Principles of Communication Systems - Part II
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Lecture – 39
Capacity of the Gaussian Channel – Part I

Hello, welcome to another module in this massive open online course. So, in this module let us look at another fundamental property or fundamental result for the Gaussian channel which is the capacity characterized the capacity of the Gaussian channel.

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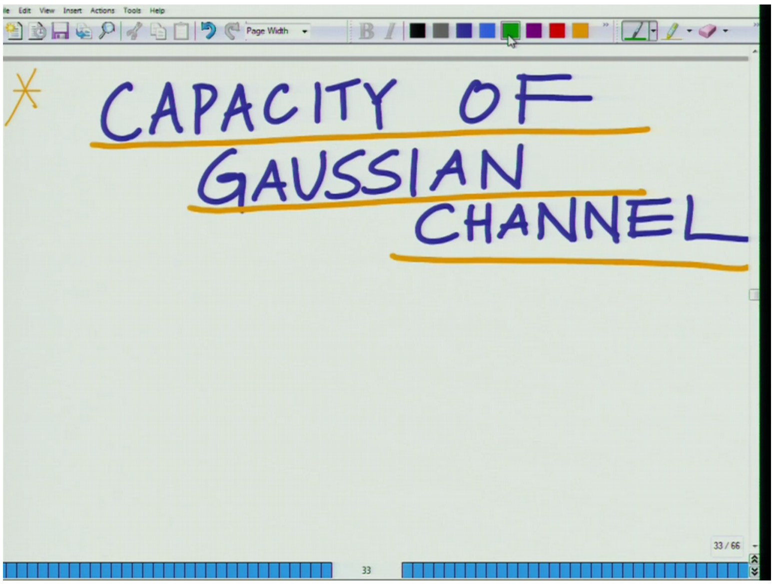

So, what we are interested in doing in this module is to look at the capacity of the; we will look at the capacity of the Gaussian channel. So, what do we mean by Gaussian channel.

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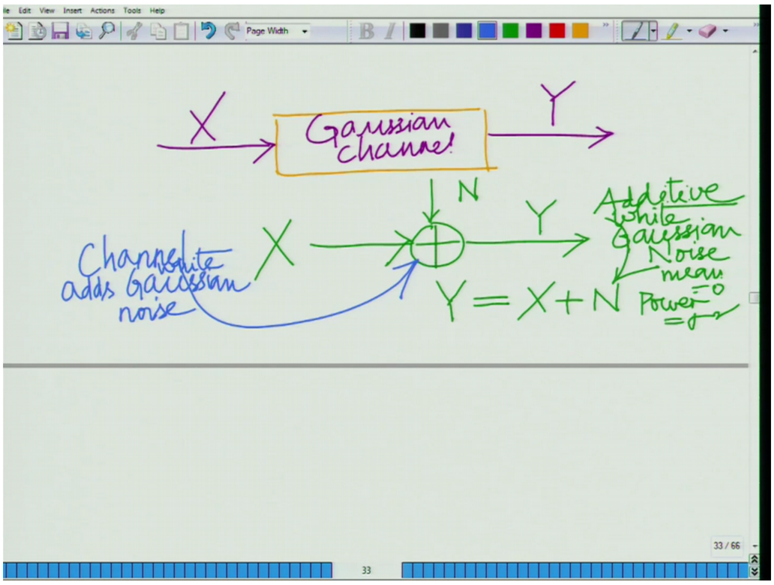

This is one of the most as we have seen several times in this course this is one of the most relevant practically relevant and so we have an input we have an output and what do we mean by Gaussian channel is that the input and the output are related by the addition related by the addition of white Gaussian noise.

So, we have Y equals X plus N where N is additive white Gaussian noise mean equals 0 and the power equals the variance equals sigma square. So, this is basically the channel adds Gaussian noise or white Gaussian noise channel adds white Gaussian noise.

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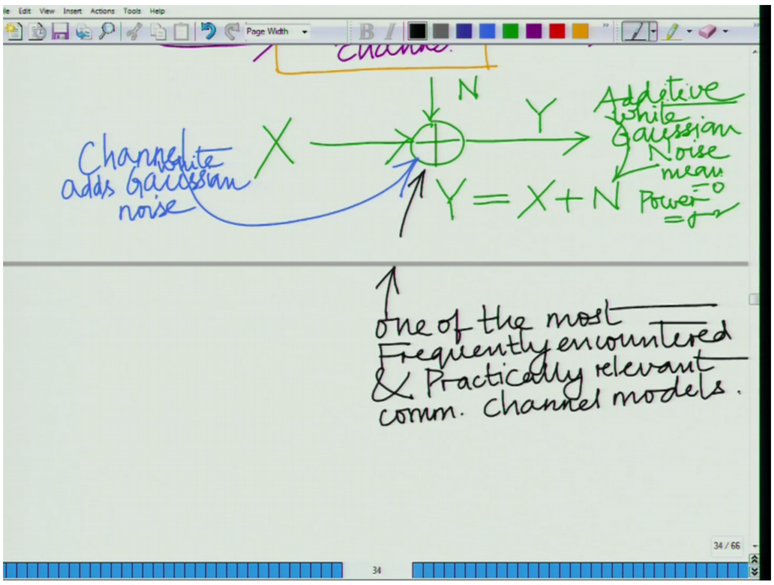

