Principles of Communication Systems - Part II
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Lecture - 08
Digital Communication Receiver, Optimal SNR, Matched Filter

Hello, welcome to another module in this massive open online course. So, we are looking at the SNR at the output that is after the processing, after the filtering and the sampling at the receiver in the digital communication system, alright.

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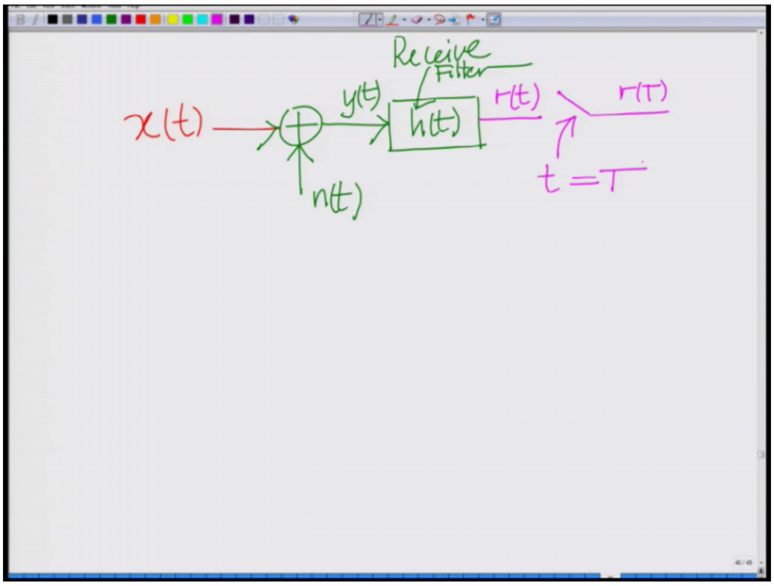

So, we have seen that our model is the following thing we have the input signal x t to which we have additive noise correct, n t this is given by followed by h t which is the filter or the receive filter, correct.

This is basically your y t which is then, so output is r t which is then sampled, correct, which is then sampled at t is equal to, which is then sampled at t is equal to capital T to yield r T, so this is sample. And therefore, now we have seen that in r t the signal. So, we have calculated the signal component of r t, the signal component of r t is, if you look at the signal component of r t that is basically your.

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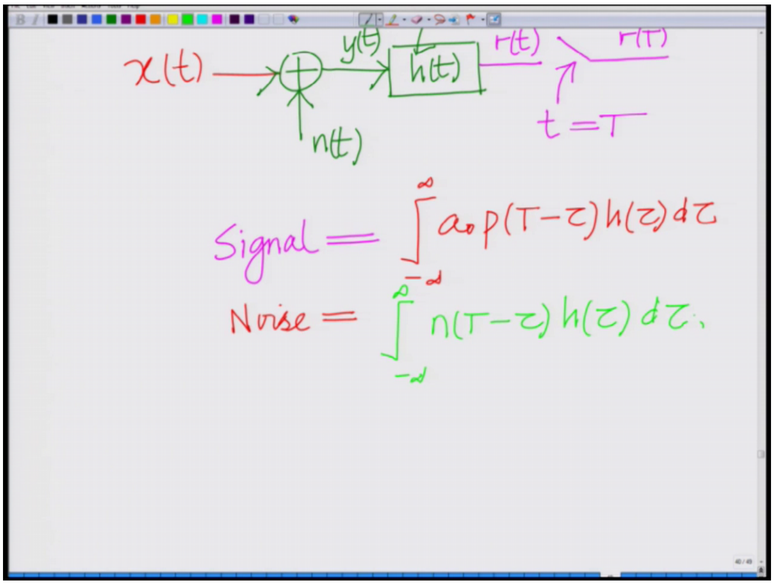

So the signal, at the output of the sample or that is integral minus infinity to infinity a naught p T minus tau h tau and the noise is well the noise is minus infinity to infinity n T minus tau h tau d tau.

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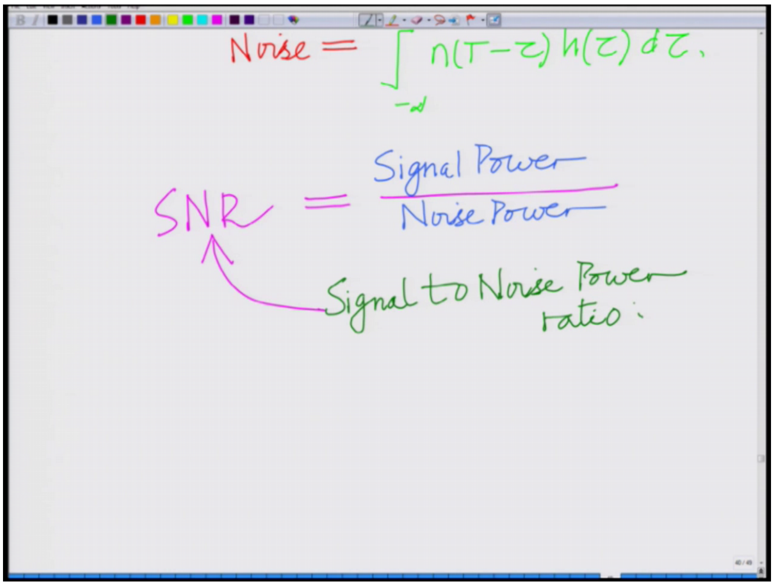

