Principles of Communication Systems – Part II
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Lecture - 23
M-ary QAM (Quadrature Amplitude Modulation) - Part II, Optimal Decision Rule, Probability of Error, Constellation Diagram

Hello, welcome to another module in this massive open online course. So, we are looking at M-ary QAM that is M-ary quadrature amplitude modulation, and we are about start over discussion on the receiver and receive processing and the bit error rate for M-ary QAM, but before we do that yesterday we derived the value of amplitude A.

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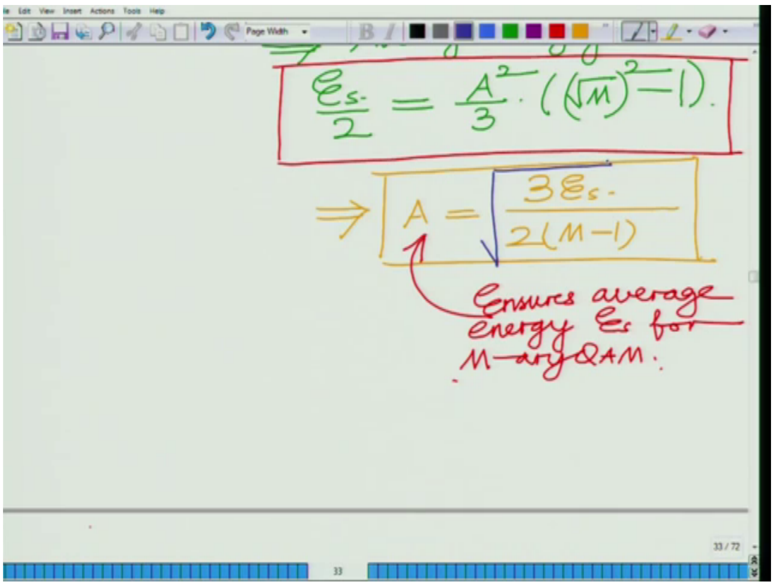

So, let me just correct this amplitude A is square root before we proceed further and just correcting this as missing a square root. So, this is simply square root of 3 a like ES by 2 M minus 1. So, that is fairly obvious.

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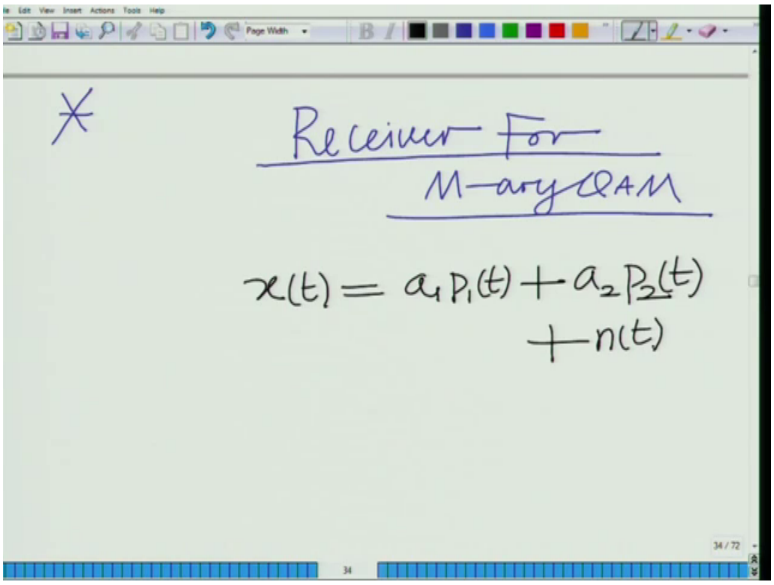

So, let us now start with the receiver of course, again we have seen this many times before, the receiver for M-ary QAM, and in M-ary QAM remember that the transmitted waveform x t equals a 1 P 2 t plus a 2 P 2 t, plus n t which is the noise.

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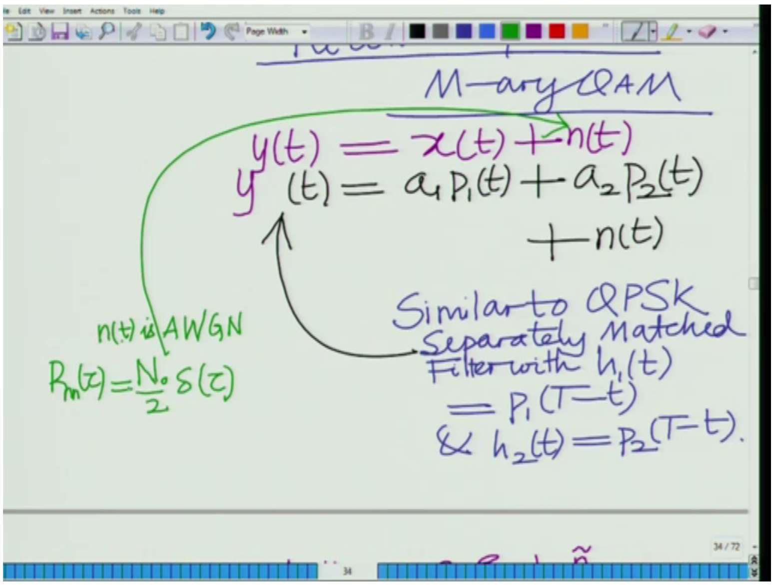

Now, similar to QPSK we matched filter it similar to QPSK, we separately match filter is separately matched filter with well h1 t equals P 1 t minus t and h 2 t equals P 2 t minus. So, you separately for the detection of a 1 all right at the receiver of course, we using matched filter the matched filter to process the received signal, but we have remember there are 2 independently modulated signals that is a 1 on the in phase a pulse a 2 on the quadrature alright.

So, therefore, we have to individual we have to separately matched filter x t once with filter matched to P 1 t that h1 t equals P 1 capital T minus t, and once with the filter matched to P 2 the pulse shape P 2 t which is also which by the way is orthogonal to P 1 t. So, h 2 t equals P 2 capital T minus t. So, we have 2 separate matched filtering operation, similar to again nothing difference similar to what you seen in quadrature phase shift keying.

So, since P 1 h P 1 t is orthogonal to P 2 t therefore, when you matched filter with h1 t that extracts the symbol a 1 and the component with respect to P 2 that is the component symbol a 2 the contribution will be 0. Similarly when you matched filter with respect to p h 2 which is matched to P 2 that is P 2 capital T minus t the contribution of a 1 becomes 0 because again P 1 is orthogonal to P 2. So, this is all these points we had seen in the our discussion for QPSK that is quadrature phase shift keying.