This is one of the as I have already mentioned it several times before its one of the most practically relevant as one of the most frequently encountered and practically relevant communication channel models and what do we mean by the capacity of this channel remember by capacity we mean the fundamental rate at which or the maximum rate at which information can be transmitted across this channel with an arbitrarily low at an arbitrarily low probability of error.

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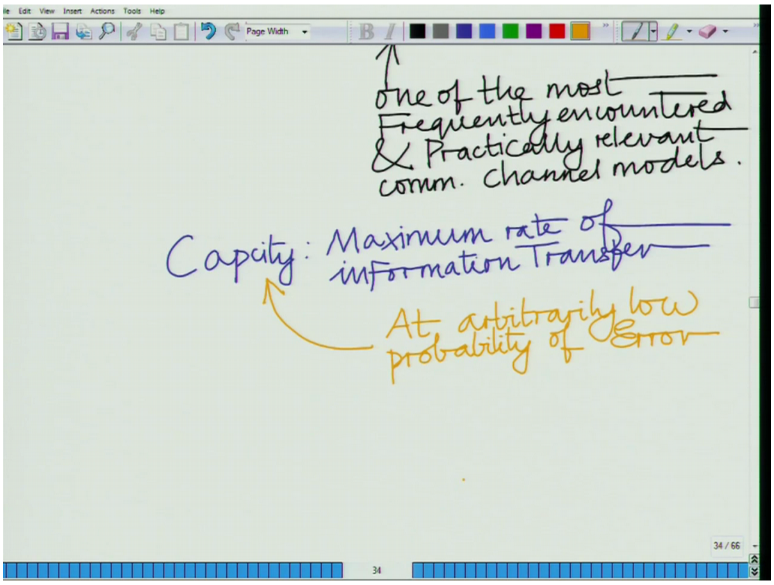

So, the capacity has a fundamental relevance in communication system because it gives us. So, the capacity this is of fundamental relevance because capacity that gives us this is basically maximum rate of information transfer maximum rate of information transfer at an arbitrarily low at an arbitrarily low probability of error. So, we have the capacity at which we can the maximum rate of information transfer at an arbitrarily low probability of error.

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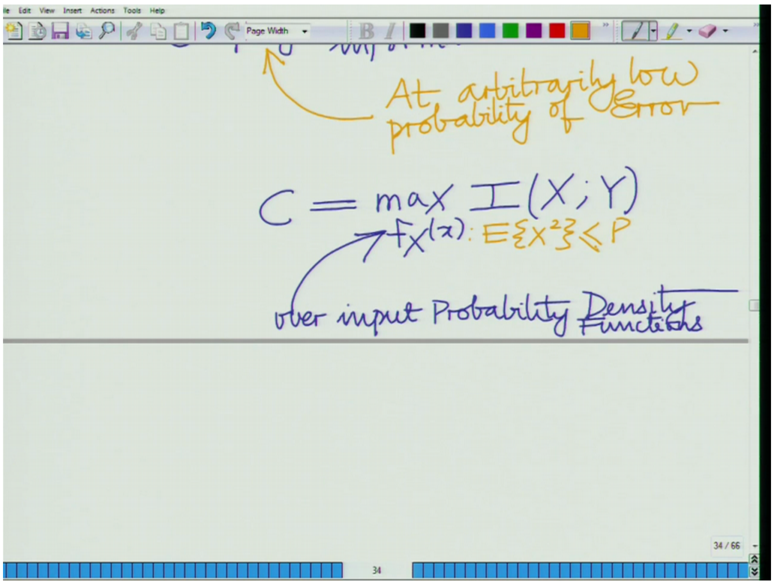

And we have seen that the capacity from the fundamental result by Shannon is given as the maximum of the mutual information that is given by the maximum of the mutual information which is still valid here and except and the maximum has to be over the input probability density functions that is we have to maximize this mutual information over input. However, we have an additional constraint here that is the input power is limited.

