the characteristic impedance Z_0 okay. So all impedance are normalized I can actually obtain or write this Γ in a slightly different fashion in the sense that Γ_L can be written as $Z_L/Z_0 – 1/Z_L/Z_0 + 1$ all I did was to take this equation and then divide both sides by Z_0.

And now I will introduce a normalized notation, okay or a notation to denote normalized impedances this notation of having an over bar denotes the normalized impedance so I have Z_L bar – $1/Z_L$ bar + 1 okay which could further be equal to $\Gamma L_r + J \Gamma L_i$ right, so we will come back to this later, this is the load reflection coefficient but on a transmission line I can always find that there is some incident wave which is propagating along the positive Z direction or it is propagating from source to the load.

And then there is a reflected wave right which is propagating backwards and we can define the reflection coefficient at any point on the transmission line along you know at any point Z along the transmission line by calculating the ratio of the reflected voltage to the forward going voltage so V_i^+ would be the forward going voltage, V_r^- would be the reverse going voltage or backward traveling voltage, what are these expressions.

We know that reflected voltage can be expressed as the incident voltage which is multiplied by the load refection coefficient Γ_L right this is actually $V0^{+} e^{+j\beta z}$ because this is traveling in the backward direction and V_i^{+} is the incident voltage which we can write as $V_0^{+} e^{-j \beta z}$. Because this is travelling along +j or in the forward direction, please note here that we have consider lossless transmission lines, clearly we can remove this V_0^{+} and V_0^{-} and now cancel that and what you get is the load reflection coefficient Γ_L multiplied by this parameter $e^{-j 2\beta z}$ okay.

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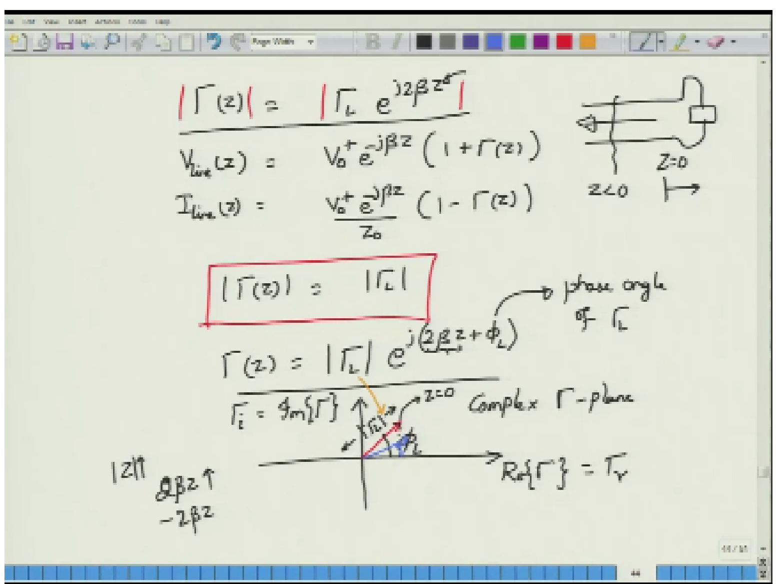

Next we write this one Γ at any pint on the transmission line is given by the load reflection value or Γ_L multiplied by $e^{-j 2\beta z}$ infact the line voltage that is voltage on the transmission line is simply given by $V_0^{+} e^{-j \beta z}$ which is the forward going voltage plus 1+ Γ (z) as we have seen in the earlier modules, I can also write the line current, lying current will be $V_0^+ e^{-j ß z} / Z_0 (1 - \Gamma (z))$ this also we have derived in the earlier module okay.

Look at this Γ (z) if I imagine that Z is changing you know Z is varying here in this expression, what would happen to Γ (z) clearly Γ (z) is varying but what will happen to magnitude of gamma, magnitude of Γ will be equal to the magnitude of the reflection coefficient at the laud itself because when you take the magnitude on these two I mean to this quantity you will see that the magnitude (j2 ß z) will go away okay.

