Principles of Communication Systems – Part II
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Lecture – 27
Definition of Entropy, Average Information/ Uncertainty of Source, Properties of Entropy

Hello. Welcome to another module in this massive open online. So, in this module alright, let us start looking at a new topic that is entropy which is one of the most fundamental aspects of information theory, which we will going to define shortly. So, this is entropy.

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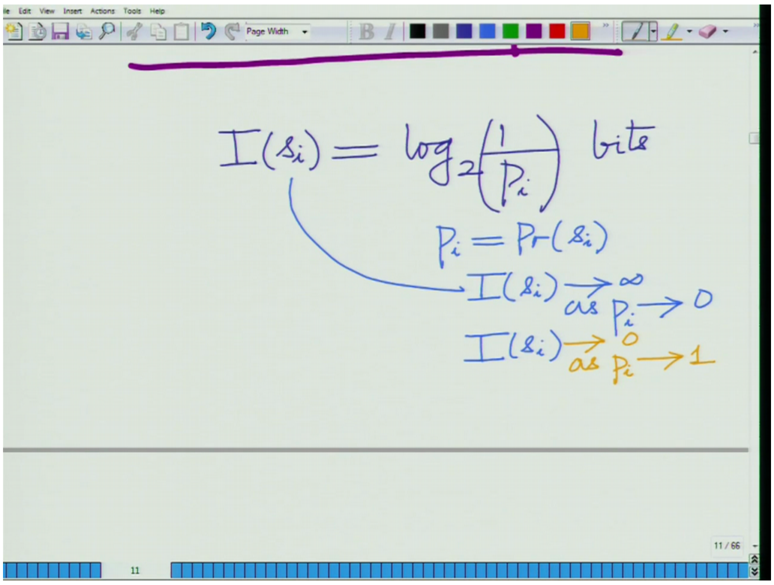

So, we have seen in our discussion that, I si the information per symbol with the corresponding to source alphabet S I is log 1 to the base log 1 over Pi to the base 2 bit is correct right. And well Pi we have seen this is nothing, but the probability of the symbol S I and we have seen that Isi this quantity tends to infinity as Pi S the probability as base equally Pi tends to 0. And I si tends to 0 that is for certain events that is when the event occurs with probability 1 I si tends to 0 information tends to 0 alright. So, frequently occurring event have less information rare events have higher amount of information associated with them.

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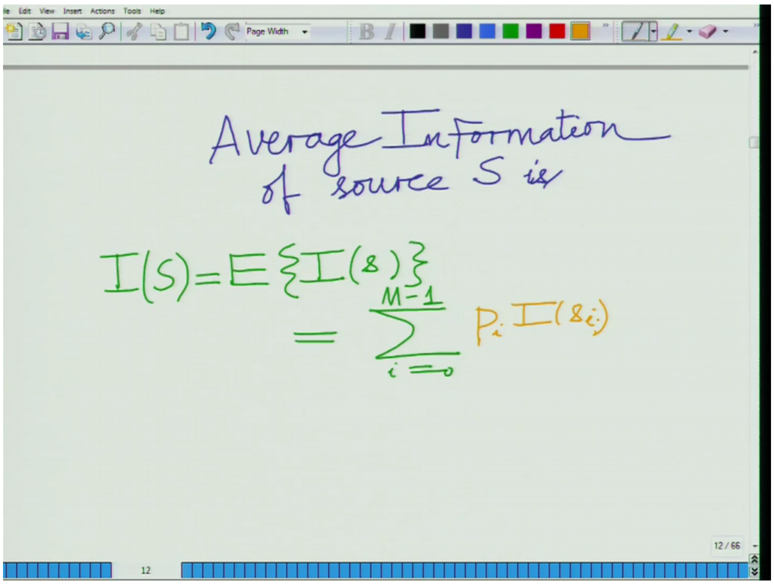

Now, the average information the average information of the source can now be defined as the average information of source S well average information of source S is simply the expected value this is I of S which is expected value of I of well expected value of I of S which is basically nothing, but well this is summation i equal to 0 summation i equal to 0 to m minus 1 log 2. So, this is Pi, log 2 or let me write it this way. This is Pi times the information that probability of symbol i times the information associated with symbol i.