So, what we are doing here is we are trying to find the capacity the capacity is given by the maximum maximization of the maximum of the mutual information correct maximize the mutual information; however, unlike the case of the discrete. So, as here we have an additional constraint for the continuous source that the power of the source has to be limited by this quantity mean similar to a practical wireless communication scenario in which we have a constraint that they transmit power right each transmitter right for instance when you have a telephone system or we have a mobile phone the mobile phone for instance is limited by the battery, alright. So, there has to be a power source in any communication system and the communication system and that power source limits the amount of power that can be used by the transmitter.

So, naturally it is a very meaning full constraint because if obviously, if you can transmit and in finite power then you cannot get transmit at any rate that one wishes to because if there is no power constraint then one can in theory one can transmit and arbitrarily high power and overcome the effect of any noise. But the challenge is to overcome the limitations of noise imposed by the channel using a battery which supports transmission at a finite power or with a maximum certain maximum power. So, that is the key aspect here.

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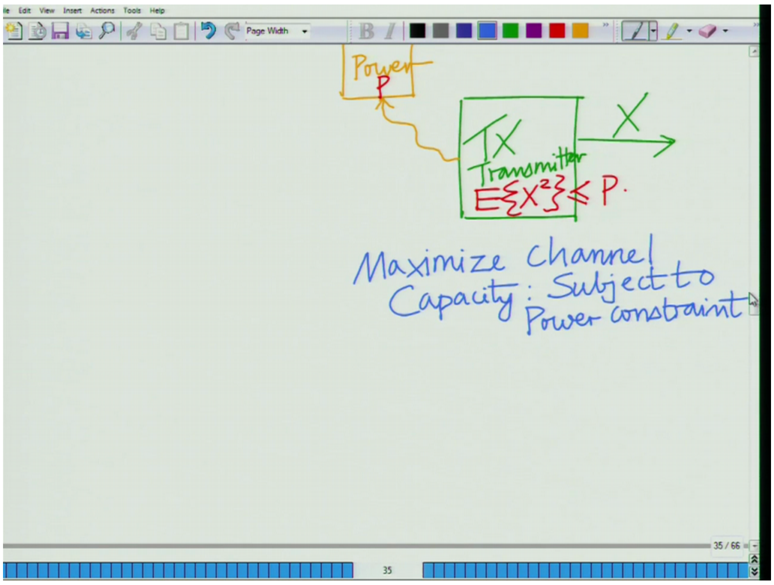

So I have a source or let us say I have a transmitter correct at transmitting a signal X; however, this is basically connected to a power source which supplies the power. So, this transmitter let us say the power is P. So, my transmit power that is has to be limited and remember transmit power is nothing, but the expected value of the X square.

So, the transmit power has to be limited by T. So, I have to maximize capacity. So, we would like to maximize capacity of the channel, but subject to subject to the power constraint maximize the power capacity, but subject to the power constraint that is the total transmit power is limited by this quantity. Now let us proceed by doing that.

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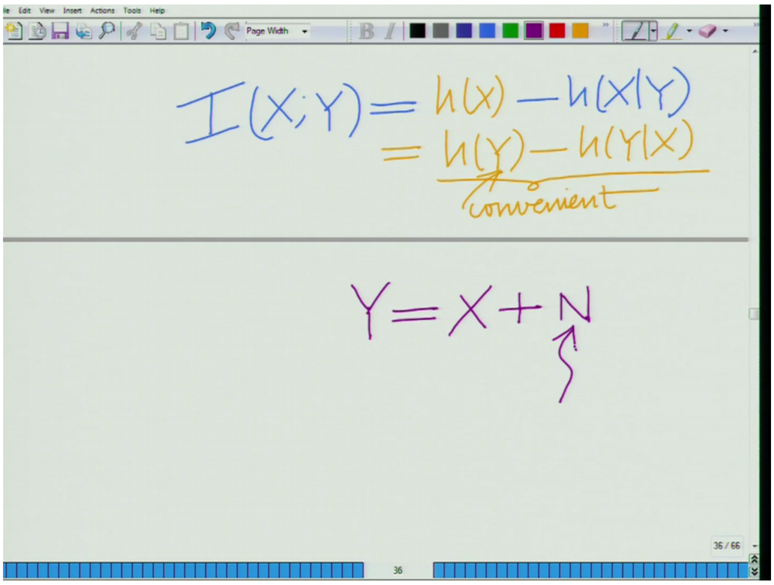

Now, first in order to do this first realize that from our definition of mutual information the mutual information is nothing, but h X minus h X given Y this is the mutual information. So, this is the entrope; this is let us write in a slightly different form of course, we know it is also equal to no harm because this is also correct. So, this is h X minus h X given Y equals h Y minus h Y given X this is particularly convenient for us as we are going to see shortly this is a particularly convenient form and h Y given X of course, h Y is the differential entropy of Y you must have you are all you must already be familiar with these quantities h Y given X is the differential entropy of Y given X.