It will be equal to 1 whatever the phase angle of Γ_L it might be that would also go away because of the magnitude operation and what you end up is a very interesting relationship which tells you that the magnitude of the reflection coefficient at any pint on the lossless transmission line will be equal; to the magnitude of the load reflection coefficient, how about the phase? If you go back to the phase relationship we know that Γ (z) can be written as magnitude of the reflection coefficient I can go back to that. And $e^{-j\ 2ß z\ +\ Ø_L}$ this $Ø_L$ is the phase angle okay of the reflection coefficient Γ_L now how do I represent this particular complex number on the complex Γ plane, well let us first draw the complex Γ plane okay on the x-axis or on the horizontal axis we write the real part of Γ okay and I do not keep carrying around this Z factor I just assume that whenever I speak of reflection coefficient I actually mean reflection coefficient on the line.

Therefore I can write this as real part of Γ which we can denote by Γ with a subscript of R similarly on the y-axis I will have the imaginary component written of the reflection coefficient which we will denote the imaginary part as Γ with a subscript I, now let is began by considering what happens at Z = 0, please remember that Z = 0 corresponds to the load location, correct? Z = 0 corresponds to the load location and Z axis is actually measured positive from the load.

But when you go backwards that would actually correspond to negative values of Z that is the values of Z here are all negative values of z, $z < 0$ here so please note that this is our convention of representing the coordinate along the transmission line, okay. Now first at Z = 0 whatever we obtain Γ (0) that would be just the reflection coefficient of the load itself. Which will have some amount of magnitude okay.

Let us say this is the magnitude that I have and I can write an arrow indicating the length of the arrow will be equal to magnitude of Γ_L okay, so the length of this fellow will be equal to Γ_L also note the angle that is made here with respect to the real axis that angle will be the phase angle $Ø_L$

okay. So what we have successfully done. Is to represent the polar form of the reflection coefficient at the load in the complex Γ plane.

So this is the complex Γ plane, okay. If you project this line onto the real and imaginary axis you will actually end up with the value of Γ_{LR} and Γ_{LI} here okay which is very obvious so we will not write that one at this point, okay. What happens as Z changes? Remember as Z increases right or other Z goes from 0 and then has you go away from the load. So as you move away from the load towards the source Z value increases in magnitude that its sign is actually minus so what it means is that, 2 ßz will increasingly become larger in magnitude okay. As the magnitude I have said increased that is to move away from the load 2βs at increases that it will increase in the negative direction that is you will actually get -2βz, right. So the value increases but you get a -2βz so your total angle here which would be changing so this becomes negative here z increases or assuming away from the load towards the source this becomes negative, this negative value plus some may be initially we will assume that this is Φ_L if positive, so positive plus some negative value the total value of this angle is actually shrinking, right. So you started from here which corresponded to the reflection coefficient at the load.

But now after travelling a certain distance z we end up with the same magnitude but at a different angle, okay. So let me expand this particular graph.

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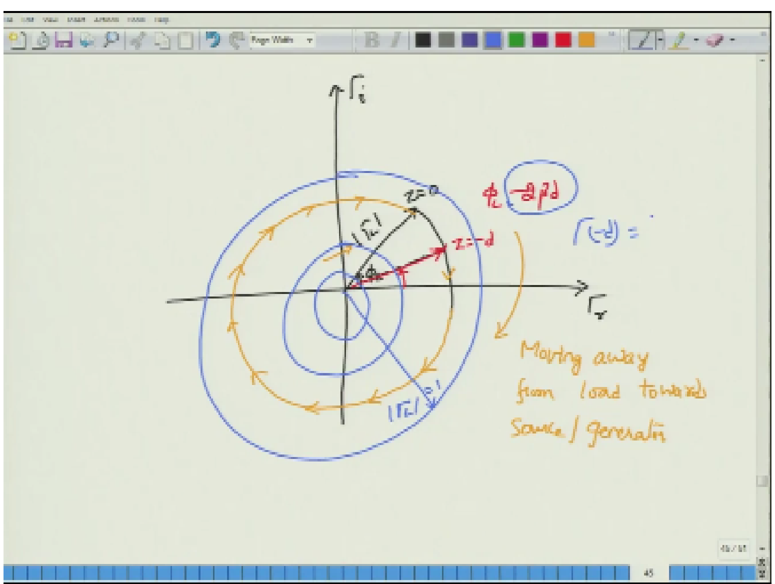

Because it is a kind of interesting to look at, so this is the complex Γ plane, this is Γ_r and this is Γ_i and we started of with the amplitude here are the magnitude of Γ_L at this point with an angle phase Φ_L here, as z increases so this was actually at z=0 at sum value of z away from the load right, towards the generator your reflection co-efficient would remain the same magnitude but its angle is now different.

