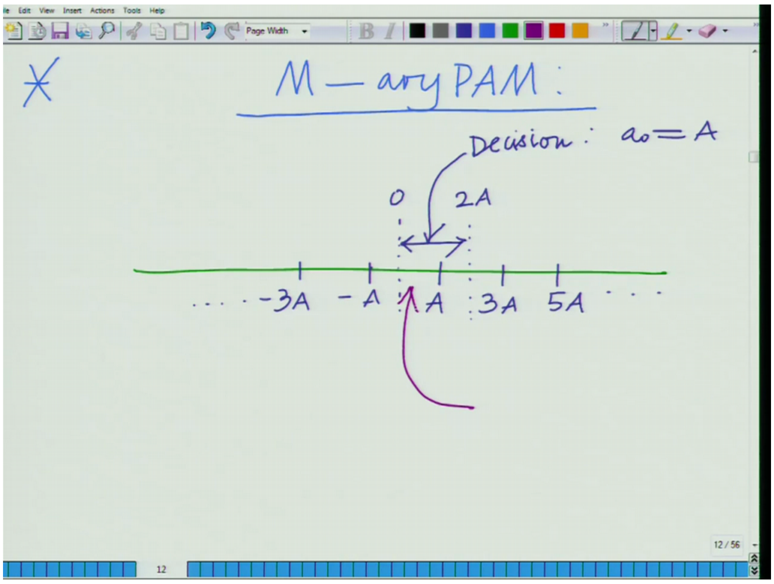
Principles of Communication Systems – Part II
Prof. Aditya K. Jagannatham
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

Lecture - 21
M-ary PAM (Pulse Amplitude Modulation) - Part II, Optimal Decision Rule, Probability of Error

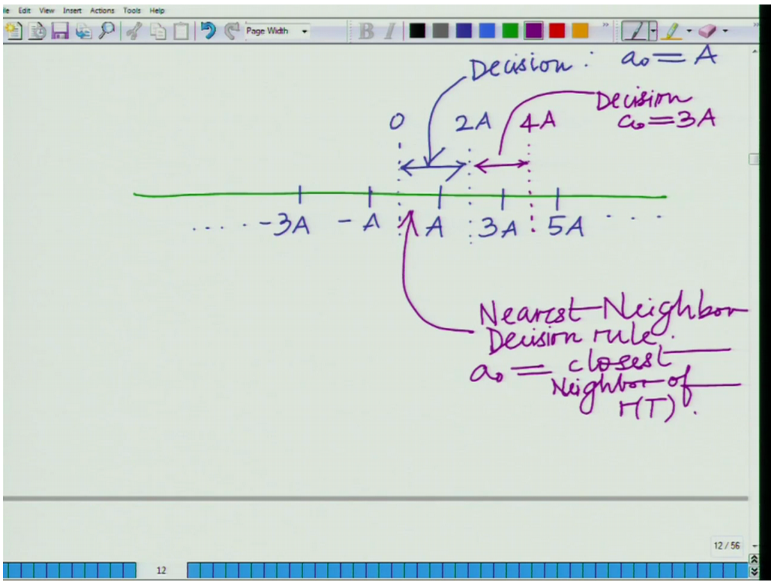
Hello, welcome to another module in this massive open online course. So, we are looking at M-ary PAM: that is M-ary pulse amplitude modulation.

(Refer Slide Time: 00:22)



And we have looked at the optimal decision rules for M-ary PAM that is the set of cases optimal decision rules and what we have seen is the following thing for instance for all interior points A; for all the interior points, we use the nearest neighborhood that is if r T lies between in this region that is 0 which is the midpoint of minus A comma A or 2 A which is the midpoint of A comma 3 A, if it lies in this region that is if r T lies in this region, the decision should be basically that A naught; the decision is that A naught is equal to A. This is the decision and so on. We have also called this as the nearest neighbor decision rule.