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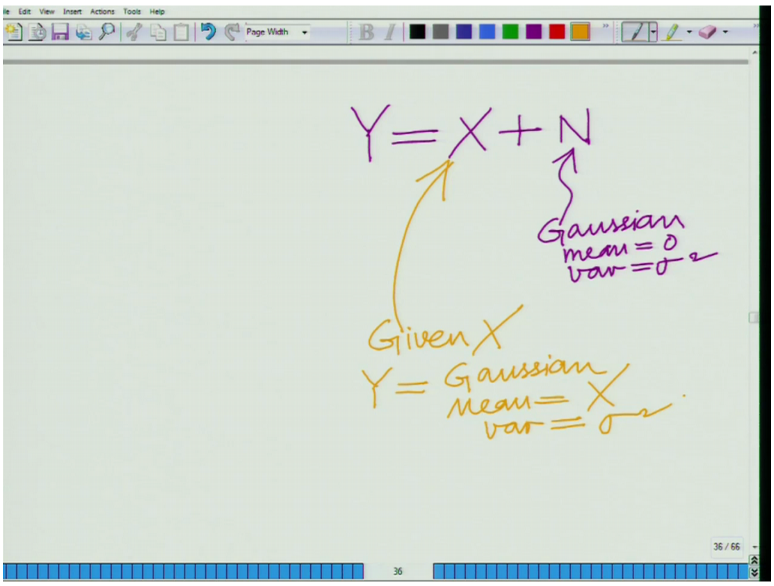

Now, if you can look at if you can go back and look at our channel modal Y equals X plus N; N is Gaussian remember the notation is N is Gaussian mean equal to 0 variance equals sigma square now with the addition of Y now given Y basically you have Gaussian noise N. Now given X it means X is a constant and this constant X is being added to the Gaussian noise of a constant is added to a 0 mean Gaussian random variable all it does is it simply shifts the mean right the mean of this Gaussian noise N is 0 the addition of this constant X simply shifts the mean to the X and the variance remains unchanged. So, given X now you have to realize this is something very important given X Y equal to Gaussian mean equal to 0 variance I am sorry mean equal to X it simply shifts the mean is equal to X variance remains unchanged.

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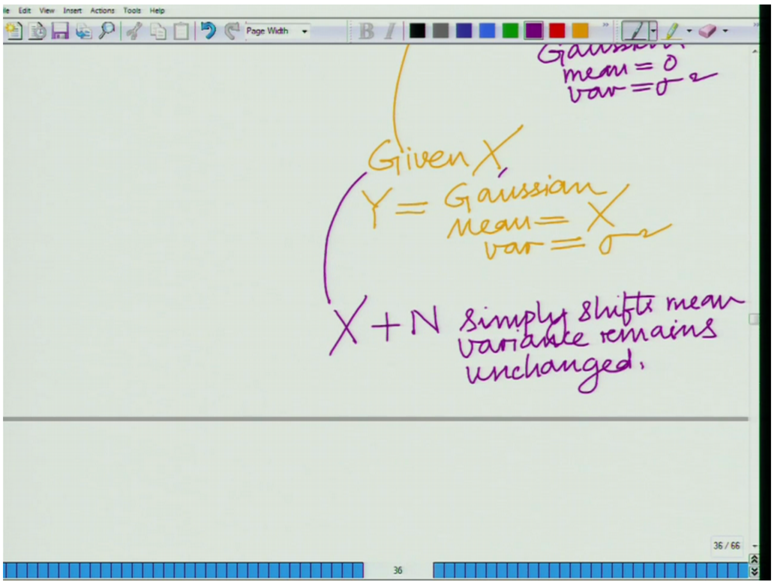

So, given X given X; X plus N simply shifts mean variance the variance remains unchanged.

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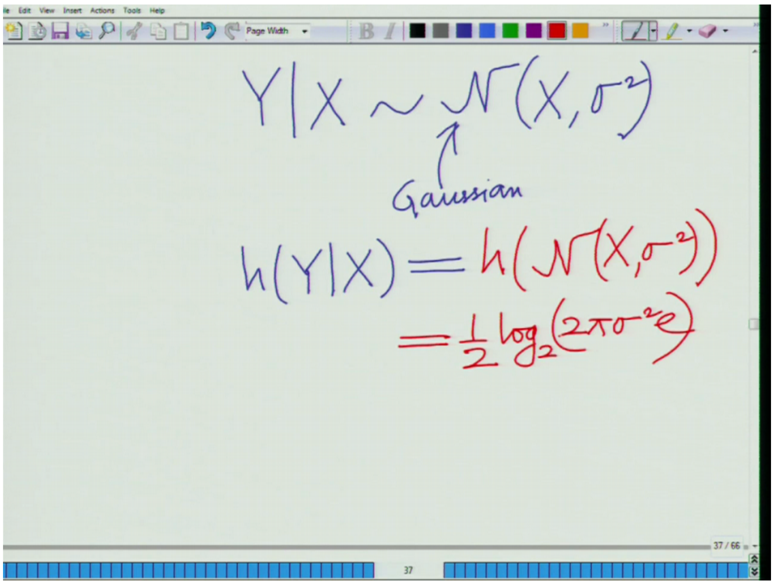