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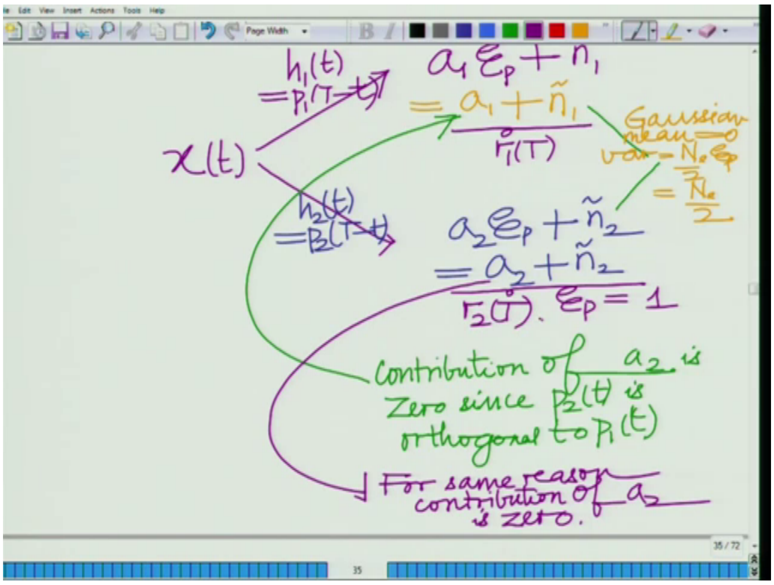

So, therefore, what we have here is basically x t you separately matched filter with h1 t equals P 1 t minus that gives a 1 E p plus n 1 tilde and E p is equal to 1. So, this is simply a 1 plus n 1 tilde; now notice that in this contribution from a 2 is 0; when a matched filter with respect to h1 t 1 t, h1 t minus t, contribution of a 2 is 0 since P 2 t is orthogonal to P 1 t.

Similarly, here when you matched filter with h 2 t equals P 2 t minus t you get a 2 E p plus n 2 tilde this is a 2 plus n 2 tilde of course, this is because energy of the pulse E p is normalized to 1 and again here contribution of now here contribution for the same reason, contribution of for same reason a 2 is 0, same reason this is a P 1 t is orthogonal to P 2 t.

And further if n 2 n t, let us assume that the received signal this received signal is I am sorry this is not the transmit this is y t which is x t plus n t, remember y t is we are assuming a w g n channel we can look at other channels also, but we have restricted our self to the simplest and most general case of an a w g n channel. So, far so n t is additive white Gaussian noise with n t is AWGN with auto correlation R n and tau equals N naught by 2 delta tau.

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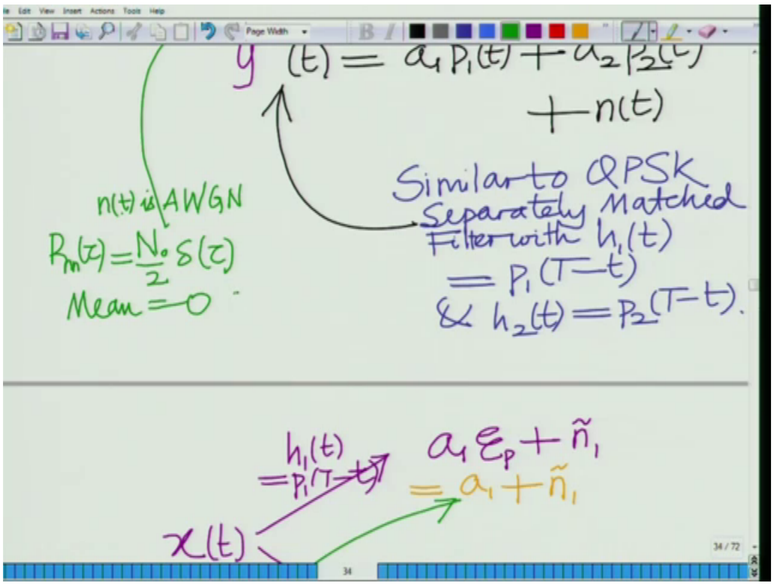

And the mean equal to 0 then n 1 tilde and n 2 tilde these are we have seen both Gaussian mean equal to 0 and variance equals n naught by 2 E p again E p equals 1. So, this is n naught by 2.

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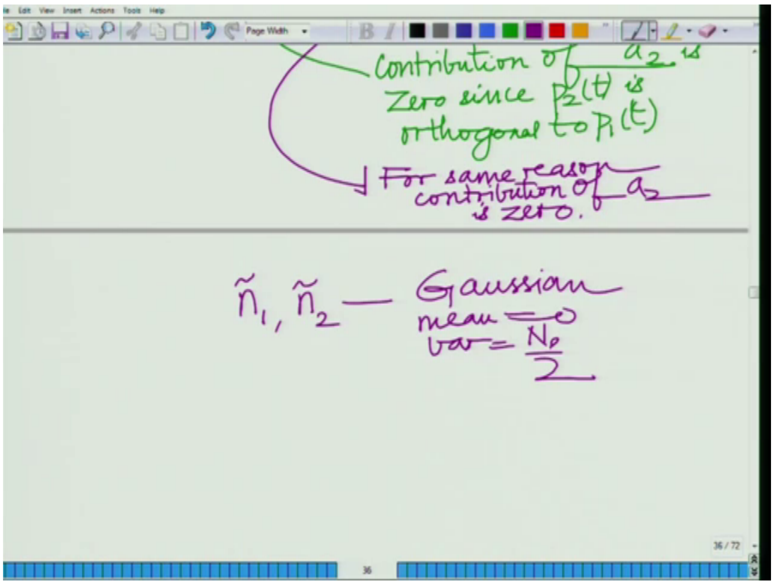

So, variance, so n naught so, n 1 tilde these are Gaussian mean equal to 0 variance equals n naught by 2. Now, therefore, and further now this we can call this as a 1 plus n 1 tilde, we can call this statistic as r 1 t that is followed matched filtering and followed by sampling at t equal to capital T, this we can call this as R 2 t.

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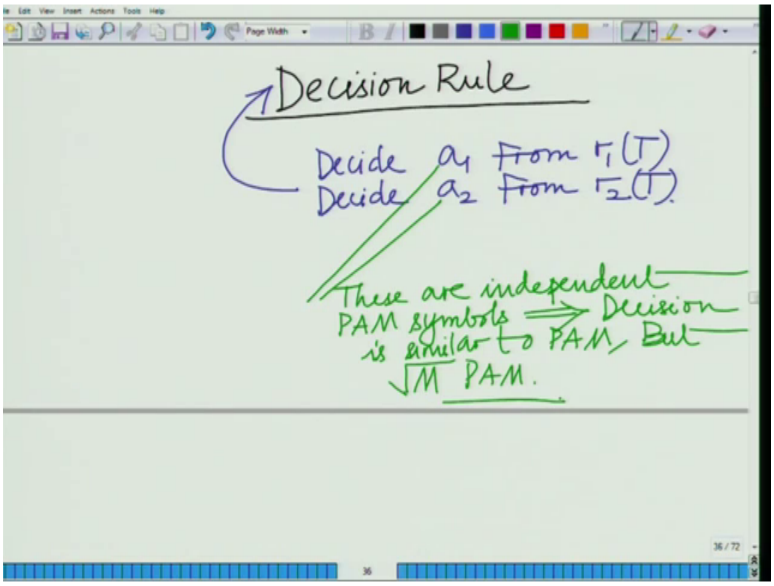

Now the decision rule for each r 1 t now what is the decision rule? Decision rule that is decide a 1 from r to t, and decide a 2 from a 1 from r 1 t, a 2 from r 2 t and remember these are PAM symbols. So, decision rule is similar to PAM these are independent PAM symbols, which implies decision is similar to PAM.

But remember these belong to square root M-ary PAMs, but square and this is important square root M-ary PAM. So, we are going use a same decision rule that we have used for PAM that is pulse amplitude modulation, but each of these belongs to a constituent square root M-ary PAM.

