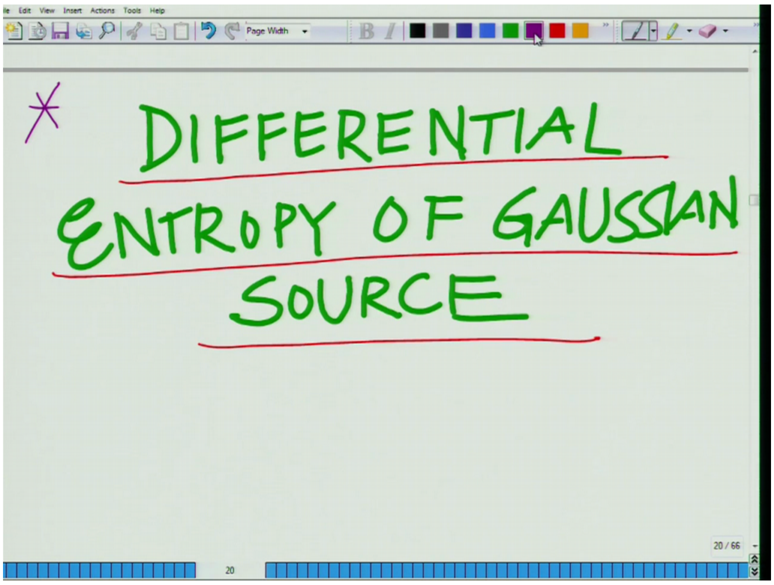
Principles of Communication Systems - Part II
Prof. Aditya K. Jagannatham
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

Lecture – 37
Differential Entropy of Gaussian Source, Insights

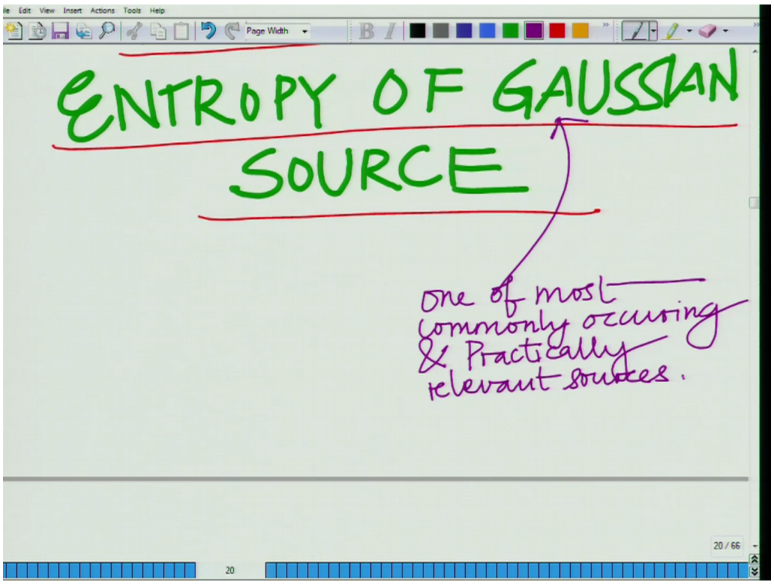
Hello, welcome to another module in this massive open online course. So, we are looking at the concept of differential entropy for a source which can take continuous set of values alright. So, source alphabet which can take a continuous set of values right now let us look at the differential entropy of a Gaussian source that is a source output which follows the Gaussian which has the Gaussian probability density function and this is very important because this is as we are going to see later this is one of the; I mean this is one of the most commonly occurring such source or one can say that the source output at the output one of the most commonly occurring probability density functions for the source output and has a fundamental role to play in the context of information theory as we are going to see later.