And then we can calculate the signal to noise power ratio we said that the important metric in a digital communication system is the signal not the signal or the noise individually, but the signal to noise power ratio because we want to maximize the signal (Refer Time: 02:53) power while minimizing the noise power. So, the signal to noise power ratio now the SNR that is the; what is the SNR, that is basically your ratio of the signal power divided by the noise power this is nothing, but your signal to noise power ratio this is known as a signal to noise power, signal to noise power ratio.

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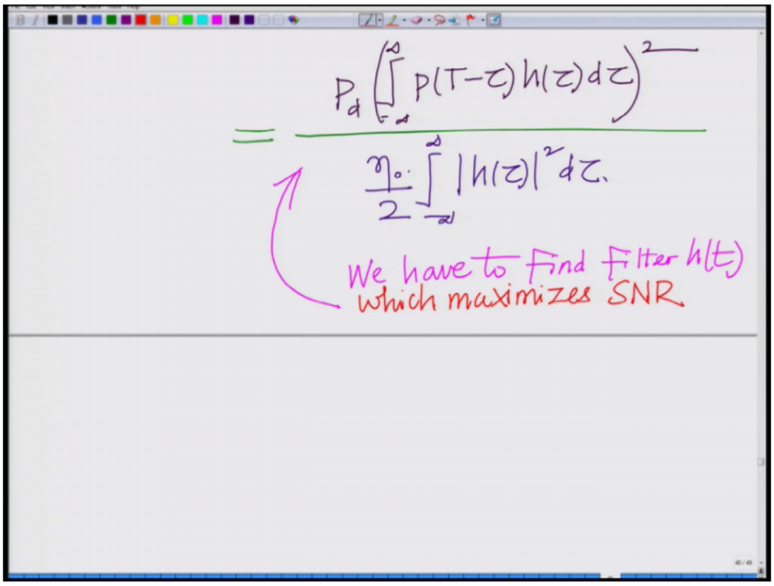

And we have calculated the signal power and noise power independently. We have calculated both these terms you have seen signal power is P d, where P d is the power of the information symbols times integral minus infinity to infinity correct. P T minus tau remember P T is the pulse times h tau d tau whole square divided by; also calculated the noise power the noise power is eta in fact, the noise power is eta or eta naught by 2 where eta naught by 2 times delta is the power spectral density or eta by 2. So, eta naught by 2 minus infinity to infinity magnitude h tau square d tau this was something interesting that it is proportional to the energy of the filter.

So, we are assumed that the noise power spectral density is eta naught by 2, we had assumed that the noise power spectral density let me just quickly confirm that that is eta naught by 2 delta tau. Again, basically this is this thing is the signal to noise power ratio; this is the signal to noise power ratio. Now we have to find the filter which maximizes the signal to noise power ratio, we have to find filter h t which maximizes the, now we have to find the filter h t which maximizes the SNR and for that we will use a property. So, basically our idea is to we said the SNR is an important metric and the higher the SNR the betters that is maximum signal power minimum noise power. So, we would like to maximize the signal to noise power ratio. For this we would use the property the following property.

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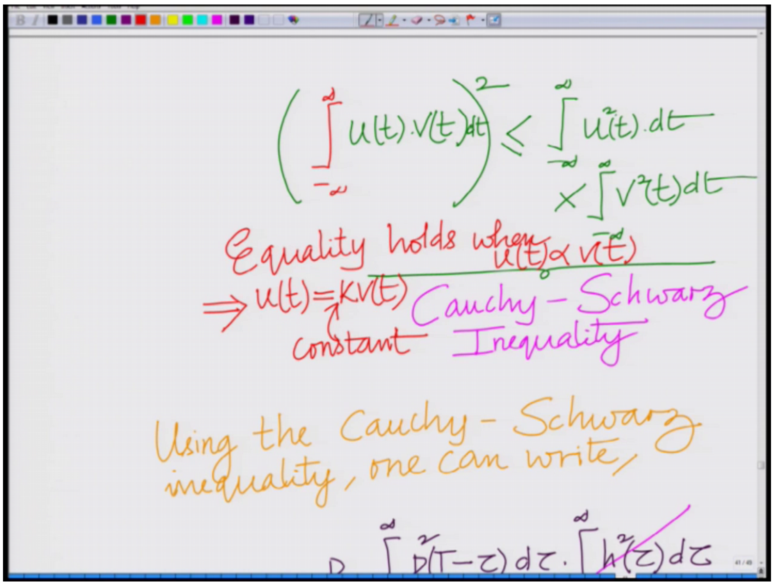

That is integral minus infinity to infinity if we have two real functions x t and y t or let us use the terminology u t and v t since x t and y t are confusing - u t and v t whole square this is less than or equal to integral minus infinity to infinity u square t times d t this is u square t, d t into integral minus infinity to infinity v square t d t. Integral minus infinity to infinity u t v t d t whole square is less than or equal to integral minus infinity to infinity u square t d t times integral minus infinity to infinity v square t d t this is termed as the Cauchy-Schwarz inequality.