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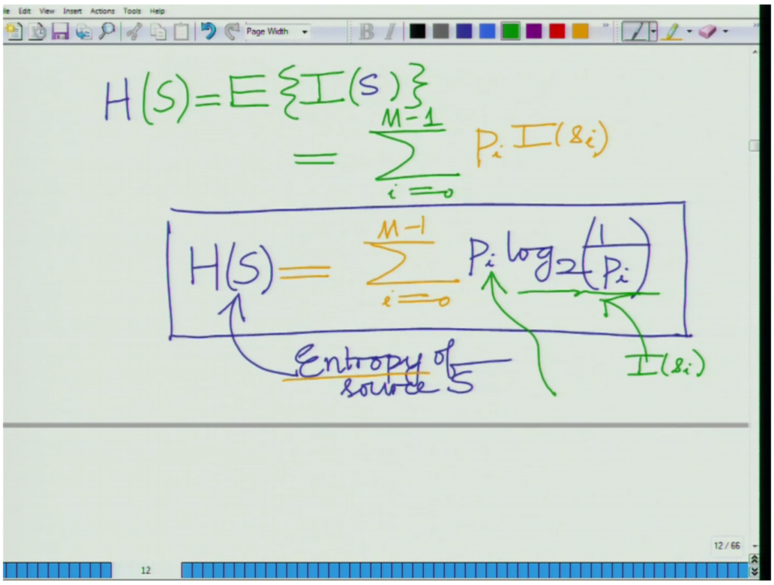

So, we are computing of weighted average. So, this is equal to Pi log to the base log 2 to the base 1 over Pi. And this quantity it is expected value of s this quantity is termed as H of S H of S equals Pi log 1 over Pi to the base 2. This is termed as it is a fundamental quantity of information theory is termed as the entropy. This is termed as the entropy of sources.

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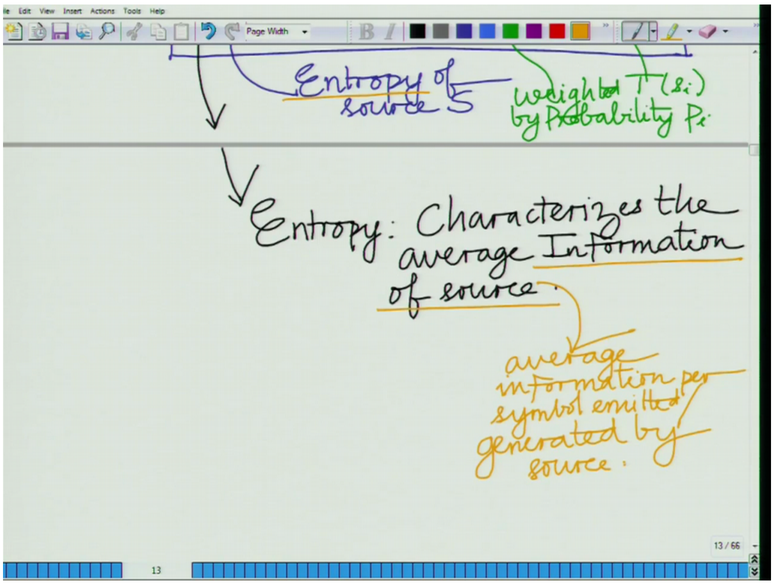

So, this is termed as the entropy of source S. So, it is basically information of symbol i this is your information of symbol i weighted by the probability Pi, weighted by it is probability of occurrence. So, it is the expected value as a average information of the source which is obtained by weighing the information of each symbol S I that is I si with it is probability Pi and summing over all the alphabets summing over the entire source alphabet, alright.

So, this quantity is termed as the entropy of the source and as we have said is a fundamental quantity in information theory. It can characterizes the average information per source average it characterizes. So, this characterizes entropy characterizes or quantifies average information content, average information of the source average information of the source as in the information average information per symbol one can also think of average information per symbol emitted by the source. Emitted or generated emitted or generated. This can also be thought of as entropy can also be thought of as average uncertainty.