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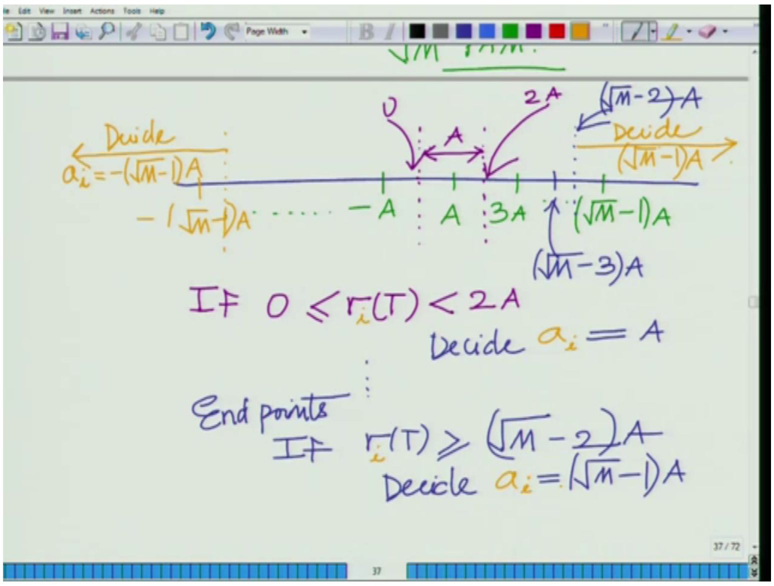

So, the decision rule corresponds to that of a square root M-ary PAM. Now we have already seen what the decisions are for a square root M-ary PAM for instance in a square root M-ary PAM we have seen we have A, 3 A correct this will be well 2 i plus 1. So, square root i equal to square root M by 2 minus 1. So, twice square root M by 2. So, this will be square root M minus 1 A minus A so on, this will be minus square root M minus 1 times A and we have already seen that the thresholds for instance, this will remain same if it the midpoint here is this is 0 the midpoint here between A and 3 A is 2 A, and if it lies between these 2 thresholds then decide A that is if for instance r 1 t less than 2 A or it is greater than equal to 0.

If r 1 t less 0 less than equal to r 1 t less than equal to 2 a then decide a 1 equals A. So, this we already seen this before decide a 1 equals A and similarly we have other decision rules. And towards the end points if for instance now you look at this, this will be square root this point will be square root M minus 3 A and the midpoint here is square root M minus 2 A.

So, if r T is well if it is greater than or equal to square root M minus 2 A, decide a 1 equals square root M minus 1 A. Because this entire decision region there is nothing to the right of square root M minus with this entire decision region corresponds to a 1 equals or decide a 1 or a 2 in fact, this is valid both for square root of M minus 1.

Similarly, if it is less than square root M minus 2 A, if you look at this decision region decide a 1 this is well a decide either a 1 or a 2, we can generally say a i equals square root of or minus square root of or rather minus square root of a i. As in fact, as I was saying you can make this very general you do not need to restrict this to either a 1 or a 2 you can simply use r i is 0 greater than equal to 0 is equal to r i T less than 2 A, then decide a i equals A if r i T greater than or equal to square root of M minus 2 A decide well a i equal to square root of M minus where a M minus 1 is and further i equals either 1 or ok.

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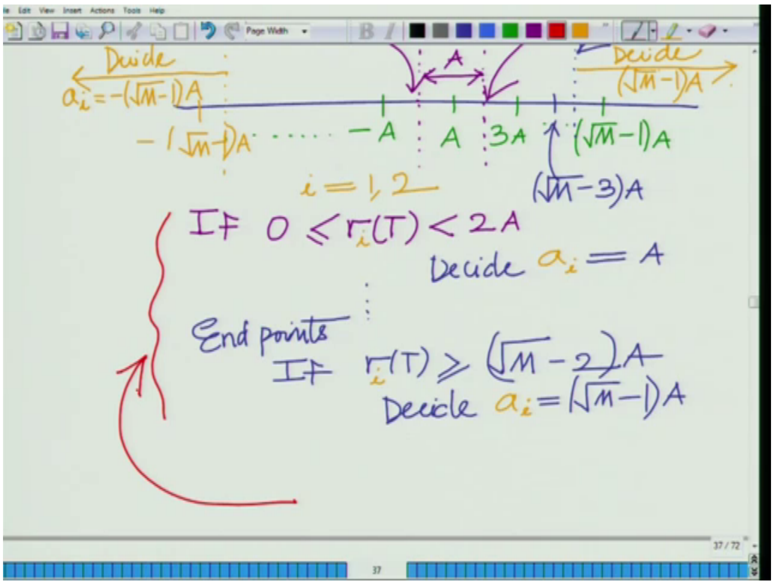

And similarly you can formulate the similarly one can formulate the rest of the cases.

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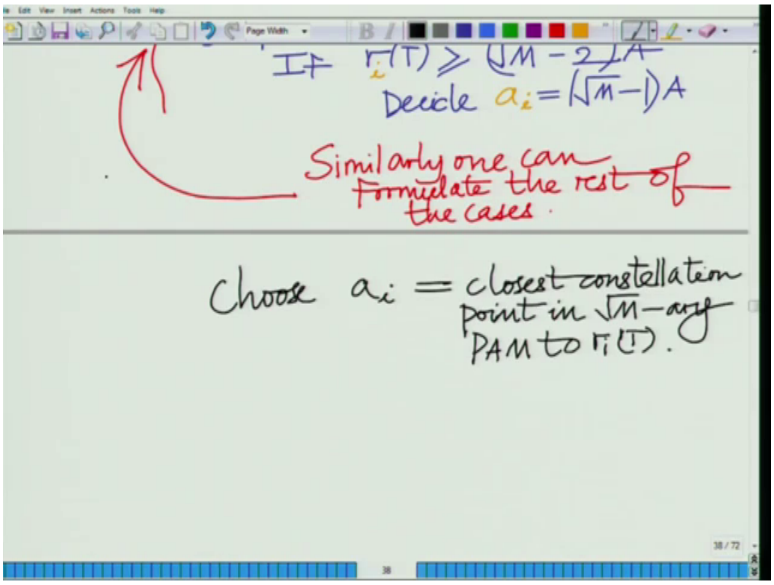

Remember the decision rule corresponds to a set of cases. Similarly one can formulate the rest of the cases that is corresponding to minus a right for a instance if you look at the level minus a you look at the midpoint on both sides. So, minus a lies between the midpoint on the right is between minus a and a 0 midpoint on the left between minus a and minus 3 a is minus 2 a. So, it lies between the received symbol r i T lies between minus 2 a to 0, then you have to decide that the transmitted symbol a i equals minus a and so on. And remember we also call this as the nearest neighbour decision rule that is a very powerful idea, that is basically depending on the closest constellation point you are assigning r r i to r i t.

Basically you are assigning a i to the closest constellation point to r i T that is the whole point. It is a very simple if you look at it is very intuitive that is you look at r i T, look at the closest constellation point in this square root M-ary PAM to r i T and choose a i as basically the closest constellation point to r i T. So, that is basically the idea let me just summarize this. So, choose a i equals closest constellation point in square root M-ary PAM to r i T choose the closest constellation point to r i T that is the basic idea.

Now, the probability of error; remember the probability of the error for each PAM.