(Refer Slide Time: 01:00)



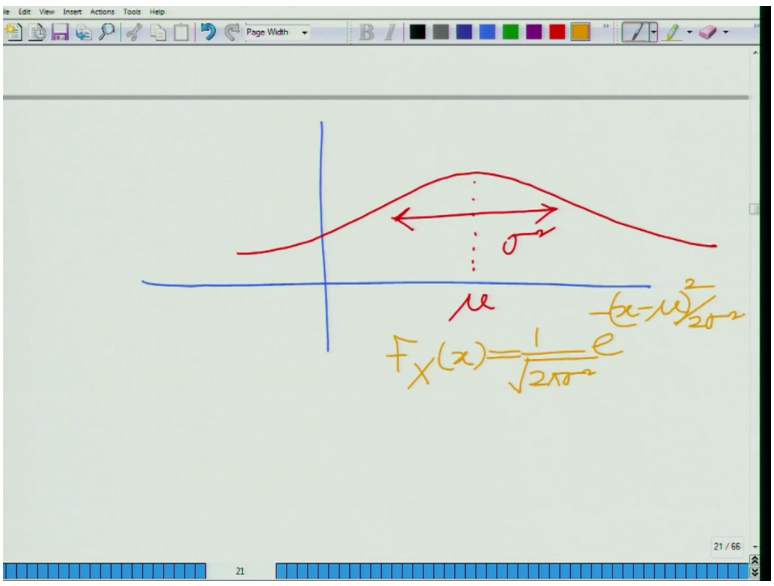
So, we are going to look at the differential entropy differential entropy of a; this is the differential entropy of a Gaussian source.

(Refer Slide Time: 01:42)



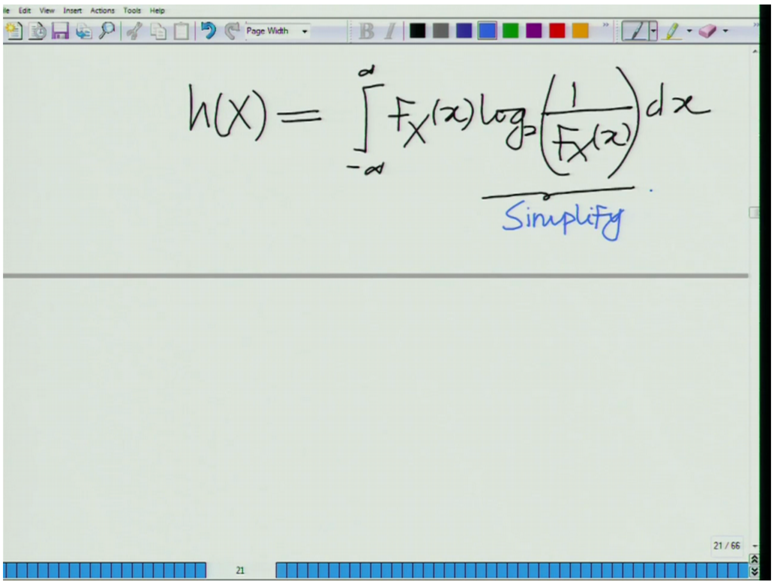
And this is important because this is one of the most is a fundamental relevance in the context of information theory and its one of the most commonly occurring practically relevant source practically relevant source in the sense frequently this is Gaussian distribution naturally occurs; occurs very frequently in nature. The source output source probability the probability density function of the source output frequently of and be re frequently represented using at the Gaussian probability density function alright. So, it is important to characterize the differential entropy of such a Gaussian source.

(Refer Slide Time: 02:39)