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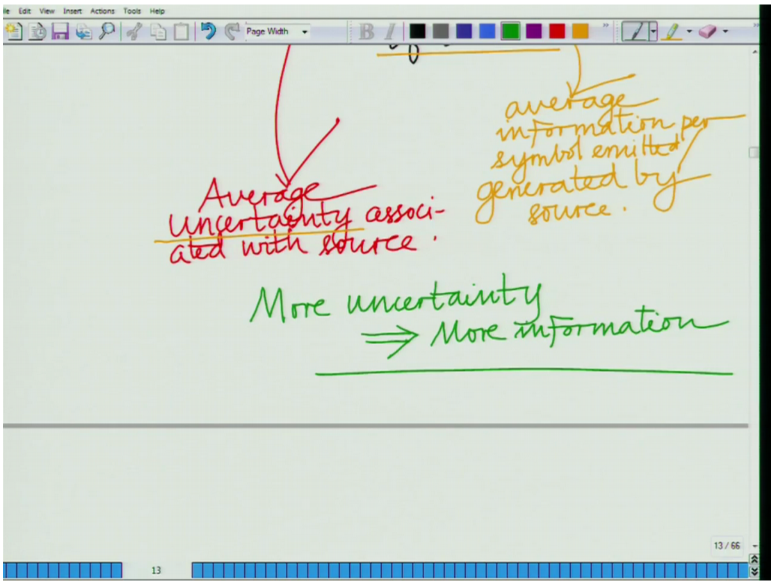

It is the average the average uncertainty associated with the source. So, more uncertainty also implies basically more randomness or so more uncertainty.

So, more uncertainty more uncertainty implies more information. So, you can also be thought of as the average uncertainty right. So, this symbol what is uncertain about the next symbol that is going to be generated about the source all right. There is which means that there is greater uncertainty, which captures the fact that that is created information in the symbols generated by the source. If this if there is a greater certainty if we know for sure what the or if we show or if you know with great certainty what the next symbol that is generated by the source is going to be; that means, on an average there is less information in the symbols or less information content in the symbols that is generated by the source.

We are going to see this also of course, as we proceed through subsequent aspects or subsequent examples of entropy.

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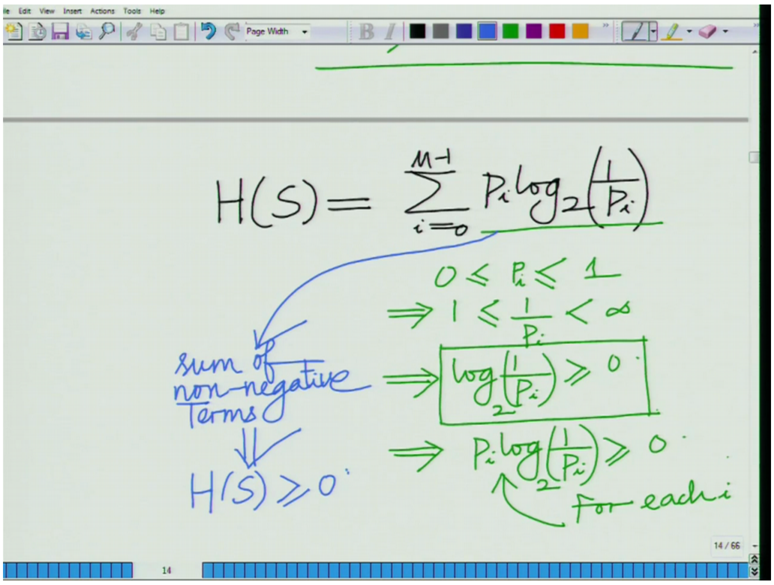

Now, let us start by looking at some simple properties of entropy, if you look at the definition of entropy which is Pi log on or p an observed that well Pi greater than equal to 0 less than equal to 1 over Pi less than or equal to 1 this implies well one less than or equal to 1 over Pi less than. So, 1 over Pi is always basically greater than or equal to 1 which implies log 1 over Pi to the base 2 is greater than or equal to 0. Since 1 over Pi is greater than equal to 1. So, Pi is greater than equal to 1 log to the 2 to the base 1 over Pi. Log to the base 2 1 over p is greater than equal to 0. So, Pi is greater than equal to 0 correct log 1, over Pi to the base 2 is greater than equal to 0, this implies both of these are non negative quantities.