Therefore if you look at Y given X this is distributed as Gaussian remember N denotes Gaussian or normal with mean X and variance sigma square therefore, h of Y given X the first inter interesting property you will observe is h of Y given X is h of Gaussian source with mean X variance sigma square the difference entropy of the Gaussian random variable or a Gaussian source with mean and variance sigmas with a given mean and variance sigmas as we have seen it is independent of the mean it does not depend on the mean it is simply half log to the base to 2 pi sigma e correct which does not depend on the mean X. So, this is a constant. So, basically this is simply equal to depends only on the variance to refresh your memory we might as well look at what we have derived previously.

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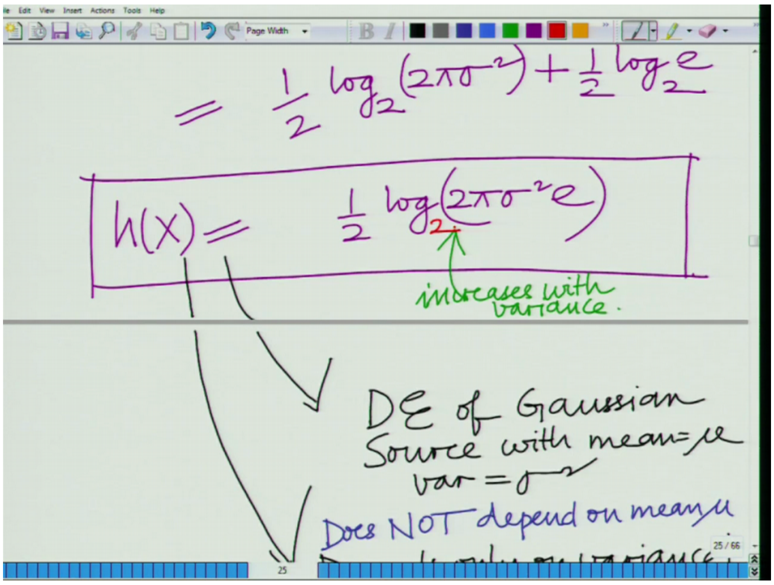

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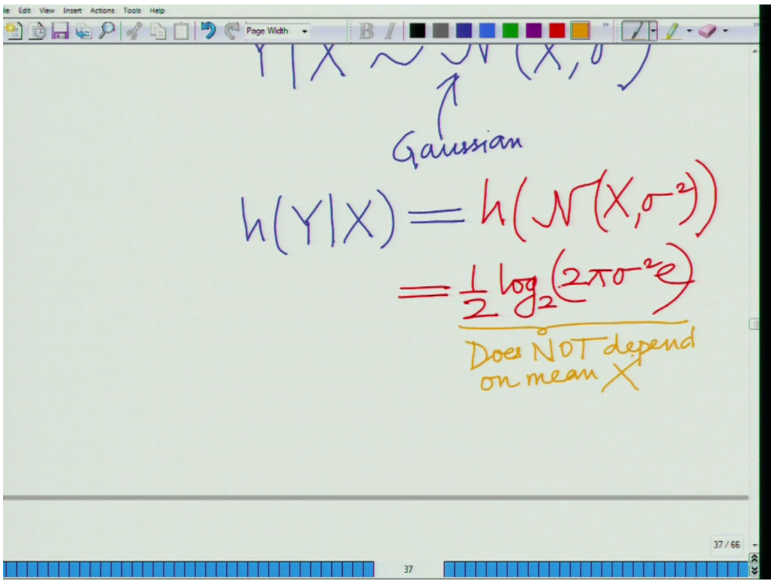

That it is half log to the base 2 log to the base 2 2 pi sigma square it does not depend on the does not depend on the mean we have seen that before does not depend on mean X.

So, given X the differential entropy of Y that is differential of Y given X is simply half log to the base 2 2 pi sigma square t now what about h of Y now let us look at Y.