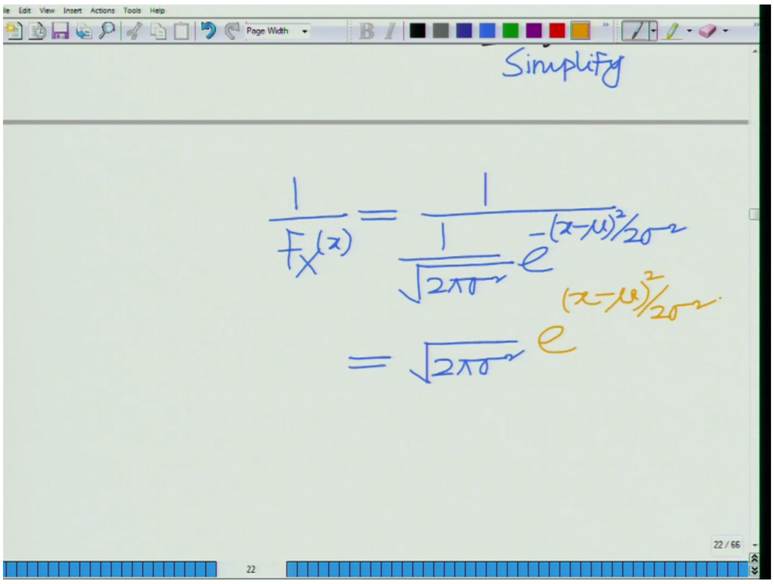
And as we have known and as we have seen several times before no need to repeat it the Gaussian probability density function is characterized by bell shaped curve which has the peak at the mean and the spread is proportional to the variance we have the probability density function that is 1 over square root of 2 pi sigma square e power minus X minus mu square divided by 2 sigma square.

(Refer Slide Time: 03:20)



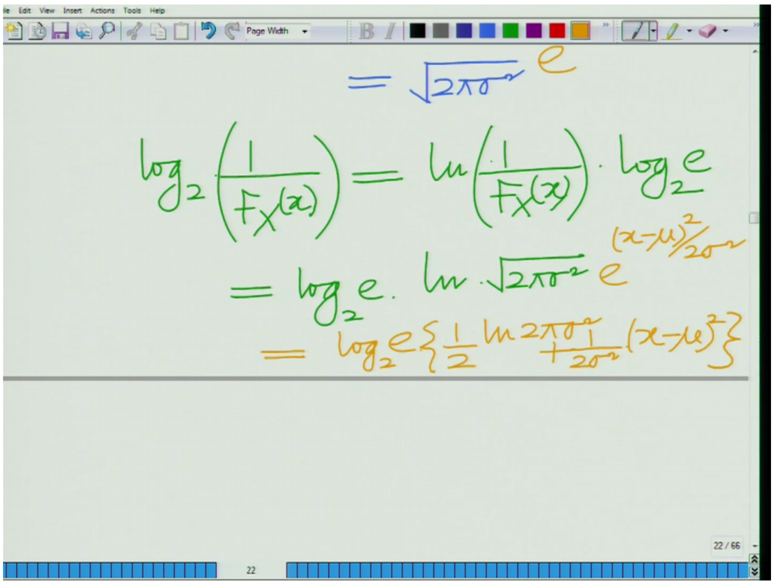
The differential entropy of this source is basically h X equals minus infinity to infinity log to the base 2 1 over F of X d x.

(Refer Slide Time: 03:56)



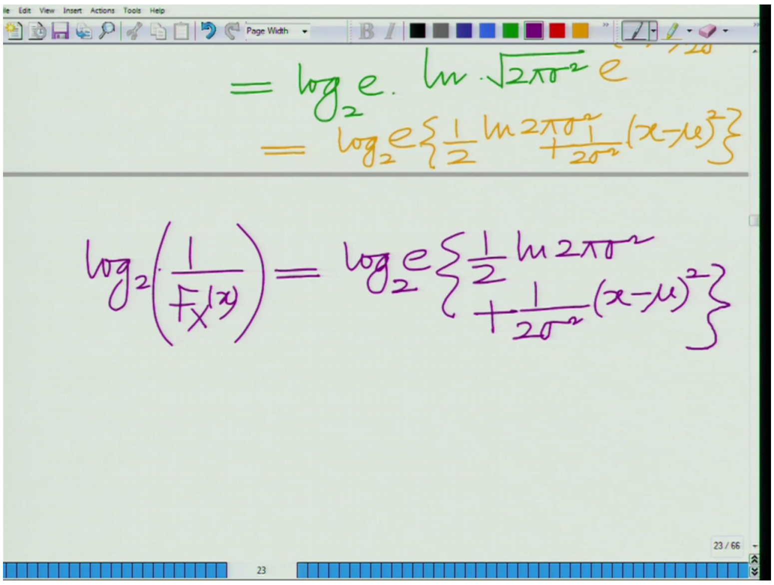
Now, let us simplify this quantity log to the base 2 you would like to simplify this quantity and then substituted back here. So, let us look at 1 over F of X x that is 1 over 1 over e square root of 2 pi sigma square e power minus X minus mu whole square divided by 2 sigma square take the reciprocal this is simply square root 2 pi sigma square e raise to positive X minus mu square divided by 2 sigma square.