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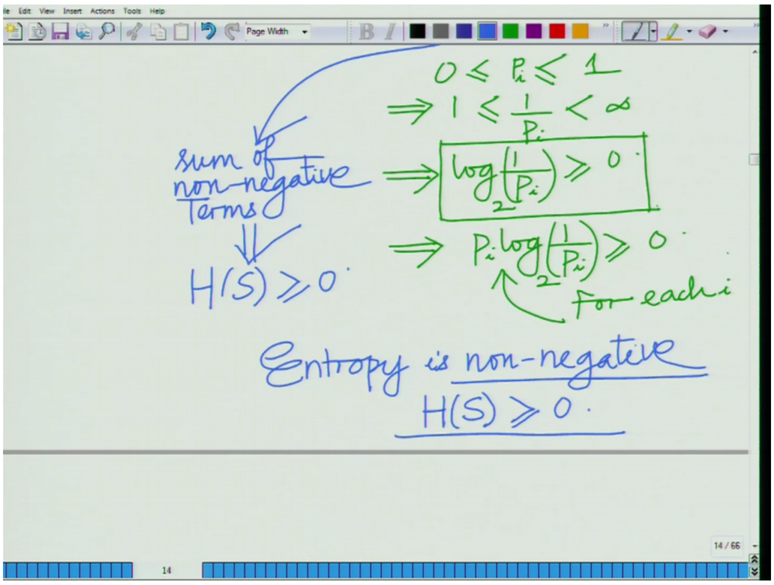

So, therefore, Pi log 1 over Pi to the base 2 is greater than or equal to 0 for each i. So, this is basically this entropy is the sum of non negative terms, implies H of S is also greater than or equal to 0. So, entropy is non negative correct. Entropy can be zero, but it cannot be negative. In fact, we will see examples where entropy is 0. So, entropy has to be greater than equal to. So, this also means that the information content of a source cannot be cannot be negative. It is non negative either 0 or greater than 0 entropy of a source the information content of a source has to be greater than or equal to 0.

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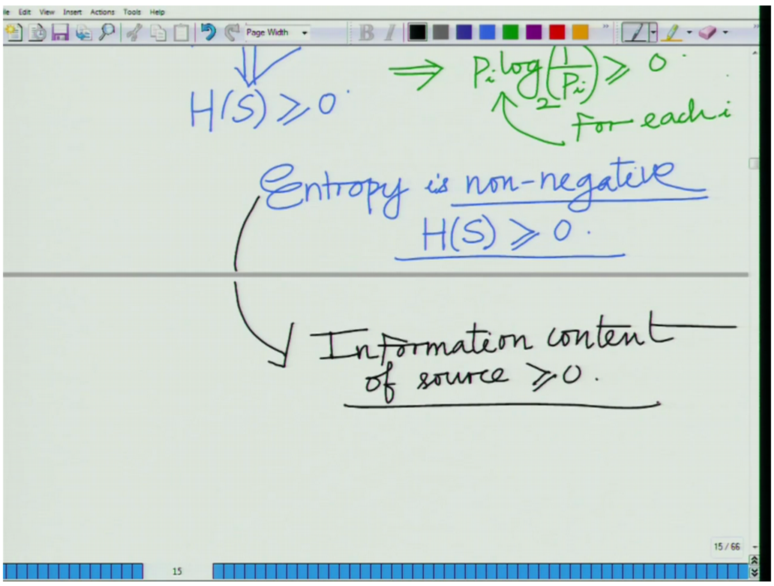

So, this basically means that the information content, information content of source is greater than equal to 0.

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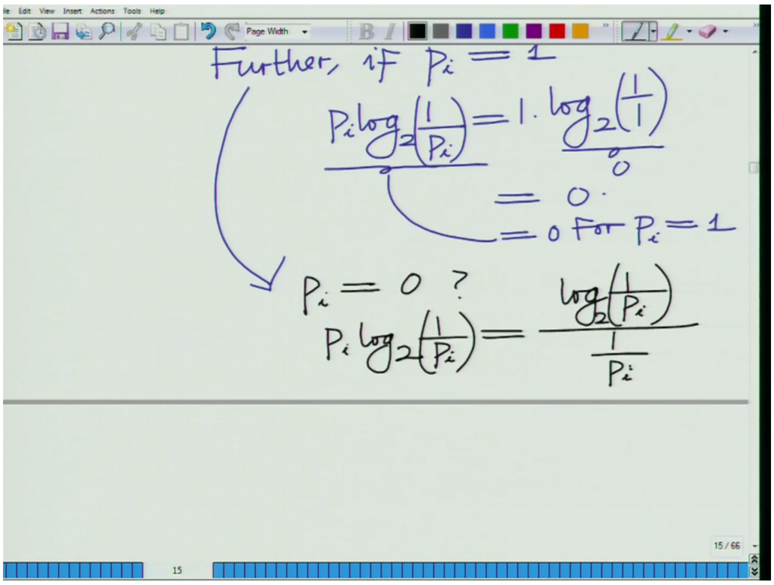

Further observe that if Pi equal to 1, then Pi log to the base 2 or Pi equal to 1 times log to the base 2 1 over 1 this is 0. So, this is equal to 0.