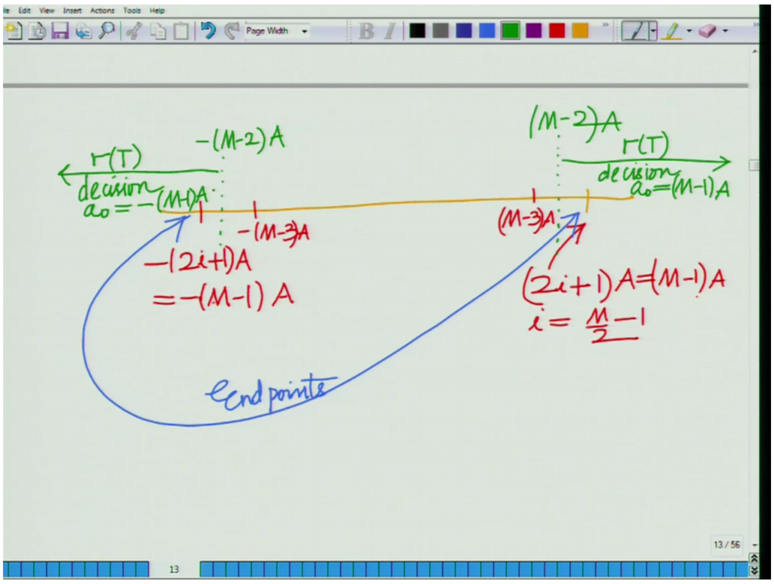
(Refer Slide Time: 01:38)



This is your nearest neighbor decision rule that is A naught equals closest neighbor the closest neighbor of r T this is the nearest neighbor decision.

Similarly, if it lies between 2 A, r T lies between 2 A and 4 A, in this region, the decision is A naught equals 3 A that is the symbol of the level 3 has been transmitted now at the end point something interesting happens when you look at the end point.

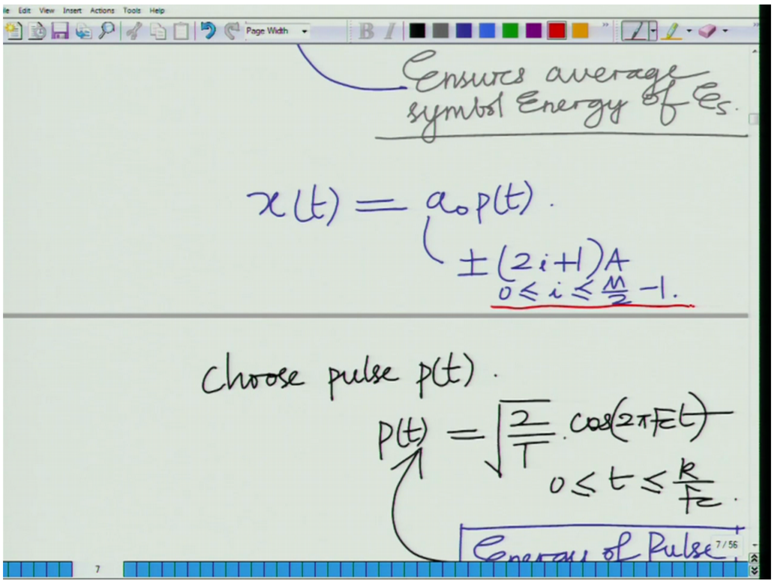
(Refer Slide Time: 02:38)



Now, there are 2 end points, one corresponds to there are 2 end points, this corresponds to of course, 2 i plus 1 times A where i equals M by 2, remember i lies in the range 0 to

M by 2. So, this is equal to 2 M by 2; I am sorry, i is equal to M by 2 minus 1, correct, 0 less than equal to i less than equal to M by 2 minus 1. So, the maximum value that I can take.