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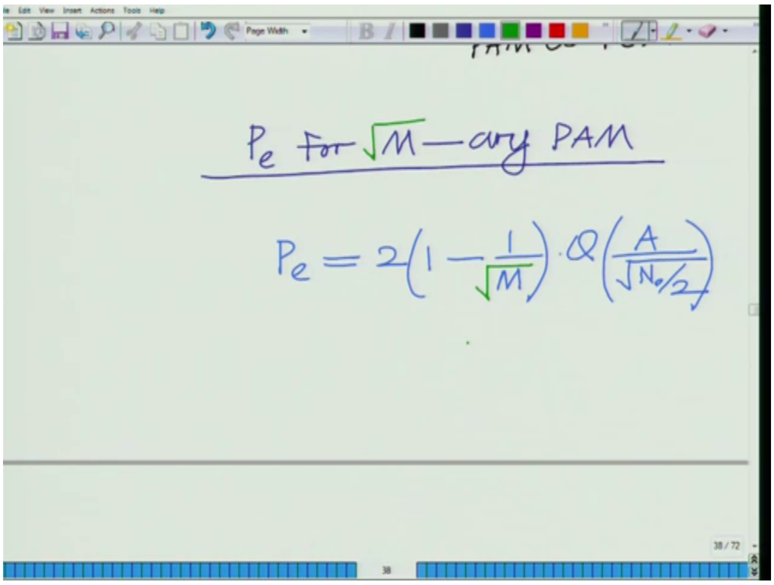

Let us recall probability of error first for M-ary PAM, probability of error for M-ary PAM is well probability of error equals twice 1 minus 1 over M correct Q A divided by square root N naught over square root of N naught over 2 except now we are dealing with the square root M-ary PAM. So, this is the probability of error for each individual square root M-ary PAM. So, this is the probability of error for a 1 comma a 2 which belong to square root M-ary which belong to square root M-ary PAM.

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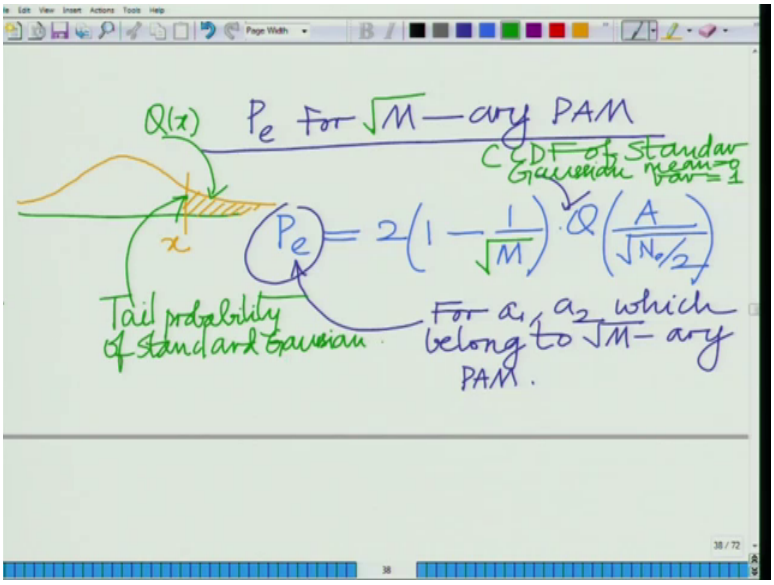

So, P e equals twice 1 minus 1 over square root of M Q A divided by square root of N naught by 2 and Q is the Gaussian I do not need to keep repeating this again Q is the C C D F complimentary cumulative distribution function of the standard Gaussian random variable. Standard Gaussian that is mean equal to 0, I am not repeating this explicitly variance equal to unity that is the tail probability of the standard Gaussian.

And at this stage I think all of you should be more than familiar with that, that is if you look at Q of x this area is Q of x this is tail probability of standard Gaussian. Tail probability of the standard Gaussian random variable and further for M-ary QAM we have derived it as a as given symbol remember we said given symbol energy E s.

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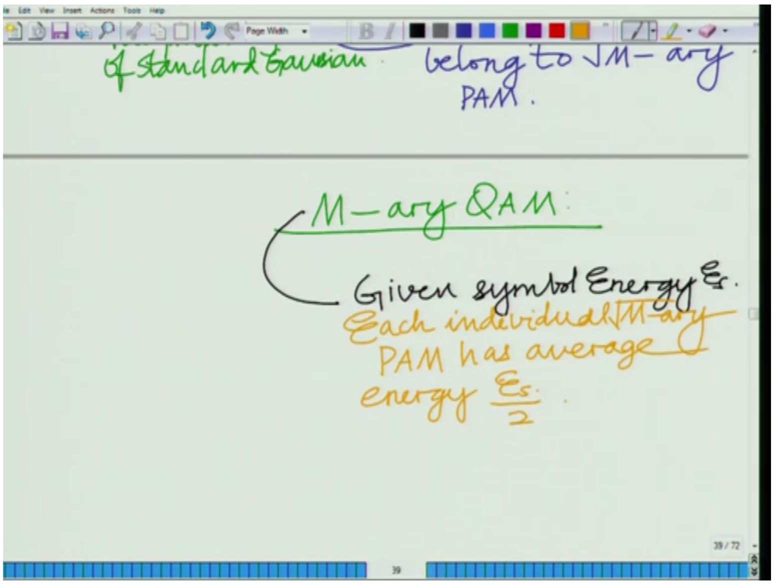

Each individual square root M-ary PAM has each individual square root M-ary PAM has average energy ES by 2.

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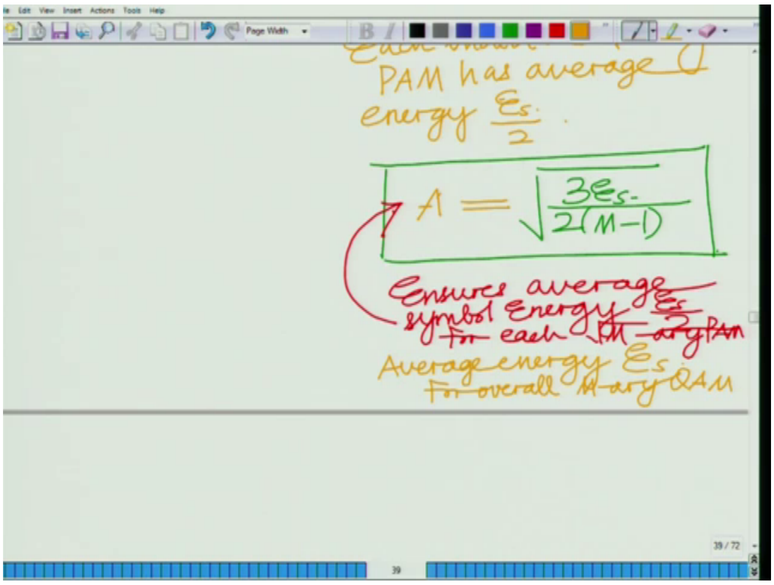

Which means and for that we have derived the condition the corresponding condition is amplitude A equals square root 3 ES by 2 M minus 1 this ensures average symbol energy ES by 2 or ensures average energy ES for the entire PAM e a average energy ES by 2. So, this ensures this value of a ensures average symbol energy ES by 2.

For each square root M-ary PAM and average energy ES because average energy ES by 2 for each square root M-ary PAM, the total average energy will be the sum of the component average energies that ES for the overall M-ary QAM. So, only thing left now is to substitute this value of A.