The amount that you have rotated from z=0 to sum z=-d would actually be equal to -2βd, because the total angle now is given by Φ_L-2βd. So what is important to note here is that as you move towards the source as you move away from the load and towards the source okay, you will be moving along this particular circle whose magnitude would remain constant and it would be equal to $\Gamma_{L,}$ so magnitude remains constant and it would be moving successively along or successively on, in the clock wise direction as you move away from the load.

So this is moving away from load towards source or in the older terminology this was called as generator, okay so this is the movement along the transmission line away from the load and towards the generator, so you would actually move like this, right okay. So this is the complex Γ plane and you can actually consider or think of different circles for example I might have a different value of Γ_L so that would correspond to a different circle, then I would have another circle then I would have one circle and so on.

What is the maximum radius of this circle, the maximum radius of this circle would have to be equal to 1 why because, the maximum value of mod Γ_L can only be equal to 1, so your magnitude of Γ_L will be equal to 1 and that would be the largest circle that would be possible, okay. Along with you know initial value of phase angle that is given you can move a certain distance which is actually equal to -2βd.

In order to find our or in order to land at the new value of the reflection co-efficient Γ and from that new value of the reflection coefficient maybe for example in this case this is the new value of Γ at –d you can use the relationship Γ(-d)= is equal to or so since you already know the relationship for Γ(Z) which was given by this magnitude and this expression.

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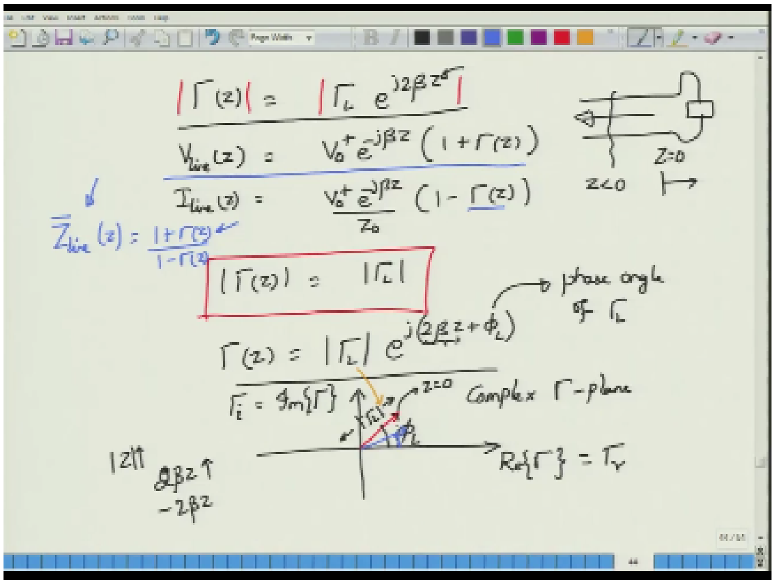

You can also go back to this line impedance or the line voltage equation in fact find out the line impedance of this one which would be the just the ratio of these two multiplied by Z_0 and obtain a simplified equation over here which will relate Γ(Z) and the impedance that is what I am trying to tell you is that given the line impedance sorry, given the line voltage, given the line current we can find out what is the line impedance at point on the transmission line and you would see that the normalized line impedance would be equal to 1+Γ(Z)/1-Γ(Z).

Because you simply divide this two left equations you know and then that ratio this Z_0 will go on to the top this would be the line impedance and that line impedance will be divided by Z_0 in order to normalize the line impedance, okay. So if you know Γ(Z) at point on the transmission line and you can now know how to do that one by following this particular arc, okay you can figure out what would be the line impedance. However this process is still a little TDS because it involves a formula. So what this PH smith came up was a very interesting scenario okay, in that what he considered.