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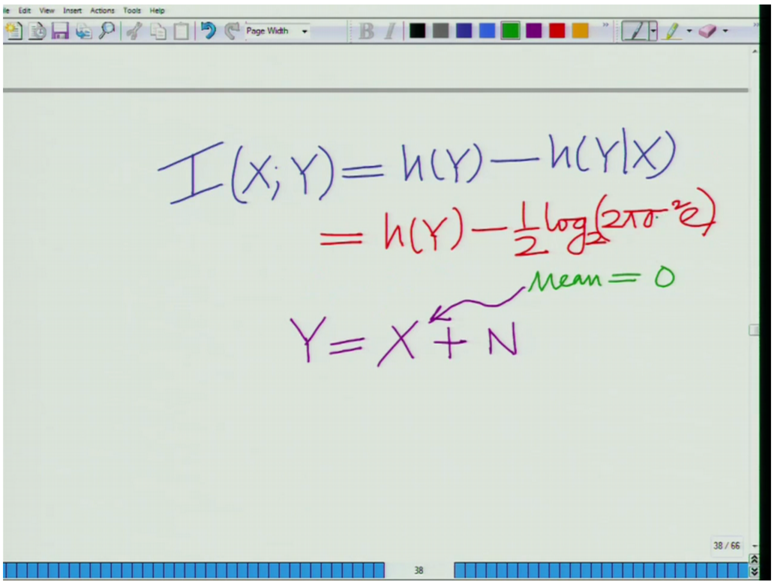

So, I of Y I of, so the mutual information between X comma Y now boils down to well it is h of my h of Y minus h of Y given X which is now h of Y minus h of Y given X is half log to the base 2 2 pi sigma square e now what about this quantity h of Y. Before we look at Y let us look at what is Y y equals X plus N let us assume that X is also 0 mean although this is let us relax this and assume that X is also 0 mean.

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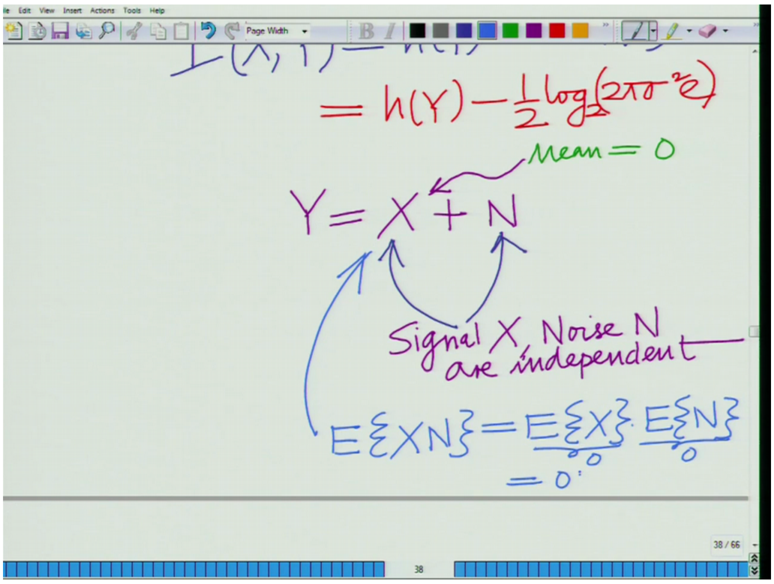

And further an important property in the communication system is that the signal X and the noise Y are independent because the noise is added by the channel or at the receiver exits the transmit symbols all right. So, these 2 arise from fundamentally different processes one is the signal the other is the noise and the signal and noise these come from different sources correct. These are there are different mechanisms which give rise to the signal and noise and therefore, these 2 random variables are independent.

So, that is not one of the fundamental assumptions these are independent now as a consequence of that what we are going to have is that expected value of X and N if these are independent X and N are independent that is equal to expected value of X into expected value of N.

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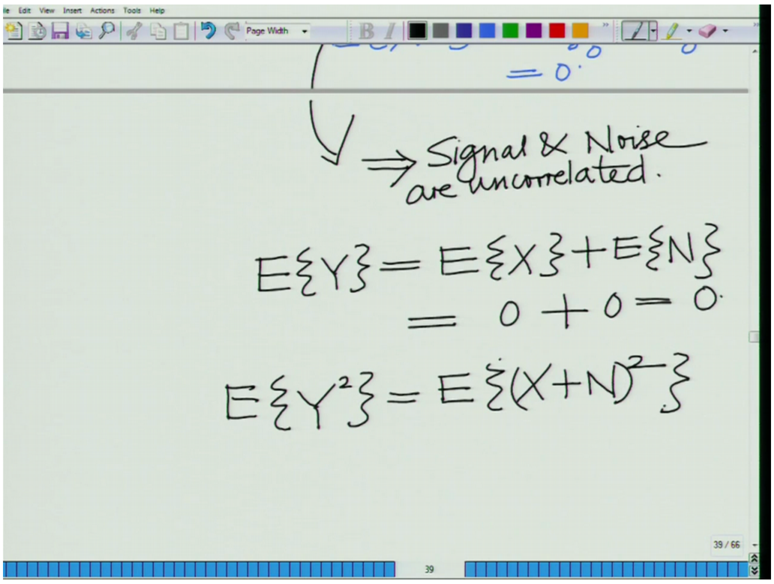

This is 0 this is 0. So, this is equal to 0. So, this also implies that they are uncorrelated implies signal and noise are uncorrelated. So, independent implies that signal and noise are now, therefore, at this point if you look at this Y is equal to expected its well let us look at expected Y expected Y is of course, Y is X plus N. So, this is expected X plus expected N which is equal to 0 plus 0 equal to 0.