More precisely this is the Cauchy-Schwarz inequality for two functions or for the integral of two functions. Now how can we use this property? Now look at this if we look at the numerator I have integral minus infinity to infinity p T minus tau h tau d tau whole square. So, this is less than or equal to if apply the Cauchy-Schwarz inequality I get this less than equal to minus infinity to infinity p square t minus tau d tau times integral minus infinity to infinity h square tau d tau. So, I can write the SNR using the Cauchy-Schwarz.

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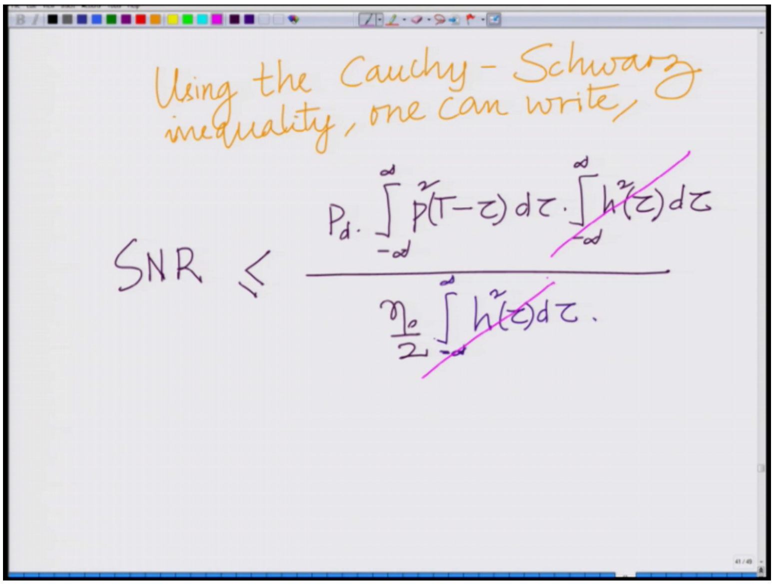

Now using the Cauchy-Schwarz inequality one can write the SNR using Cauchy-Schwarz inequality. You see that the numerator SNR is less than or equal to well SNR is less than or equal to P d integral p square T minus tau d tau into integral h square tau d tau divided by the denominator remains the same eta naught by 2 integral minus infinity to infinity, infinity to infinity magnitude or you can simply write considering all quantities real that is filters are real h square tau d tau.

Now you observe something interesting that this numerator integral minus infinity infinite h square tau d tau denominator integral minus infinity to infinity h square tau d tau cancel, therefore the maximum value. So, SNR is less than or equal to this quantity which is P d minus infinity to infinity something very interesting p square t minus tau d tau divided by eta naught by 2.

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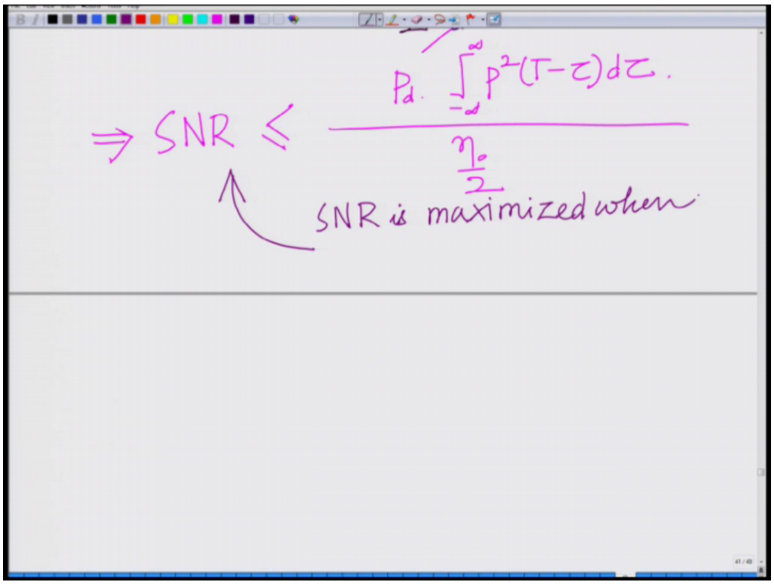

So, the SNR is less than or equal to this quantity implies the SNR is less than or equal to this quantity, we are using the Cauchy-Schwarz inequality and we have demonstrated since the numerator is less than or equal to that therefore, the SNR is correspondingly less than or equal to the appropriate quantity.