(Refer Slide Time: 04:41)



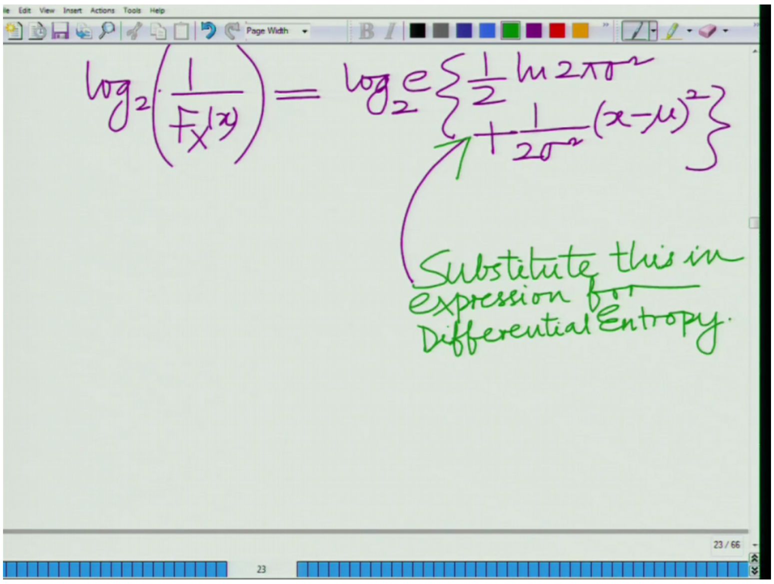
And if you take the log to the base 2 of 1 over F of X x that is basically log natural the natural logarithm of 1 over F of X x times log e to the base 2 and you can see from this the natural logarithm well that is log e to the base 2 times the natural logarithm of square root of 2 pi sigma square e raise to well X minus mu square by 2 sigma square which is log e to the base 2. Now log of square root of 2 pi sigma square is half 2 pi sigma square plus log e to the power of well I am sorry, this is half log 2 pi sigma square plus 1 over 2 sigma square into X minus mu whole square.

(Refer Slide Time: 06:04)



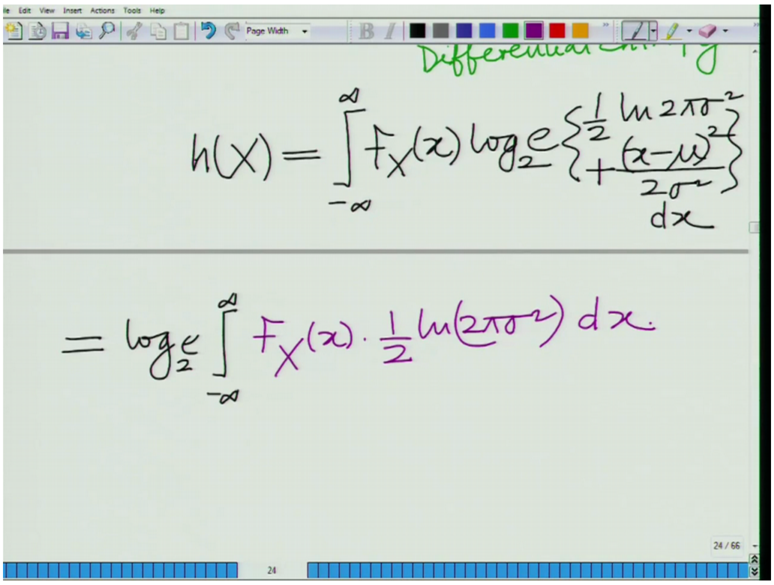
So, log to the base 2 this quantity is equal to this quantity is equal to log e to the base 2 half log 2 pi sigma square plus 1 over 2 sigma square times X minus mu square mu whole square we have simplified the expression for log to the base 2 1 over F of X and now going to substitute this in the expression for the differential entropy to evaluate the differential entropy of the Gaussian source.