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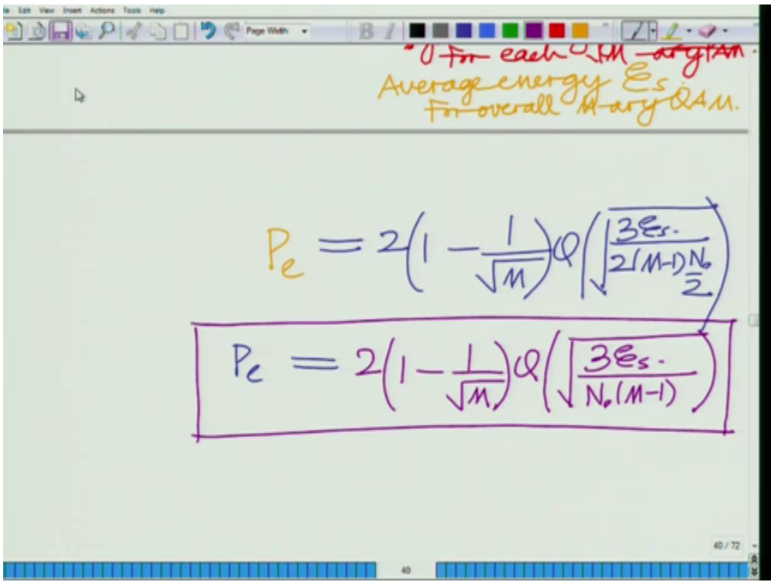

So, P e we have a beautiful result P e is twice 1 minus 1 over square root of M Q substitute the value of a which is 3 ES by 2 M minus 1 into N naught by 2 which is equal to. So, P e 2 1 minus square root of M Q square root of well, 3 ES by N naught into M minus 1.

So, this is the average energy for each constitutes average probability.

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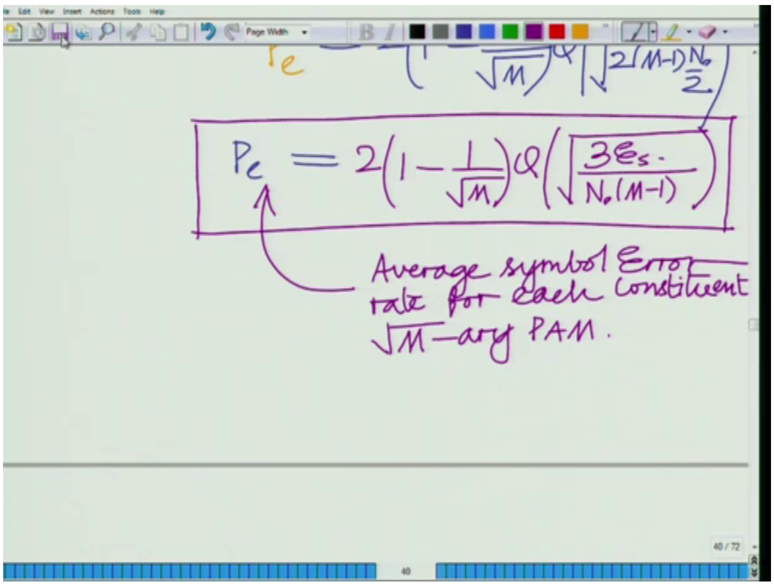

I am sorry average probability of error or average symbol error rate for each constituent average symbol energy for each constituent square root M-ary PAM, and similar to

QPSK again similar to QPSK. Now this is the average energy it is a average symbol error rate for each constituent square root M-ary PAM.

Now, the overall QAM symbol will be an error if either of the PAM symbols is in error and therefore, similar to QPSK right the overall symbol is in error if either of the component bits are in error all right. So, overall QAM is in error, if either of the constituent PAM symbols are in error therefore, the overall error rate for the overall M-ary QAM constellation can be derived as is well P e equals.

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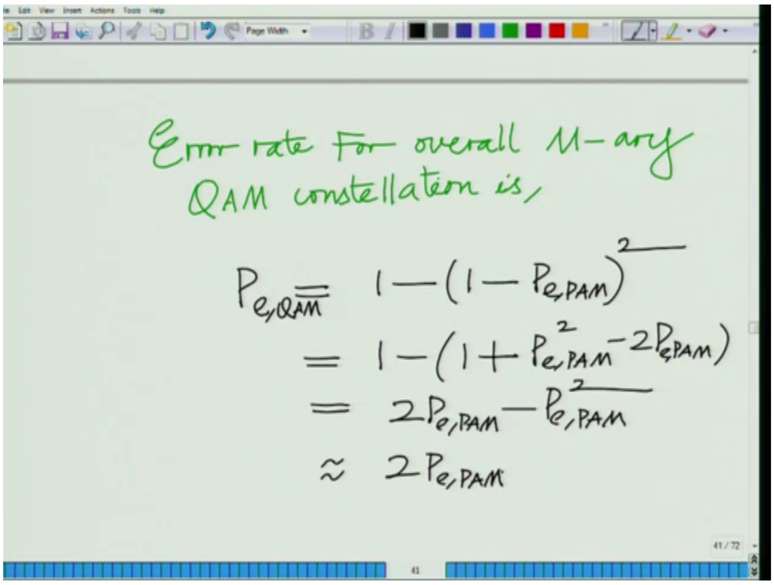

Let us write this as P e QAM that is 1 minus 1 minus remember we derived this again I am not going through the entire derivation 1 minus 1 minus the constituent PAM error rate which is 1 minus 1 plus P e PAM square minus twice P e PAM, which is equal to well twice P e PAM minus P PAM whole square which is approximately equal to twice the error rate of the constituent PAM because at high SNRP e PAM square.

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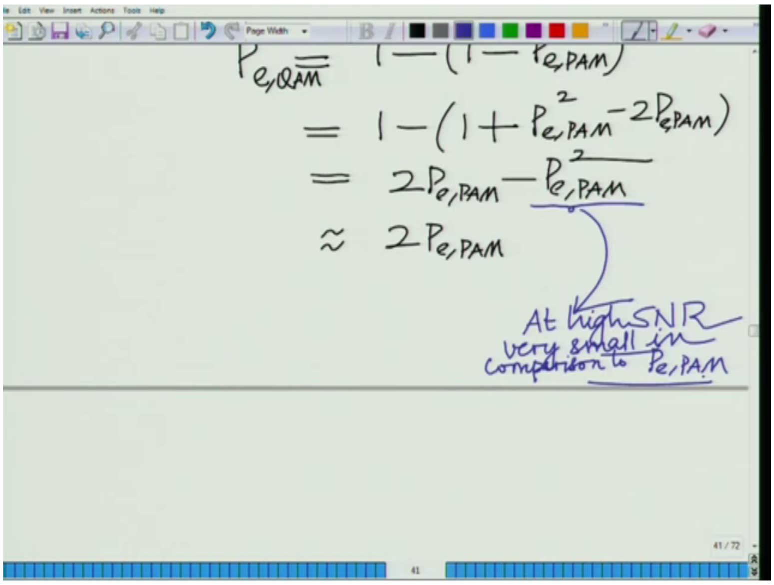

This is a square is very small, because this square term at high SNR this is very small in comparison 2 P e PAM square is very small in comparison 2 P e PAM.