So, now, naturally when is the SNR maximized the SNR is maximized when the Cauchy-Schwarz inequality holds with equality. So, the Cauchy-Schwarz whole inequality holds with equality when equality holds when or if and only if u t is proportional to v t. This implies u t is some constant k times v t. So, this k is any constant. So, the equality holds in the Cauchy-Schwarz inequality only when u t is proportional to v t that is we are looking at integral minus infinity u t times v t it is maximized when u t is proportional to v t.

Therefore, the SNR using that result we can say that the equality holds which implies that maximum is that SNR is equal. So, SNR is always less than equal to u quantity on the right, equality holds or SNR is maximized when SNR is maximized when in this we have remember our u t is p T minus tau v t is h tau, we have to have p T minus tau is proportional to h tau that implies p T minus tau equals h tau or equals k times h tau.

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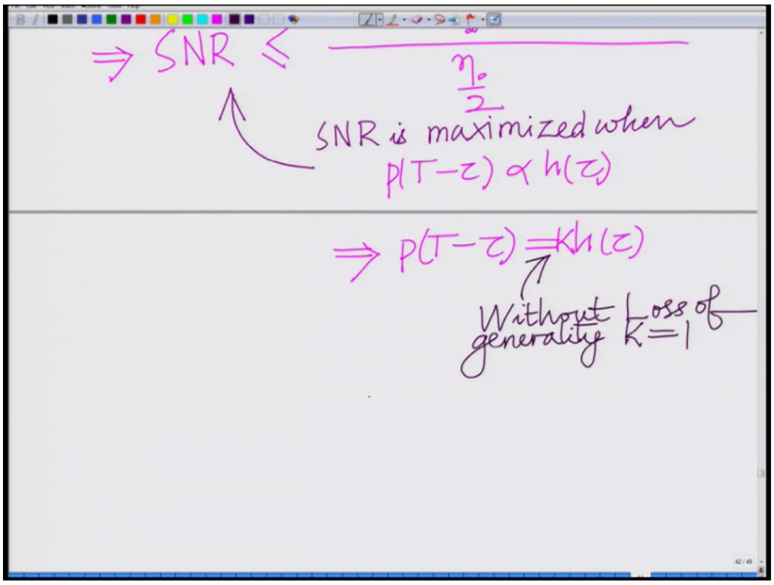

Without loss of generality we can set k equal to 1, it holds for any k without loss of generality set k equal to 1 implies we have to have h tau equals p t minus tau.

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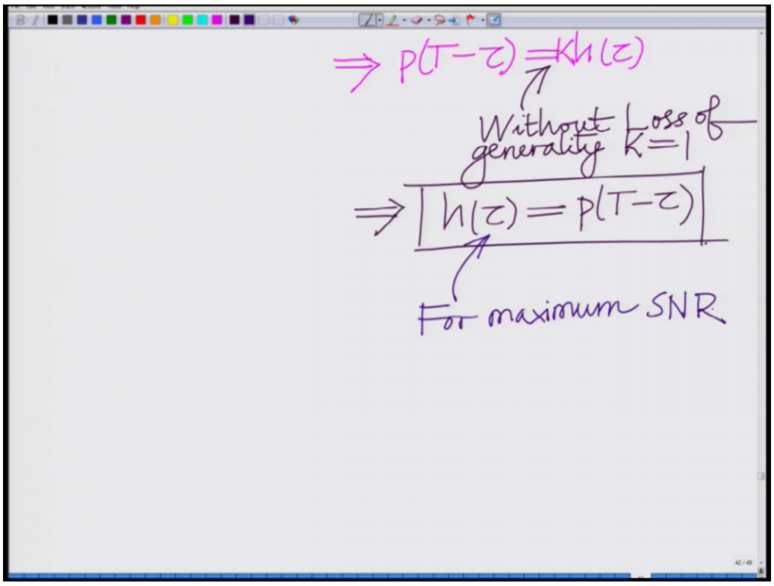

So, for maximum SNR this is the property you have to satisfy. For maximum SNR h tau equals and therefore, what you are seeing is very interestingly h tau is similar to p T, except it is p T minus tau, so h tau has to be match to the pulse. So, if you have pulse p T the response impulse response h tau or if you have pulse p tau the impulse response h tau is p capital T minus tau. So, in that sense the filter h has to the impulse, response of the filter h has to be matched to that of the pulse shaping filter, therefore this is termed as a matched filter.