So, this quantity equals 0 for Pi well for Pi equal to 1. Now what happens if Pi equal to 0 what happens to this quantity Pi equals 0 Pi equals 0, let us look at this quantity Pi log to the base 2, 1 over Pi, which is equal to I can write it as log to the base 2, 1 over log to the base 2 1 over Pi divided by 1 over p i.

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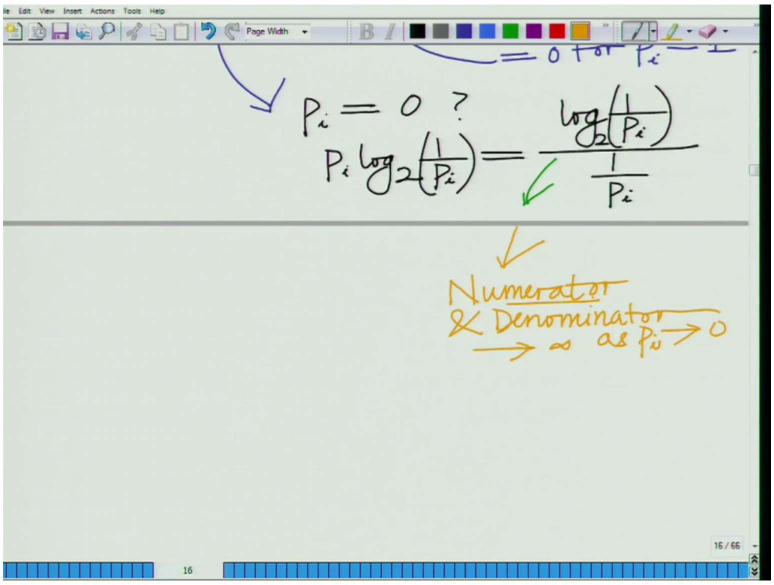

Now, you can see here that both the numerator that is if you look at this quantity here, numerator and the denominator tend to infinity as Pi tends to 0 as Pi tends to 0 both the numerator and denominator tend to tend to infinity. So, this is undefined. So, this expression at 0 in that sense it is undefined.

So, let us look at the limit as Pi tends to 0. And naturally if both the numerator and denominator tend to infinity as Pi tends to 0, we can use the lthopital's rule right which is the rule in calculus to evaluate the limit. So, correct.

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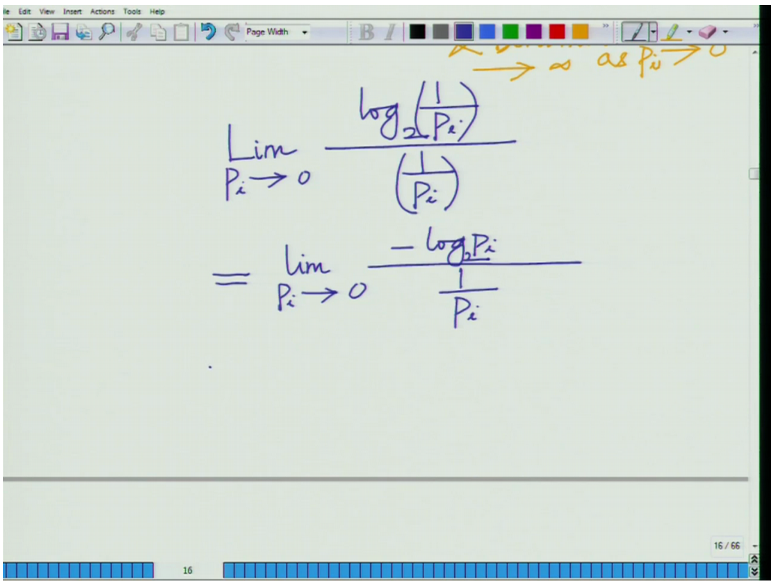

So, now let us calculate the limit of this as Pi tends to 0, that is log the base to 1 over Pi which is equal to well, I will write the numerator as minus log to the base 2 Pi 1 over Pi which is equal to.