(Refer Slide Time: 07:08)



So, now what we are going to do in expression for substitute this in the expression for the differential entropy.

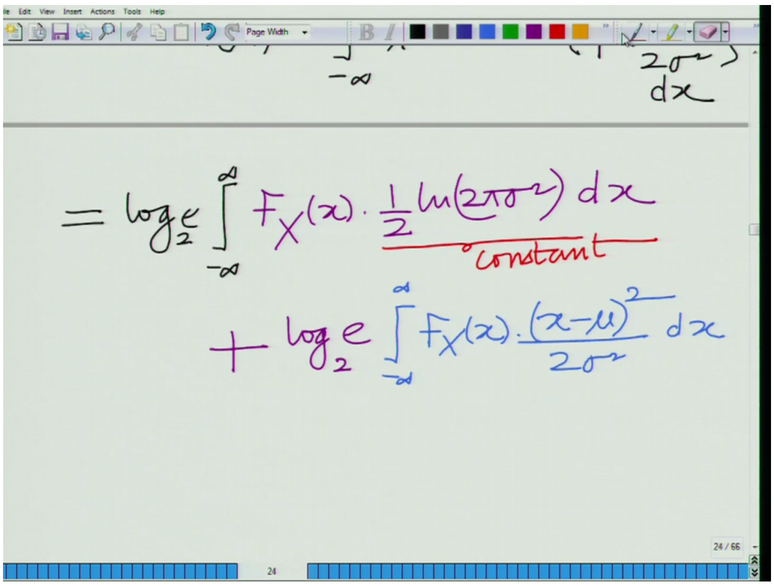
(Refer Slide Time: 07:41)



What I have is well h of X equals minus infinity to infinity F of X x times log e to the base 2 times half l n that is the natural logarithm 2 pi sigma square plus X minus mu whole square by 2 sigma square times of d x.

Now, let us look at this term by term of course, log e to the base 2 that is a constant.

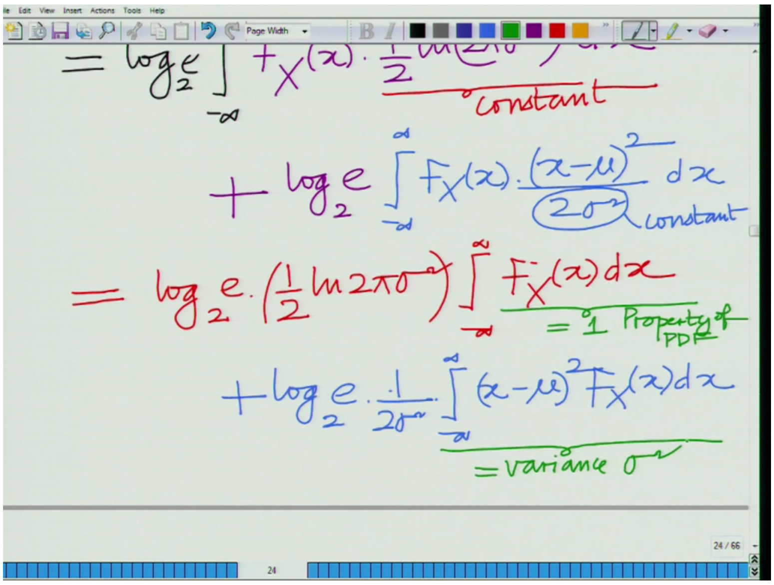
(Refer Slide Time: 08:58)



So, that comes out times F of X x lets split the term in the brackets times half l n d x plus well log e to the base 2 F of X x; X minus mu whole square by 2 sigma. So, I split the term into the bracket. So, let us split the term written it as 2 separate integrals.