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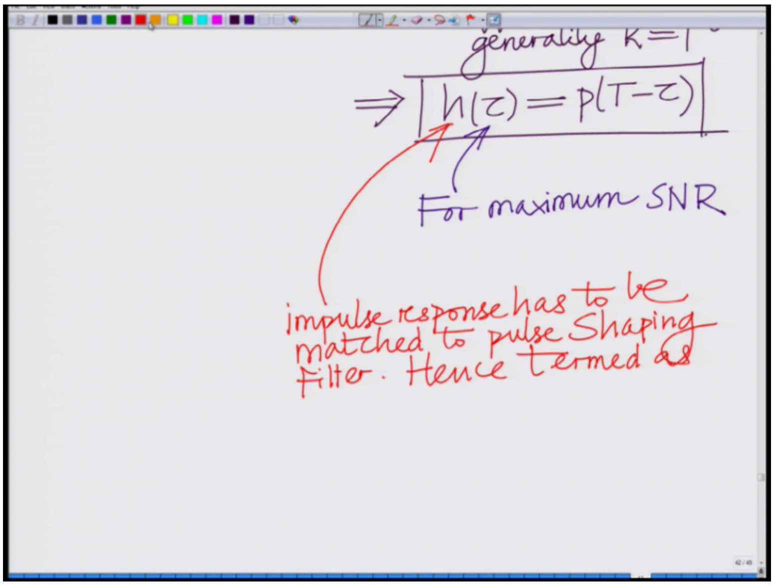

So, you can see impulse response is matched to the pulse shaping filter. Therefore, this is termed as a matched filter. Hence it is termed as a matched filter.

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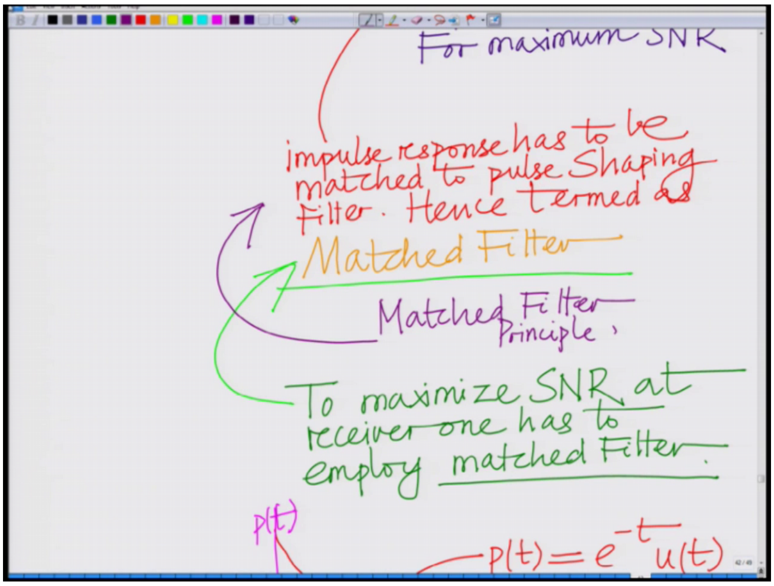

So, to maximize the SNR one has to employ matched filter at the receiver that is the whole point, one has to employ a matched filter.

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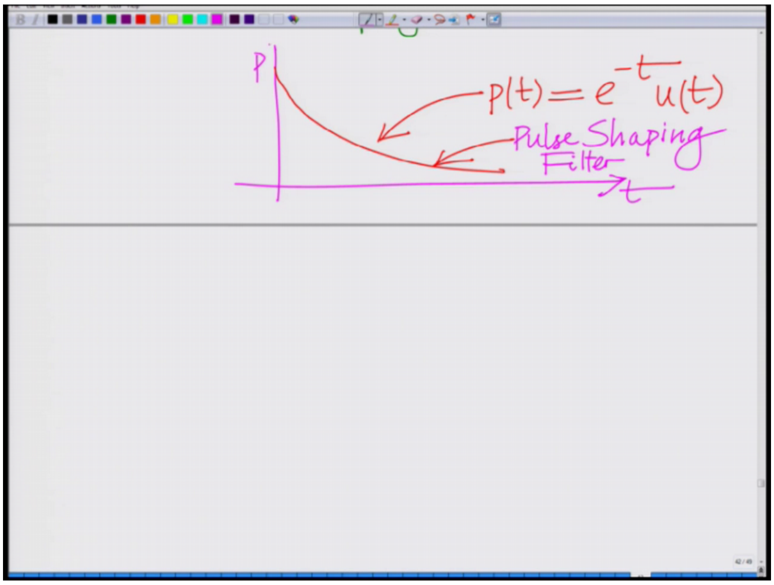

Let us take a simple example let us say this is my pulse shaping filter; we can choose any pulse shaping filter that is what we said right. This is let us say my pulse shaping filter p t equals e to the power of minus t times u t, that is e to the power of minus t if t is greater than equal to 0 this is my impulse; this is my pulse shaping filter or response of the pulse shaping filter. I can simply call this as the pulse shaping filter, this is t, and this is p t.