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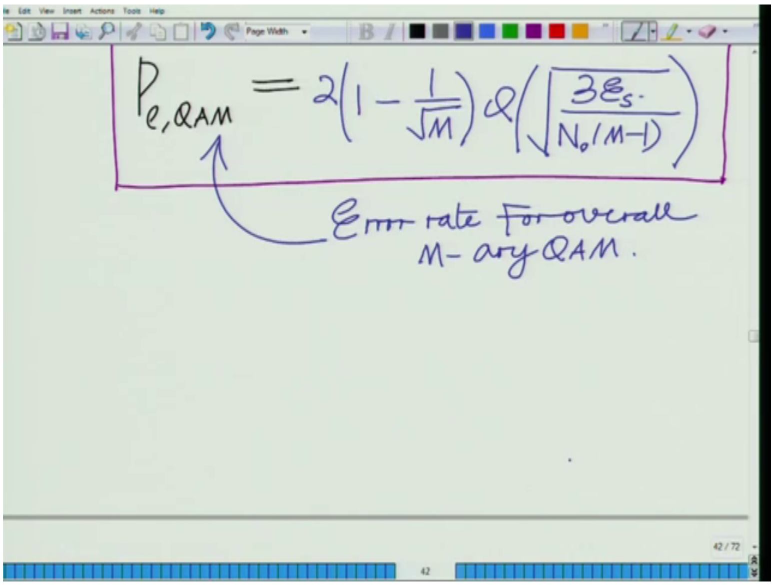

So, the error rate finally, overall error rate for the QAM for the M-ary QAM is equal to twice dropping the approximation for simplicity twice 1 minus 1 over or let me just write it; twice 1 minus 1 over have drop the approximation sign Q square root of 3 ES by n naught M minus. So, this is twice let me just write it a little bit more clearly, this is twice

1 minus 1 over square root of M Q3 E s by n naught into M minus 1. So, this is the error rate for the overall QAM.

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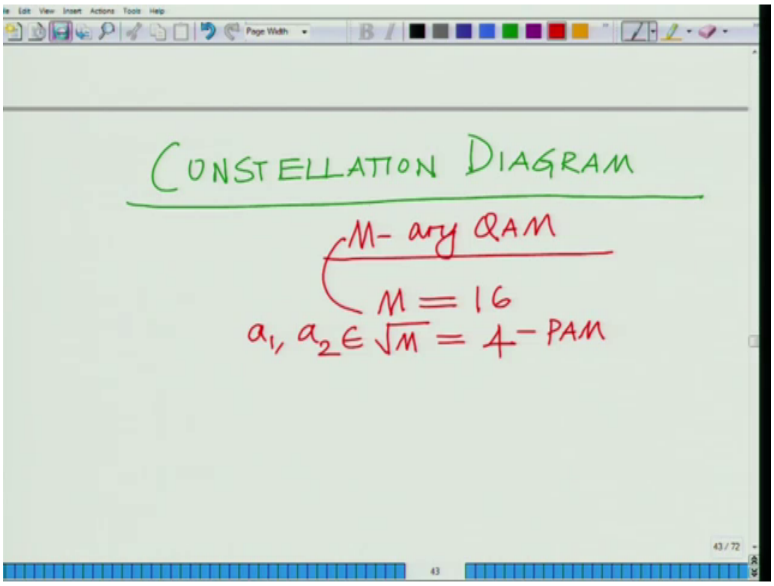

Now, before you wrap this up let us look at a simple constellation diagram for the M-ary QAM, for M-ary QAM let us consider the simple case where M equal to 16 which means each constituent square root of 4 PAM. So, a 1 comma a 2 belongs to a square root of 4 square root of 16 that is 4 QAM.

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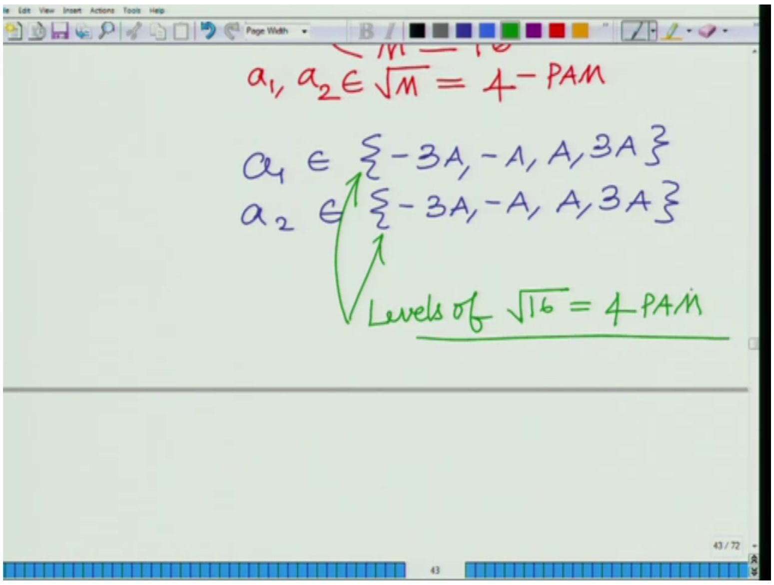

So, basically a 1 belongs to the set of 4 levels minus 3 A minus A, A comma 3 A, a 2 also belongs to minus 3 A, minus A, comma 3 A. So, these are the levels corresponding to levels of the 4 PAM or the square root of 16 PAM.

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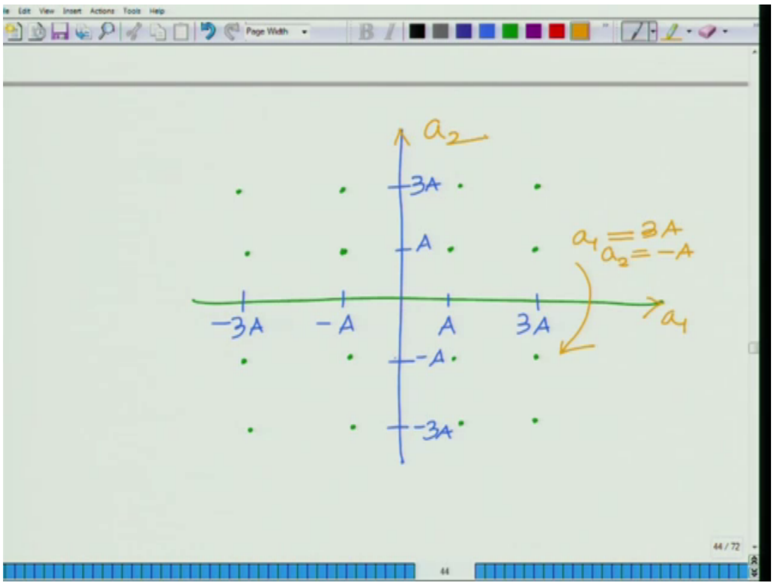

And these can be. So, I can represent a 1 on the x axis a 2 or the levels of a 1 on the x axis. So, here I have A, minus A, 3 A minus 3 A similarly on the y axis I have the levels of a 2 A, 3 A, minus A, minus 3 A and this is A. So, this corresponds to the point A,A this corresponds to the point minus A similarly 3 A comma A. So, this will have a total of course, we are considering M equal to 16. So, naturally this will have a total of 16 points in the constellation.