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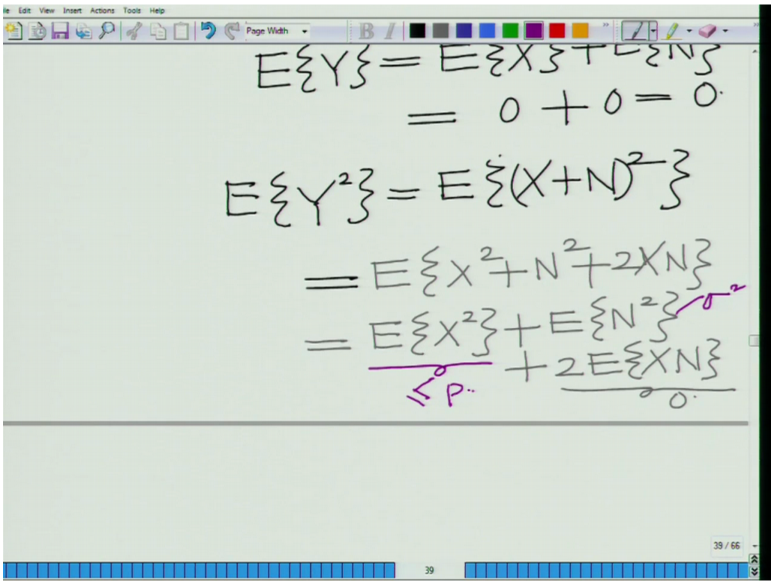

Further if you look at expected Y square this is equal to well expected X plus N whole square which is equal to you can simplify this as expected value of X square plus N square plus 2 X N, now we know what is expected x, so, this is equal to expected X square plus expected N square plus twice expected value of X comma N of course, these are uncorrelated.

So, this is 0; expected N square this is equal to sigma square now this is less than or equal to P which means (Refer Time: 18:32) if you remember the source power is limited or the transmitter is limited.

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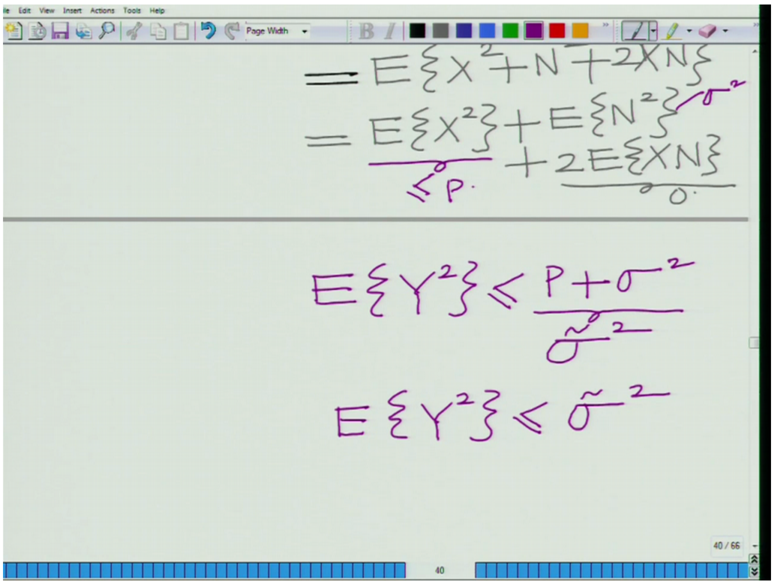

Therefore expected X square is less than or equal to P therefore, we have using this and the independence between signal and noise we obtain expected Y square is less than or equal to P plus sigma square we let us denote this by sigma tilde square. So, we have the assumption that expected Y square or we have the result not the assumption we have the result that expected Y square is less than or equal to sigma tilde square. So, that is the; so, now, we have to maximize remember we have to maximize the mutual information subjected to transmit power X P we can equally frame that as subject as maximizing the mutual information subject to the received power that is power of Y being P plus sigma square that is sigma tilde. So, naturally the transmit power is P noise power is sigma square the maximum power received it can be P plus sigma square that is sigma tilde square.