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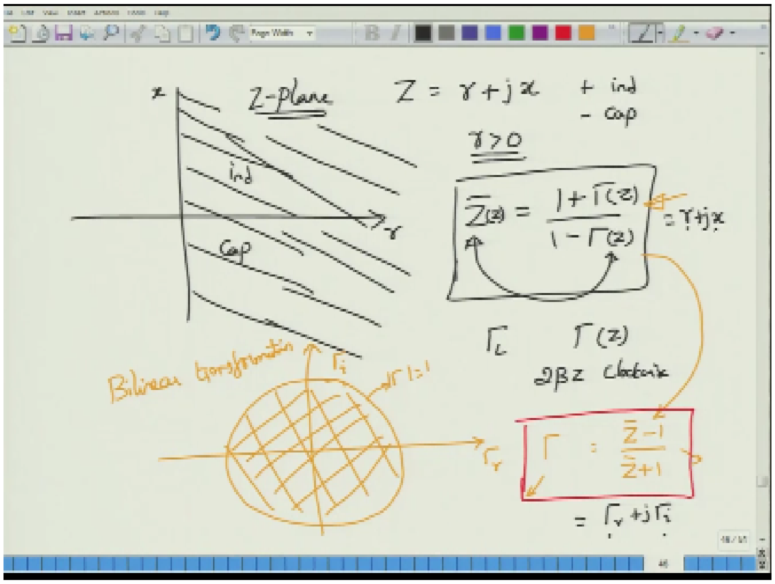

This basic quantity suppose we restrict yourself to passive notes which means that your value of mod Γ_L cannot be exciting 1 okay, and what possible values of resistance and reactants that you can find of, any impedance right the complex impedance will have a real and the imaginary parts. The real part will be the resistive part and the imaginary part will be the inductive or the capacitors susceptance, right.

So this reactants part will be there, the reactants part has a sign associated with it, if it is positive then it would inductive you know reactants if it is negative then it would be capacitive susceptance, okay. Also for the kind of loads that we consider and the transmission lines that we consider we need to always have $r > 0$, okay $r< 0$ actually corresponds to negative resistance devices and negative resistance devices do not actually exist for passive loads they can be made to exist for a active load or in active load resistance can be considered to be negative and negative resistance means actually gain.

So positive resistance means loss negative resistance means gain, okay. So for now we will restrict our self to only $r>0$, so this is my complex Z plane that is this is the complex impedance plane which can again be thought of as normalized value, okay so normalized again with respect to Z_0 of the transmission line. So this impedance is given by $1+\Gamma(Z)/1-\Gamma(Z)$ thus relating the impedance and the reflection co-efficient value.

So at any point on the transmission line, so you might be seeing lot of Z values being used so please excuse this notational in convenience I hope the context is making it clear which that we are talking about. The Z in the bracket corresponds to any point on the transmission line the Z outside corresponds to the impedance and a bar over that corresponds to the normalization that we have done with respect to Z_0 of the transmission line.

So this equation we know right, and we also know that given Γ_L we can find out what is Γ(Z) at any point on the transmission line, because this simply involves moving a distance or moving an angle by a value of -2βz or if I forget the sign then you simply need to move by 2βz value in the clock wise direction, okay. So this relationship is actually a one to one relationship why, because I know what is Γ(Z) I can find out what is the value of normalized impedance.

Also the normalized impedance will be completely this region right, so this entire region is possible values for Z(Z) okay, the complex Z value which we will write it as r along the real axis and x along the imaginary axis, so this is r and x here would be the inductive capacitances or sorry, inductive reactants and this one would be the capacitive susceptances. So what is Smith did he is to consider this Z plane and by knowing this relationship okay, this relationship can be inverted to write one more relationship I will just write that one in a few minutes.

But then he converted this entire rectangular region which is to the right side of the complex Z plane this entire right half into a circle okay, on the complex Γ plane where the axis was Γ_r and Γ_i with the maximum value of the radius being equal to 1 signifying that the reflection co-efficient for passive loads cannot be more than 1. So how can this magic of you know the entire right half being converted into the region within the unit circle happen this magic happens by what is called as bilinear transformation, okay.

This is one method in which you know line in complex plane can be converted into a curve in the other complex plane and this is the example of bilinear transformation which relates Z and Γ, okay. You can invert this relationship and that inversion is also important because Γ(Z) can now be written as $\bar{Z}-1/\bar{Z}+1$ again I will drop the subscript or drop the argument of Z I understand that left hand side is also a function of Z the right hand side is also a function of Z, Z being the distance of the co-ordinate on the transmission line. So you can invert the relationship okay, which will relate the reflection co-efficient and the normalized impedances.