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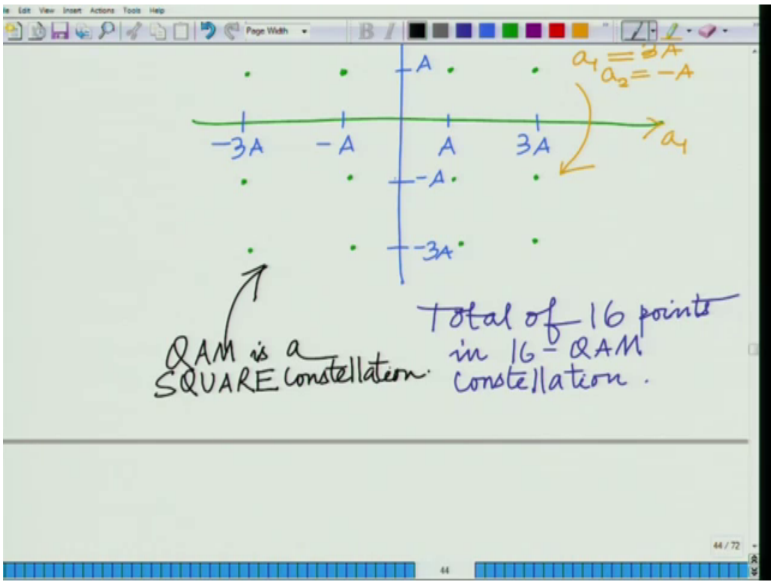

For instance if you look at this point this point corresponds to a 1 equals 3 A, a 2 equals minus A. Remember this corresponds to a 1 y axis corresponds to a 2. So, there are total of 16 points naturally this is 16 QAM constellation. So, there are a total of 16 points in the 16 QAM, observe a 1 belongs to a linear PAM that is a PAM on a line remember PAM is a linear constellation that is it can be represented on a line.

So, here we have a QAM all right which has a constituent square root M-ary PAM a 1 on a line, that is x axis co another constituent square root M-ary PAM a 2 on the line which is the y axis therefore, if you will get the net constellation this looks like a square, alright. So, this is also known as a square constellation. So, a QAM is a square constellation. So, this is a very interesting observation so you can see that the QAM is a square QAM is a square constellation that is symmetry.

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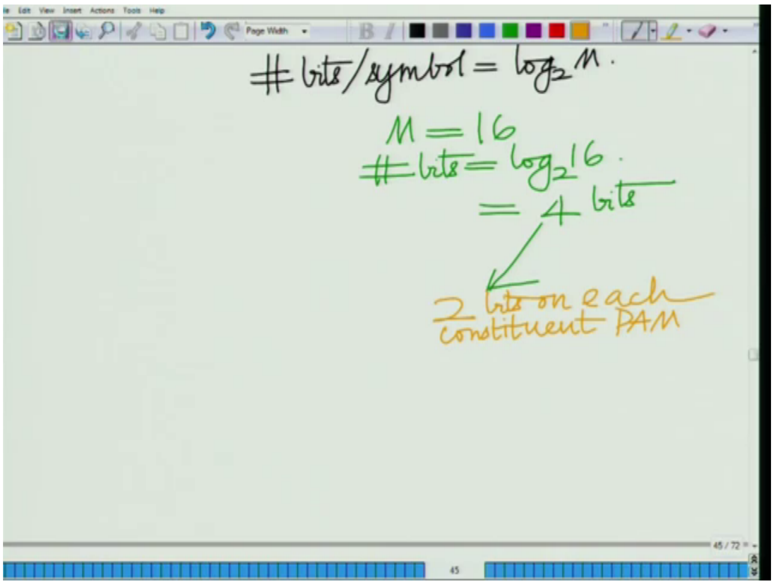

So, it is a QAM is a square constellation and of course, we have seen the number of bits equals log 2 to the base M. M equal to 16 if M is equal to 16 number of bits equals log 16 to the base 2 equal to 4 bits, 2 bits on each of course, 2 bits on each constituent PAM 2 bits on each constituent PAM. And further as I have said in the beginning of this module that QAM is a very general constellation very powerful constellation all right because you are you can transmit individual PAM symbols on both the in phase and quadrature pulses correct in phase and quadrature components and it can scale right with M equal to 16 we get 4 bits, with M equal to 64 you get 6 bits per symbol, M equal to 256 you get 8 bits per symbol, 512 1024 and so on.

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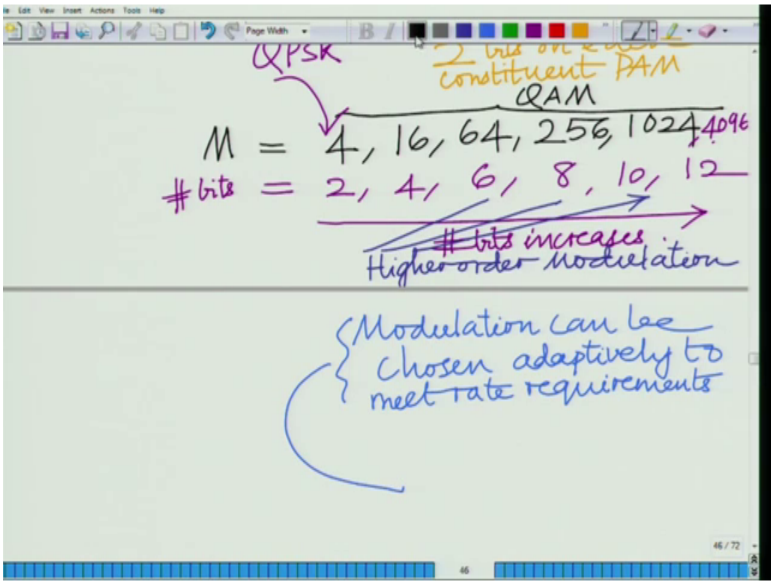

So, there are all several different possibilities. So, you have M equal to of course, if M is equal to 4 that is QPSK, 16 QAM, 64 QAM, 256 QAM, 1024 and so on. So, these are all the QAM. So, this is of course, your 4 symbols means 2 bits means one bit on each in phase and quadrature this is of course, your QPSK; and number of bits equals, 2 4 that is log to the base to of this quantity 2, 4, 6, 8, 10 of course, you can add 2, 0, 4, 8, 4, 0, 9, 6 and you will have 12 bits. So, the number of bits increases.

So, therefore, by choosing an intelligent QAM modulation to suit the channel requirements one can scale the bits you can go from 2 bits to 4 bits to 6 bits to and remember we are increasing the number of bits per symbol for the same symbol rate let us say your symbol rate of 1000 symbols per second or 1 mega symbol per second. If you choose QPSK which has 2 bits per symbol you get 2 mega bit per second. If you choose 1024 QAM which has 10 bits per symbol right you get ten mega bit per second all right. So, by choosing the suitable QAM one can scale up the data rate and that is how high data rate is achieved right high data rate is achieved in for instance 3G, 4G and 5G and also future in the future 5G cellular networks by scaling up the modulation, this is known as higher order QAM or higher order modulation.

So, these basically this is known as the higher order for instance when you go for from QPSK to all these this is known as, and modulation can be chosen adaptively correct modulation can be chosen adaptively to meet rate requirements, and this is termed as adaptive modulation which is at important scheme.

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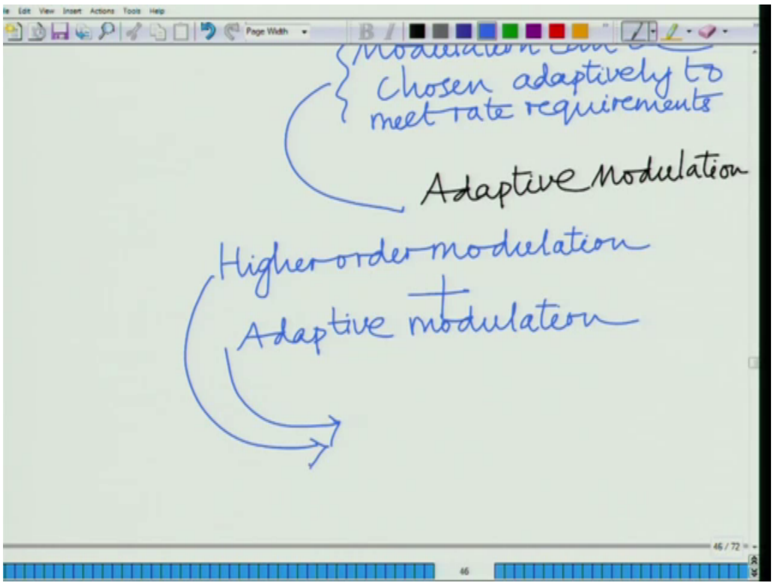

So, higher order modulation and adaptive modulation higher order modulation plus.