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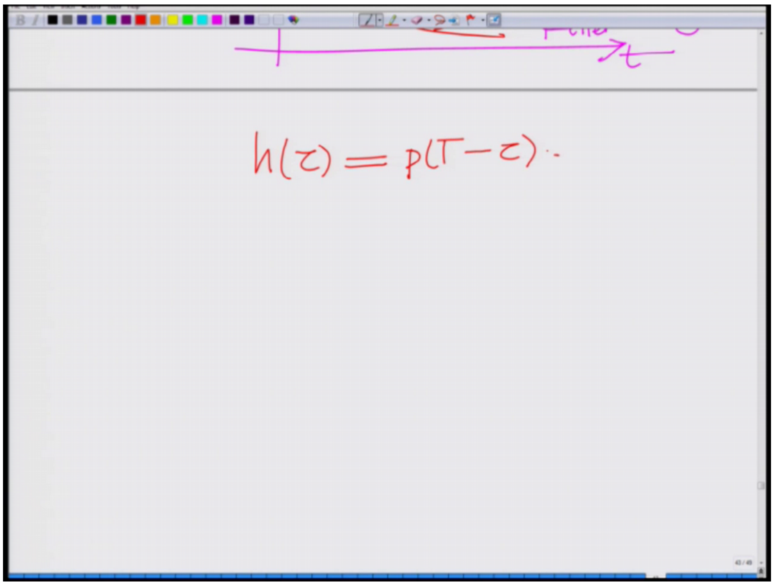

Now, h tau we have h tau equals p T minus tau. Now how do we get h tau from p T?

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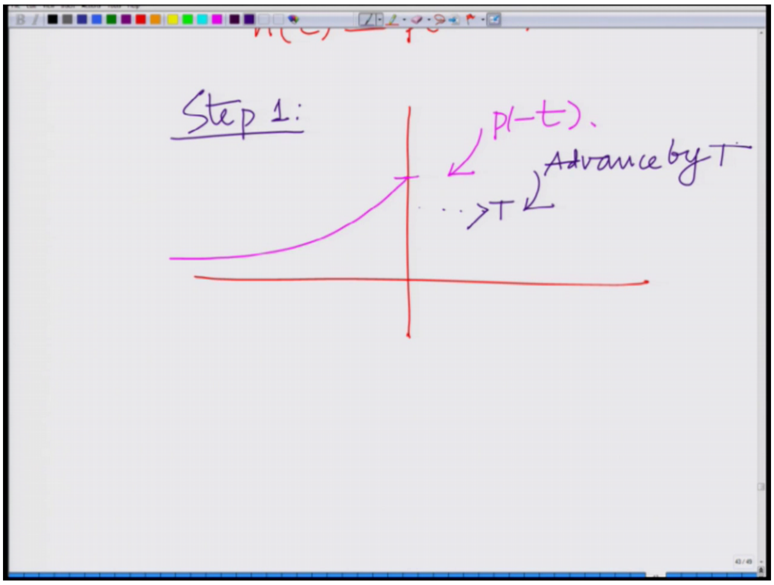

First we have to flip. So, step 1 - we have to flip. So, this is p of minus t, so first step to construct the impulse response of the optimal filter receive filter which maximizes the SNR, first you flip p t to get p of minus t then you advance by capital T then you advance this so shift. So step 1 followed by advance by T.

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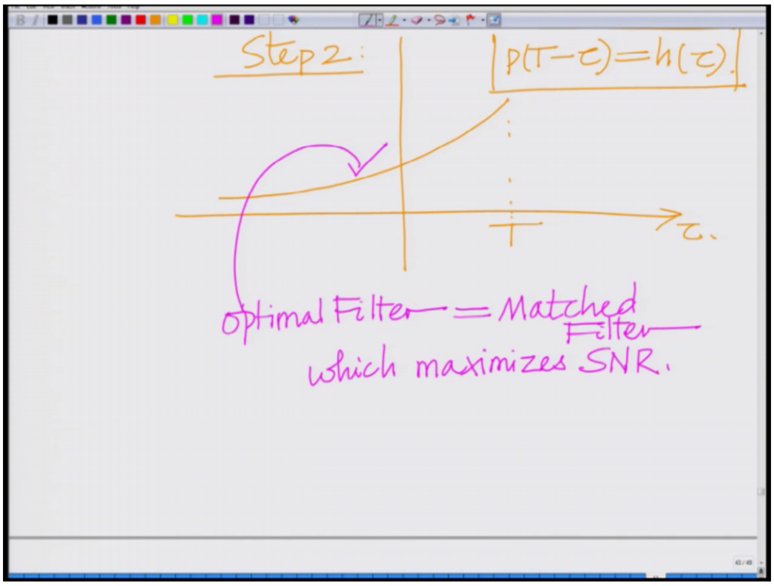

So, when you advance by T what you will get is this. So, this is p T minus tau equals h tau so this is your step 2 and this is the optimal filter, optimal filter or optimal matched filter this is optimal filter which is a matched filter; optimal filter or the matched filter which maximizes the SNR.

So, you take the pulse shaping filter p T flip it about the origin get p minus t advanced by capital T where capital T is the symbol duration or the sampling time. That gives us the matched filter and why are we calling this a matched filter because the optimal receive filter is matched is nothing but it is basically identical to that of the pulse shaping filter the only change is it has to be flipped about the origin and advanced by capital T that is the only thing. So, the optimal received filter which maximizes the SNR at the output of the filter after sampling in a digital communication system in additive white Gaussian that is also important. And remember the condition: the condition is that the noise is additive white Gaussian in nature. In additive white Gaussian noise is the matched filter this is the very important principle.