And now you can see that in the first integral this is a constant half log 2 pi sigma square this is a constant. So, this will come out of the integral and if you look at this quantity this has a very interpret we will come to this now let us let me just rewrite this so that we can look at the entire thing at once.

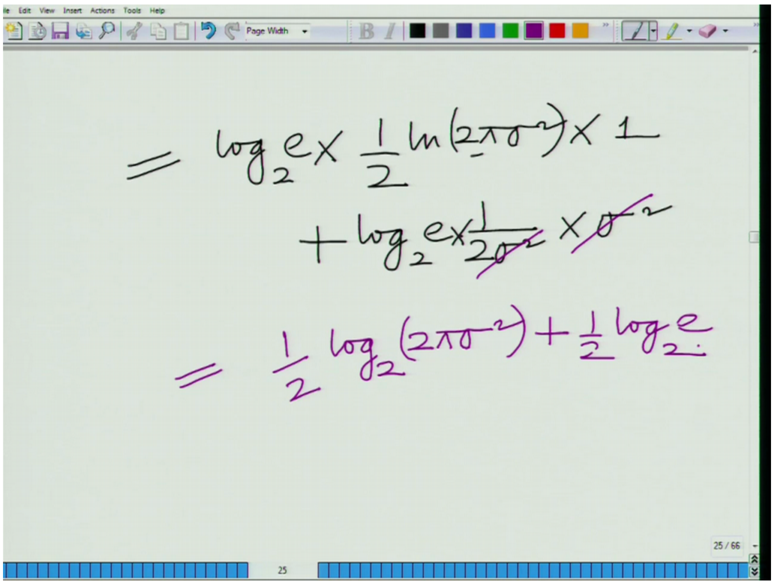
(Refer Slide Time: 10:01)



So, this is log e to the base 2 times half log 2 pi sigma square into integral F of X d x plus log e to the base 2, now in this 1 over 2 sigma square this is also a constant in the second integral. So, I can write this as 1 over 2 sigma square minus infinity to infinity X minus mu square F of X d x and you will observe 2 interesting things first this is equal to 1 because the area under this minus infinity to infinity F of X d x is nothing, but the area under the probability density function equals 1. So, this is nothing, but area under or this is nothing, but total area under pdf. So, from the; it follows basically from the definition or the properties of the pdf follows; say it follows from property of pdf.

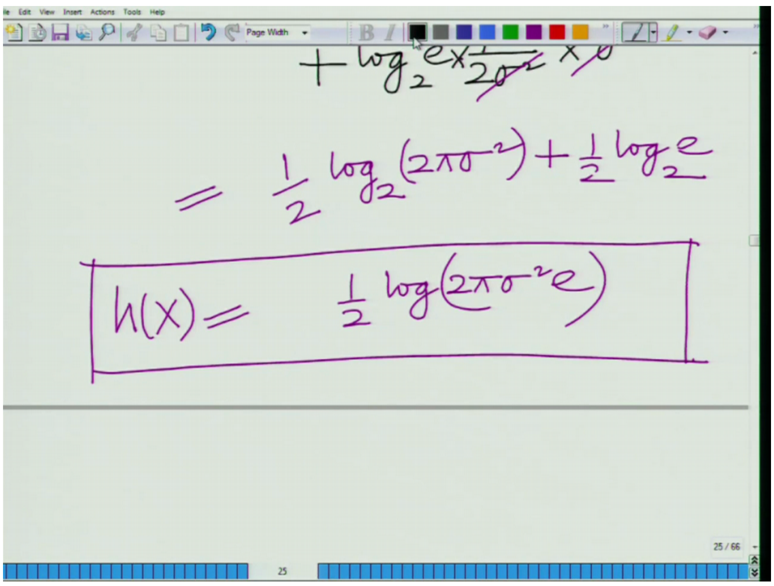
And if you look at this; this is very interesting this is integral minus infinity to infinity integral minus infinity X minus mu whole square plus where mu is the mean F of X d x this is nothing but the variance. This is simply the definition this is the variance this is nothing but this is equal to the variance that is integral minus infinity to infinity X minus mu square F of X d x that is nothing but the second center second order cent the central movement of second order that is the variance of the random variable this is for any general random variable not necessarily only for a Gaussian random variable this is the definition of the variance of a random variable.