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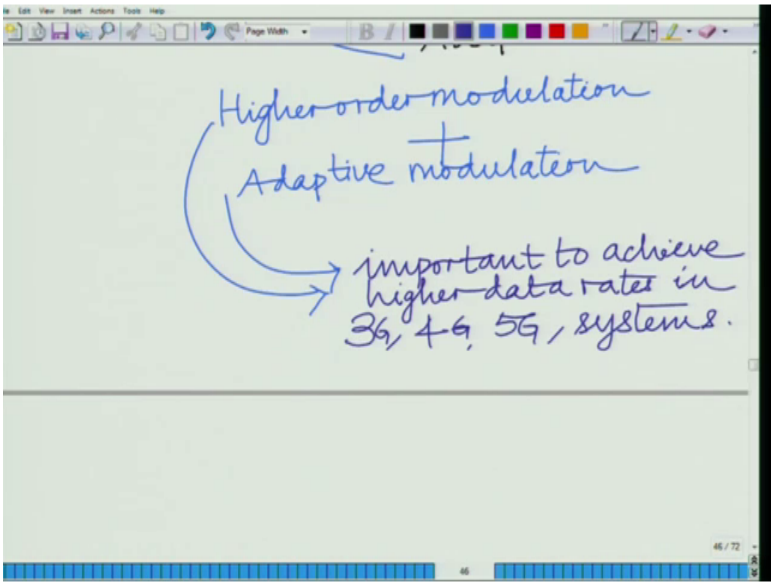

These are important in to achieve higher data rates in 3G, 4G and also 5G. In 3G, 4G and 5G so when the rates go from 100s of kilobit per second to megabit per second to tens of megabit per second to in fact, a gigabit per second right these can be achieved of course, achieved by several technologies sectoral components I am not saying that higher order modulation in is the only technology component in achieving higher data rates.

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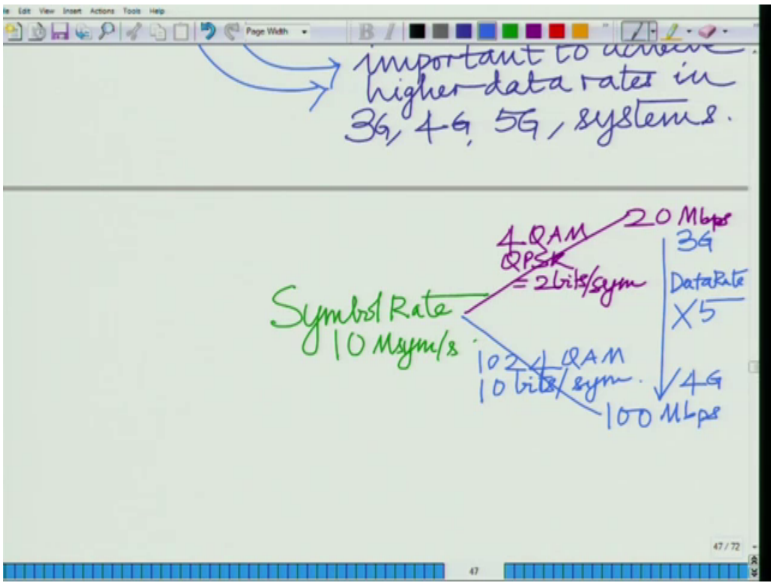

But higher order modulation usage of higher order module higher order modulation is one of the strategy is one of the main strategies that can be employed to multiply right multiplied the bit rates several times that is for the same symbol rate right for the same symbol rate for instance, if you have a symbol rate of 10 mega symbol per second if you use 4 QAM that is QPSK which has 2 bits per symbol, you get a data rate of 20 bits or 20 megabits per second 20 Mbps, and if you can succeed in using 1024 QAM, which has 10 bits per symbol you can get a data rate of 100 Mbps. So, you can see from 20 Mbps to 100 Mbps that is a factor of 5 into 5. So, that is a factor of 5 in data rate.

So, you can go from. So, let us say this can 3G data rate this can be 4G data rate. So, 3G systems typically or data rates around Mbps or at most 10 Mbps, in 4G you can go up to data you can go to data rates of 100 Mbps and in fact, more you can go to data rates of about 200 to 300 Mbps. So, it all depends on the modulation scheme use along of course, along with several other technologies, but modulation right and adaptively choosing the modulation depending on the user at user requirements. So, the modulation scheme depends on the requirement of the user right what kind of the rate the user requires and also the channel condition, if the wireless channel is able to support such a data rate. So, adaptively one can modulate on can chose an appropriate modulation right to meet the data rate requirement and it is of course, deliver the maximum data rate to each user.

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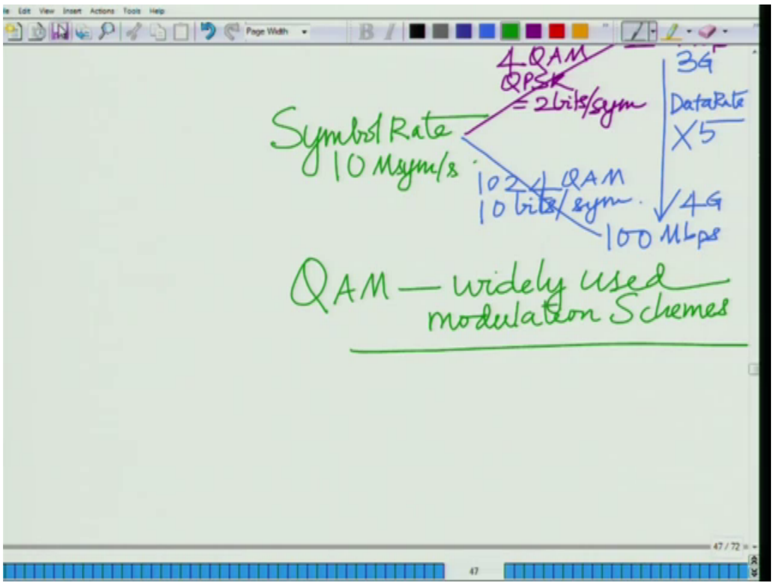

So, that increases the data rate significant as we go from 3G to 4G to 5G wireless technologies and QAM is a very general and is one of the most widely used modulation schemes. So, let me just summarize that QAM is one of the most simplest powerful and it is one of the most widely used it is very flexible one of the most widely used modulation scheme alright.

So, in this module we have covered comprehensively covered QAM that is what is the constellation what is the structure of QAM what is the structure of the transmitted waveform what is the processing at the receiver and the associated probability the probability of error for M-ary QAM and also, the importance of this QAM with respect to the modern wireless technologies all right. So, we will stop here and continue with the other aspects in subsequent modules.

Thank you very much.