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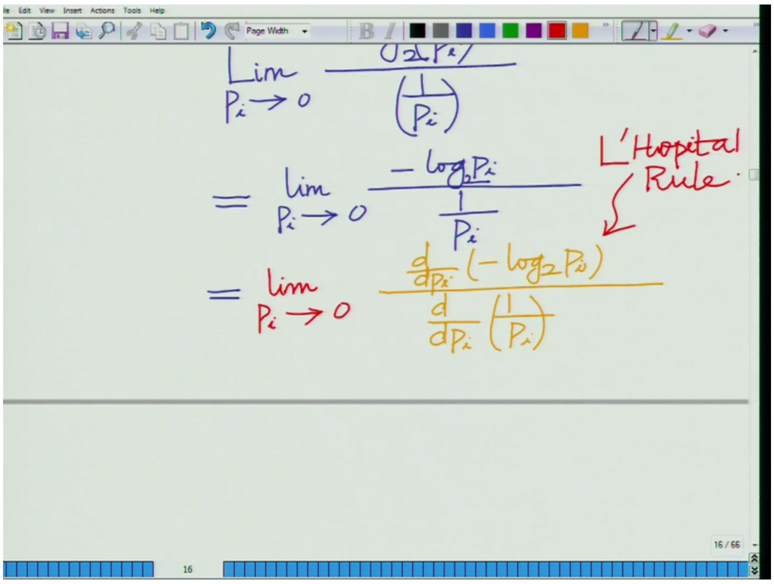

Now, again I can differentiate this I can take the derivative of the numerator since it is ill defined on substitution of 0, I can take minus log 2 to the base Pi. I can differentiate the numerator and denominator differentiate the numerator and denominator with respect to Pi. This is termed as a L'Hospital's Rule which is basically from calculus to evaluate correct the limit of precise with such quantities which are ill defined.

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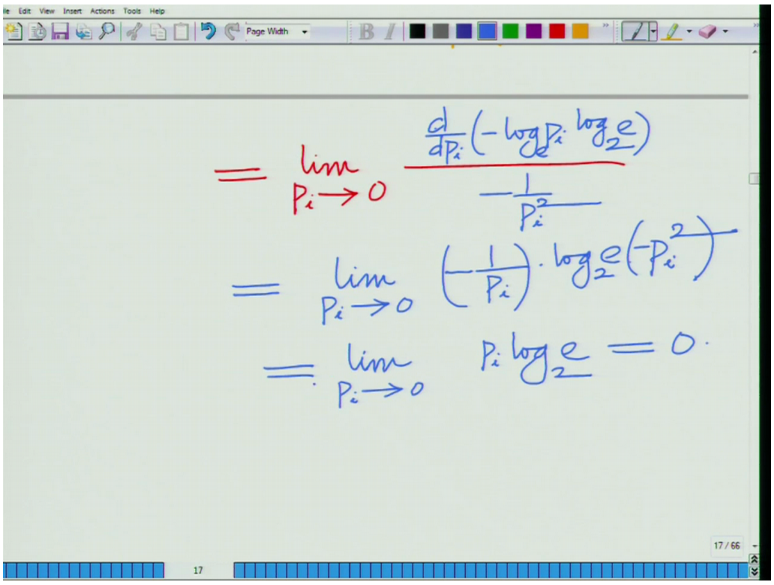

Now, if I substitute this that becomes limit Pi tends to 0, if you look at this that becomes well the numerator is well d by d minus log Pi to the base 2. I can write this as Pi 2 the base that is your traditional logarithm to the base e, to log e to the base 2 divided by well I can differentiate with respect to Pi that gives me minus 1 over Pi square. And finally, differentiating the numerator, I get minus log minus derivative of log is 1 over Pi minus log in to log e to the base 2 into Pi square from the denominator will come to the numerator. So this is basically limit Pi tends to 0, well Pi into log 2 p to the base 2 which is equal to 0. So, basically what we have established is if you look at this is limit that is.