(Refer Slide Time: 12:23)



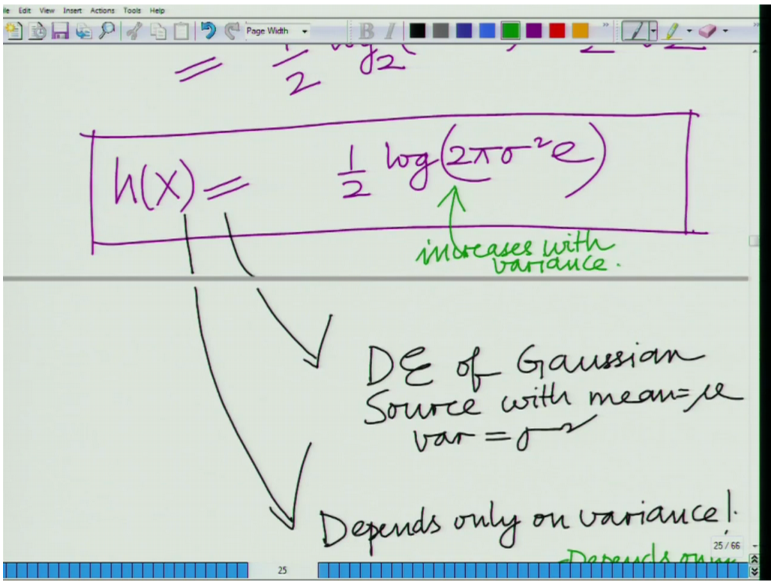
So, now I substitute these and what I have is basically log e to the base 2 times half l n 2 pi sigma square times 1 plus log e to the base 2 into 1 over 2 sigma square into sigma square and what you can see is basically the sigma squares cancel and therefore, I can like this as well log e to the base 2 into half log 2 pi sigma half l n 2 pi sigma square. So, I can write this as half log.

(Refer Slide Time: 13:36)



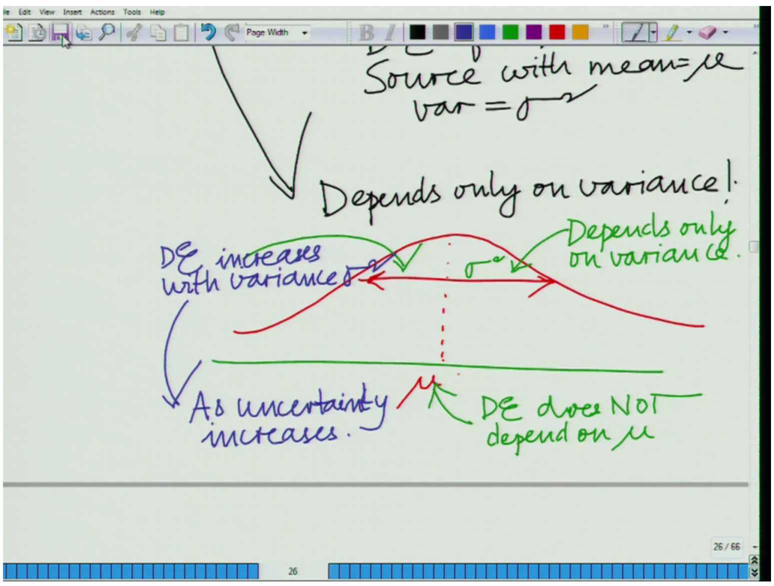
I can write this is half log to the base 2; 2 pi sigma square because log e to the base 2 into a log 2 pi sigma square to the base e is basically nothing, but log 2 pi sigma square to the base 2 plus half log e to the base 2 which is equal to which is equal to half log 2 pi half log 2 pi sigma square e.