So what is the use of these two well, I know that the complex impedance can be written as r+jx I also know that this Γ can be written as $\Gamma_r+j\Gamma_i$ in the rectangular form, right. So I can write in this particular fashion and then what I will do next is to try and relate this r with Γ_r and Γ_i and x with Γ_r and Γ_i .

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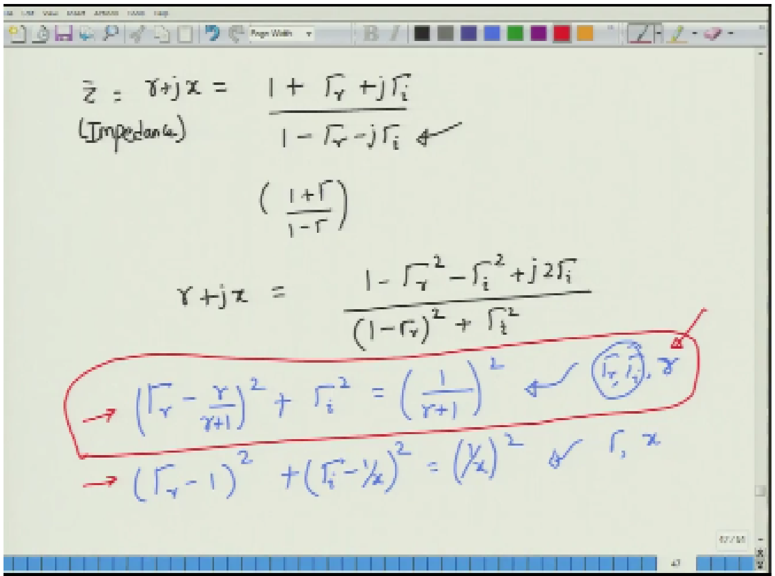

To do that we will start with this relationship that r+jx which is the complex Z value the impedance value so let me write this down, this is impedance okay except that we have written this impedance, this complex impedance in the rectangular form so this would be equal to $1+ \Gamma_r + j\,\Gamma_i$ where Γ_r and Γ_i are the real and imaginary parts of $\Gamma(Z)/1- \Gamma_r - j\,\Gamma_i$, please note that this relation is nothing but 1+ I mean $1+ \Gamma/1- \Gamma$.

Please also note that it is not the magnitude therefore do not confuse this with the standing wave ratio expressions, okay. So you can write this, you can multiply the complex numbers from the complex conjugate of the denominator and after you have done that one what you will obtain is r+jx= I will write down the expression here, now you can separate the real and imaginary parts of these right side expression and equate it to r and x.

And then do a little bit of an algebra it is little TDS but it is straight forward so do a little bit of an algebra so that we end up with the relation for r in terms of Γ_r and Γ_i and the relationship is that $(\Gamma_r - r/r+1)^2 + \Gamma_i^{\,2}$ will be equal to $(1/r+1)^2$. Similarly, equating the imaginary parts and inverting the relationship and carrying out a little bit of an algebra will give you another relationship which is $(\Gamma_r -1)^2 + (\Gamma_i -1/x)^2=(1/x)^2$.

Please note that in this relationship it is only Γ_r Γ_i and r are involved, Γ_r and Γ_i are in the complex Γ plane r is the resistance or the normalized resistance or the normalized real part of the complex impedance. In this equation you have the complex Γ plane related to x values, x being the reactants, okay. We will now in the next module actually build up that Smith chart by you know understand this equations if you remember or if you recall this equations are equations of circles.

For example, if you consider this equation which would be called as the constant r circle because this relates the value of r okay, this constant r circle values are given by or it is a constant r circles have a center of r/r+1,0 and they have a radius of 1/r+1. Similarly, this would be the constant x circles where the center is given by 1,1/x and the radius is 1/x, as r changes the center changes and the radius changes.

As x changes the center changes and the radius changes, if you put or if you plot both these equations on the same complex Γ plane you would end up with Smith chart it is as simple as that I would of course suggest that you derive this equations because they are TDS and therefore I have avoided deriving that. So in the next module we will build up the Smith chart, thank you very much.

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