In fact this is one of the most this is one of the fundamental principles that govern the functioning of a digital communication system that enable optimum power sink and optimizing or maximizing the single choice power ratio and therefore, minimizing the error rate in a digital communication receiver. Now, what is the maximum SNR? You can see from here that the maximum SNR is given by this expression P d integral minus infinity to infinity p square T minus tau d tau eta naught by 2. So, the maximum SNR for the matched filter is or you can see maximum SNR for the matched filter.

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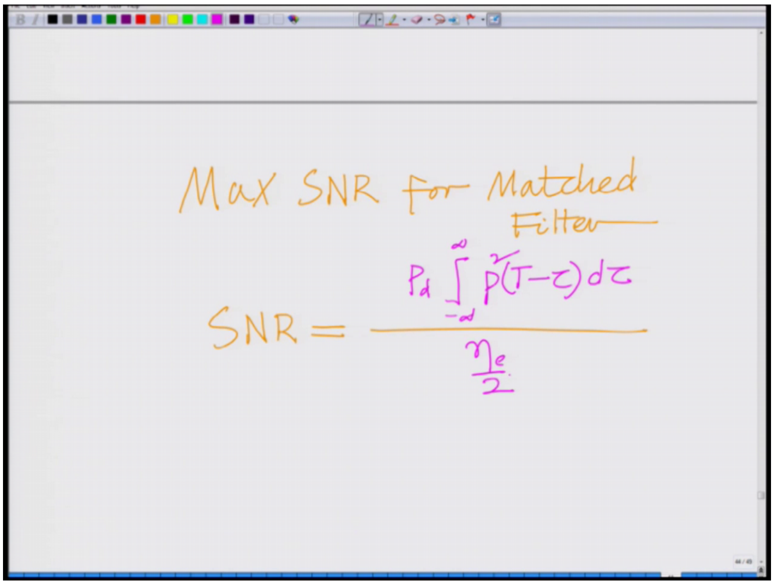

SNR equals integral minus infinity to infinity P d integral minus infinity to infinity p square t minus tau d tau divided by eta naught by 2.

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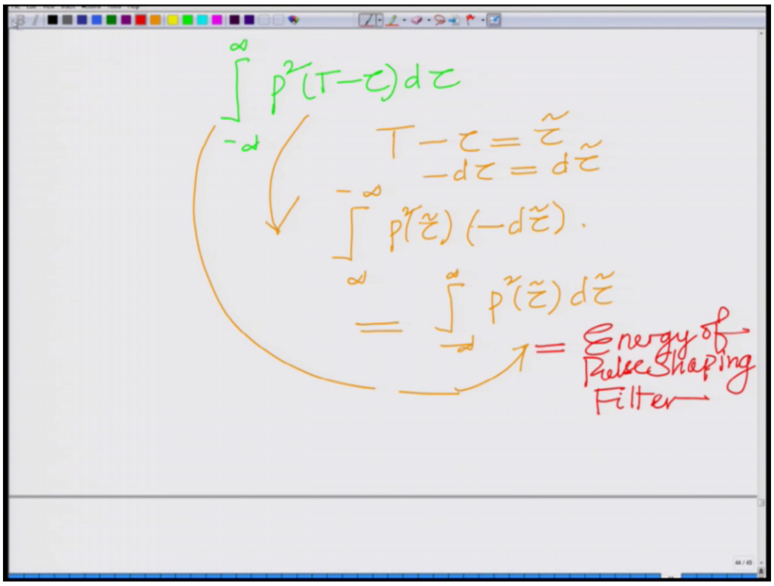

But we have seen p t minus tau is nothing, but h tau so this is P d integral minus infinity to infinity I can say h square, so p d by eta naught by 2 into integral minus infinity n square. Now you also realize that if you look at this integral minus infinity to infinity p square capital T minus tau d tau is nothing but if I set t minus tau equals tau tilde minus d tau equals d tau tilde. If I simplify this first what will happen, t minus tau equals tau tilde tau tilde will go from infinity to minus infinity p square t minus tau is tau tilde tau tilde minus d tau is d tilde minus d tilde.

But again the integrals are infinity to minus infinity, so taking the minus sign from d tau tilde this will become infinity to infinity minus infinity to infinity p square tau t tilde d tau tilde which you can see is nothing, but the energy of the filter. So, this is nothing, but if you look at this, this is nothing but equals the energy of the filter or energy of pulse shaping filter. This is the energy of the pulse shaping filter.