(Refer Slide Time: 14:00).



And this is the differential entropy of Gaussian source that is output is Gaussian probability density function with mean equal to mu and variance equal to sigma square and observe that it depends only on the variance.

(Refer Slide Time: 14:28)

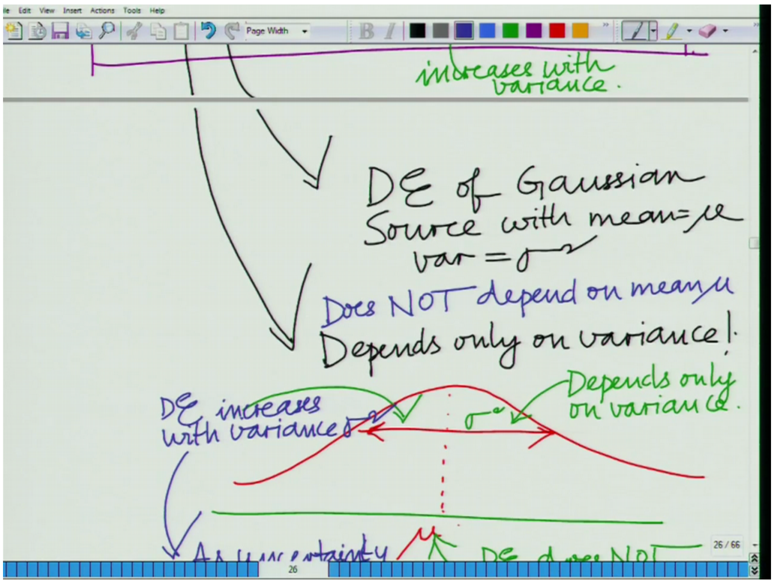


That is the something very interesting it does not depend on the on the mean mu that is we take your Gaussian source shift the output probability density function by a constant mean alright the resulting differential entropy is not different right this is the differential entropy of the source does not change it does not depend on mu it only depends on the variance sigma square and that is very interesting because if you see. So, this is mu the d e does not depend on mu it depends only on the variance it depends only on the variance that is the spread and you can see half log 2 pi sigma square this increases with variance that is what you can see is that as the variance increases of this Gaussian the randomness increases the spread increases; alright.

When its variance is low then its concentrated towards the mean alright one can say that the randomness of the uncertainty of the source is low as the variance increases the spread around the mean or the spread from the mean increases; alright. So, the randomness of the source is increasing therefore, the differential entropy is increasing or the uncertainty is increasing and entropy is nothing, but a measure of the uncertainty. So, d e increases with variance sigma square basically because as the uncertainty increases.

So, the differential entropy is you are observing 2 interesting things one is that it does not depend on the mean that is shift by a constant factor which affects only them that is of shift.

(Refer Slide Time: 17:02)



The Gaussian probability density function by a constant that is the Gaussian probability density function is just shifted in the mean the variance remains unchanged the differential entropy of the source remains unchanged it does not depend on the mean. And also you can see it depend only on the variance and it increases with the variance as the variance increases the uncertainty spread around the mean increases the uncertainty of the Gaussian sources increasing therefore, the differential entropy is increasing. So, these are some interesting aspects alright.

So, first we have derived the differential entropy of this Gaussian source which is very fundamental which has very which very relevant for as a fundamental relevance in the context of information theory as we are going to see later when we discuss about the channel capacity or the capacity of a Gaussian channel with Gaussian noise and also you can see that has an interesting property that the differential entropy does not depend on the mean and it increases with the variance alright. So, let us stop here and we look at other aspects in the subsequent modules.

Thank you very much.