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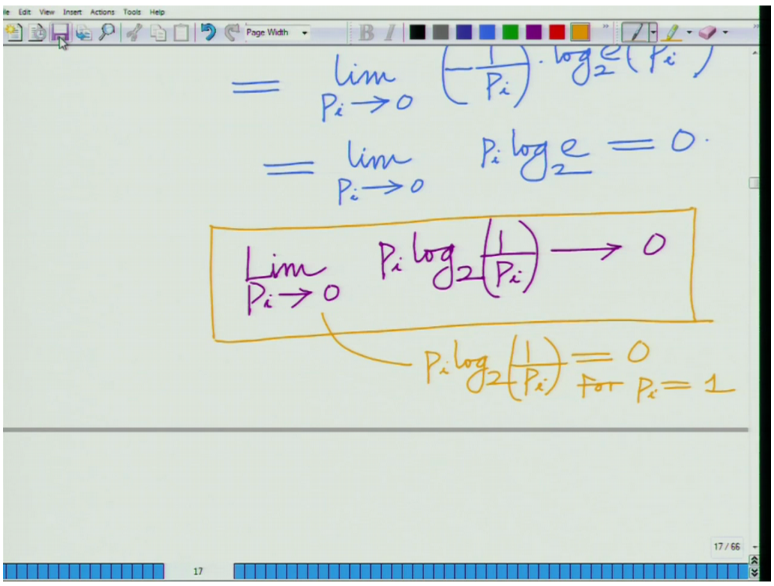

Since we cannot directly substitute Pi equal to 0 into Pi log 1 over Pi limit, p d Pi tends to 0 Pi log to the base 2 1 over Pi also tends to 0.

So, you can say. So, and also remember we have also seen before that Pi log to the base 2 1 over Pi equals also equals 0 for certain events that is for Pi equal to 1. So, we have seen that for both the what you see is that as Pi tends to 0 all for Pi equal to 1 the average information that is Pi log to the base 2 1 over Pi is basically 0 that is for both events which occur with probability one, and events which tend to occur with probabilities close to 0 or the rarest of the rare events which rarely occur which means that the probability of occurrence is close to 0 for both such events both types of such events the average information associated with 0. And this is this can be understood as follows that of course, as we have seen events which occur very frequently that is the probability close to one naturally the information associated with them is 0.

So, the average information associated with them is 0 as we have seen before, but what we have seen is that although information associated with the rare events which are probability close to 0 is very high, since they occur extremely infrequently the average information associated with them is also 0 all right. So, Pi log to the 2 log to the base 2 1 over Pi is 0 when Pi equal to 1 or when Pi is also close to 0. So, that is an important point.

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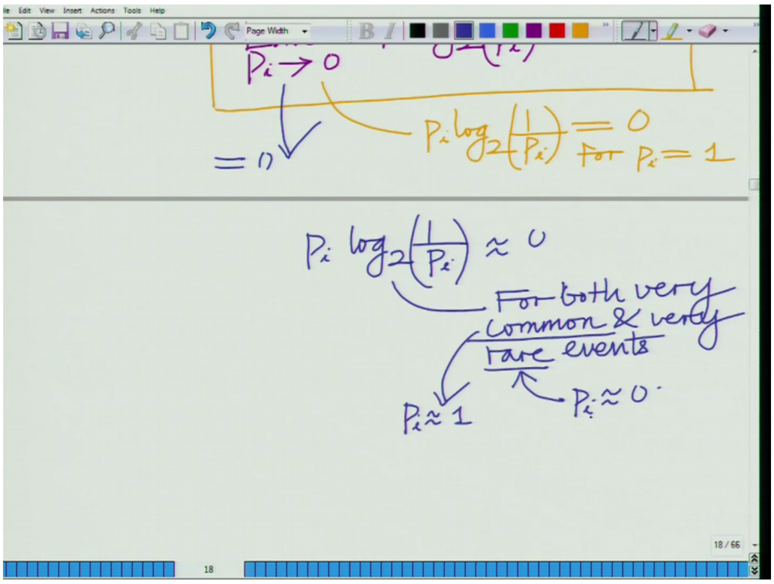

So, this basically shows that this is basically equal to. So, 1 over Pi log to the base 2 to the base. So, what we have seen is Pi intuitively this is close to 0 for both very common and very rare very common, and very common meaning, Pi approximately equal to 1, very rare meaning Pi approximately equal to 0. So, that is what we have, so all right.

So, in this module what we have seen is basically we have defined a very important quantity, which is the entropy all right which is basically the average information associated with each symbol of the source which is expected value of the information expected value of the information per symbol that is I si of the source which denotes the average information that is associated per each symbol of the source. Which also said this characterizes the uncertainty associated with the source all right, which is also basically used to characterize or quantify the information content of a particular source and we have also seen that Pi log to the base 2 1 over Pi is equal to 0 when Pi is 1 or P is very close to 0.

So, we will stop here and continue with other aspects in the subsequent modules.

Thank you very much.