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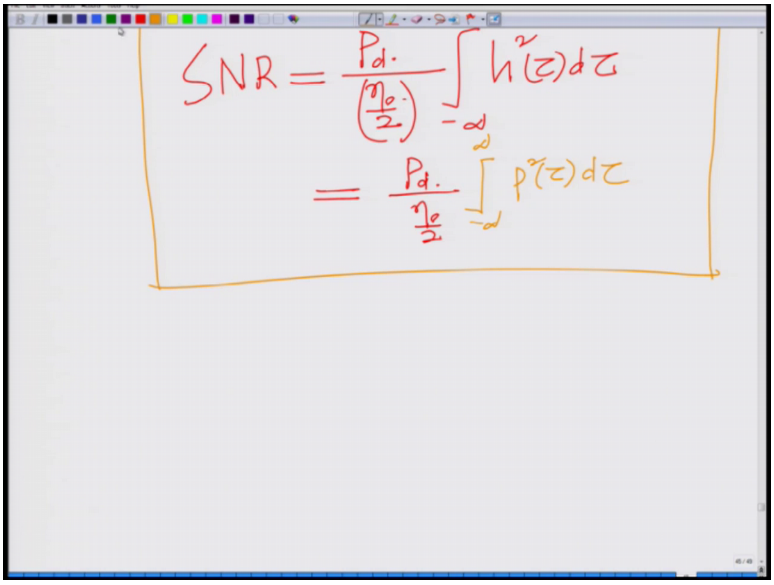

And therefore, what you can say is the maximum SNR for a matched filter equals P d where P d is the power of the data symbols divided by eta bar naught by 2 times integral minus infinity to infinity magnitude h tau square. I can generalize this, now I can generalize this you can see that even for a complex filter although we will not consider it let us to avoid any confusion let us simply consider a real filter h square tau d tau which is also p d by eta naught by 2 times integral minus infinity to infinity, times integral minus infinity to infinity and this is your maximum SNR that is it, this is the maximum SNR. This is the maximum SNR which occurs for the matched filter, for matched filter.

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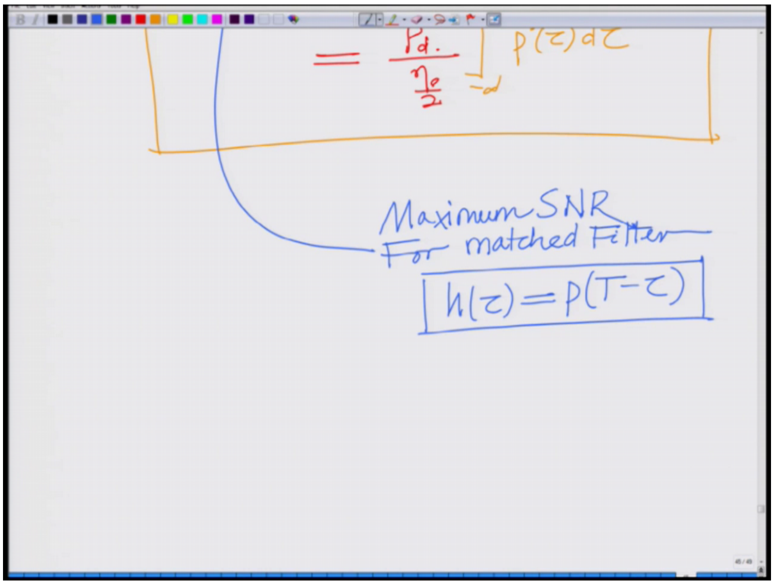

That is your h tau equals p t minus tau. All you are doing is basically it is not surprising because all you are doing is basically you are flipping about the origin and advancing by t. So, the total energy remains constant that is the total energy in the (Refer Time: 24:05), the total energy in h tau is the same as the energy in p tau which is the same as the energy in p capital T minus set up. Simply by flipping and advancing you do not change the energy, therefore the maximum SNR is now P d the power of the data symbols divided by eta naught by 2 times the energy of you can say either the energy of the pulse shaping filter or the energy of the matched filter.

The important principle here is that to maximize the SNR the receive filter the optimal receive filter is one that is matched to the pulse shaping filter in additive white Gaussian noise this is known as the matched filter. This is known as the principle of matched filter. Let us note that because that is one of the most fundamental principles. This is known as the principle of the matched filter or matched filter principle. This is known as the matched filter principle.

So, that is pretty much right which concludes a very important module I would like to add that in fact, this module is one of the introductory modules in this course and is one of the most fundamental modules I would like you to, I would like to emphasize once again because matched filtering is a very important principle in any digital communication systems since it can be employed to maximize the SNR at the receiver. In fact, matched filtering is received used in all digital communication receivers correct, that is by matching the received filter response of the receive filter to that of the pulse shaping filter. This results in a maximizing the SNR in the digital communication system and it is one of the most, if not the most fundamental principle in a digital communication system especially at the receiver of a digital communication system.

So, please go through this module again we have done this derivation through the last not just this module, but the last couple of modules we have constructed the structure of the receiver, derived the signal power, derived the noise power and finally, demonstrated that the matched filter maximizes the SNR.

So, I would like to again urge you to go over these modules since they are very fundamental in nature to understanding the working of a digital communication system.

Please go over these modules once again and thoroughly understand the principle of matched filter.

Thank you.