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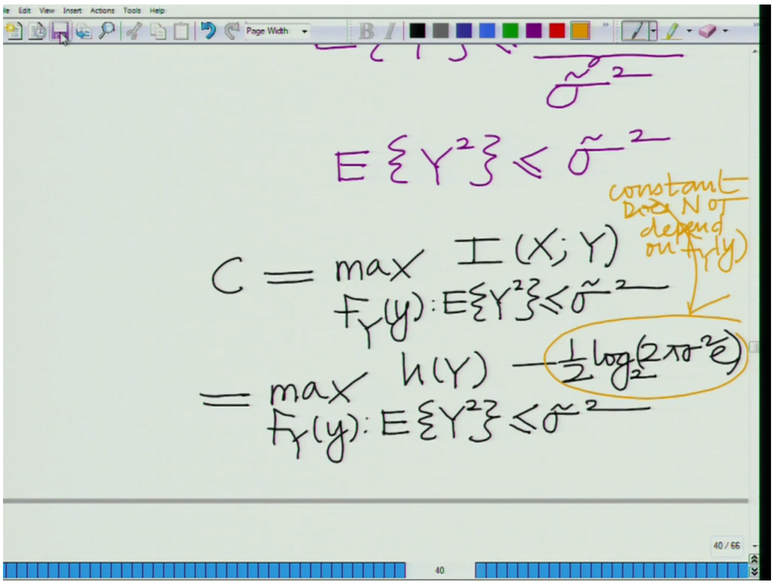

That is what I am going to do. So, I write C equal to max well now over instead of F of X choose the probability density function Y such that expected Y square less than or equal to sigma tilde square maximize the mutual information Y between X and Y. Now we have already seen very interestingly this is maximum F of Y subject to F of Y expected Y square less than or equal to sigma tilde square h of my h of Y minus half log 2 pi sigma square log to the base 2, 2 pi square.

Now if you observe this; this is your h of Y given X this is a constant; constant in the sense does not depend on it does not depend on Y or it does not depend on the input probability density function Y very interestingly h of Y given X right h of Y given X because given X Y is simply Gaussian noise Y is simply Gaussian noise with mean X variance sigma square. So, h of Y given X does not depend on does not depend on this probability with it does not depend on this quantity with which respect to which with it does not depend on sigma tilde it does not depend on this.

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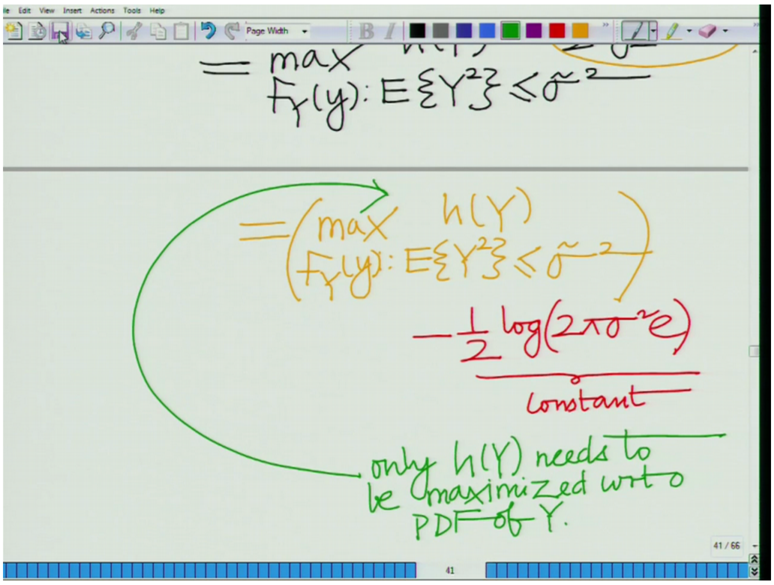

So, basically this is a constant which means the only quantity that needs to be maximized with respect to the probability density function of Y is equivalently I can write this as max F of Y with respect to the probability density function of Y expected Y square less than or equal to sigma tilde square h of Y because this is the only quantity now that has to be maximized with respect to the probability density function of Y minus this constant half log 2 pi sigma square C. So, only this needs to be maximized because half log 2 sigma square e this is a constant. So, only this needs to be maximized with respect to P D F of Y. So, that is where we are currently. So, now, we have to maximize this differential entropy of Y of the output Y with respect to the P D F that is choose an appropriate P D F which maximizes the differential entropy of this output y, but subject to the constraint with the power at the output is less than or equal to sigma tilde square that is P plus sigma square. Now, how to perform this maximization of differential entropy subject to this power constraint is something that we will differ to the next module, alright.

So, we will stop here basically where we are trying to characterize the capacity the maximum rate at which information can be transmitted across in Gaussian channel that is the channel with Gaussian noise. So, we will stop here and continue in the subsequent modules.

Thank you very much.