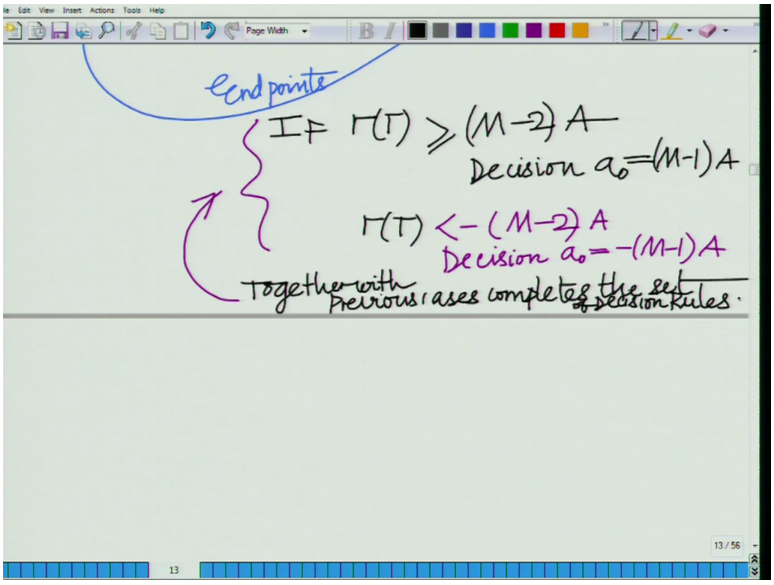
(Refer Slide Time: 03:14)



I think we should have illustrated it here, yeah, 0 less than equal to i less than equal to M by 2 minus 1. So, i is equal to M by 2 minus 1 corresponding to that this will be 2 M by 2 minus 1 that is 2 M minus 2 plus 1. So, this will be 2 M. So, this will be M minus 1. So, this will be. So, let me just write this over here this will be M minus 1 A, similarly this is minus 2 i plus 1 A corresponding i equal to M A M by 2 minus 1, this is equal to minus M minus 1 A and these are the 2 points which we are calling it as the end points these are the end points one of them is M minus 1 A the other is minus M minus 1 A. Now of course, the next point here will be M minus 3 A, here it will be minus M minus 3 times A, now if you look at the midpoint of this the midpoint of this will be M minus this midpoint will be M minus 2 A and this midpoint will be minus M minus 2 A.

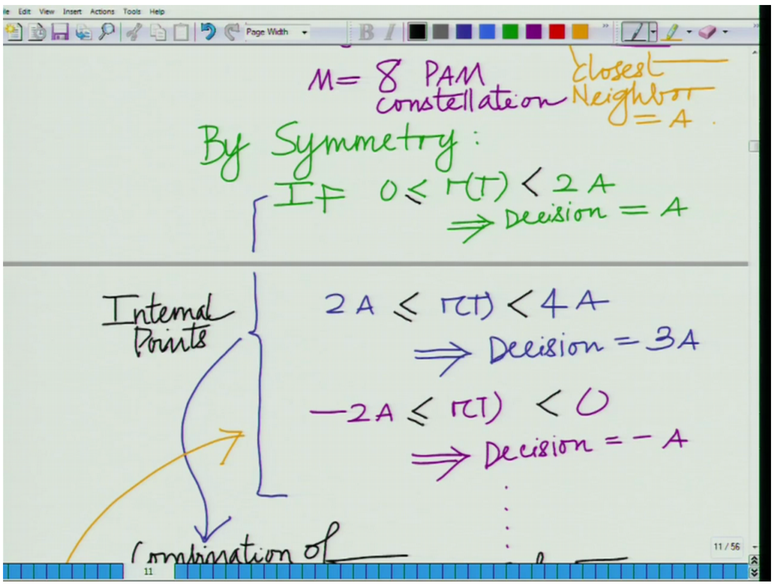
So, if r T; remember there is no point on the right of this endpoint. So, if r T lies anywhere in this region then decision is A naught is equal to M minus 1 A. Similarly there is no point on the left of this if r T lies anywhere in this region that is less than M minus 2 A if r T lies in this region then decision the decision is the decision is A naught is equal to minus M minus 1 A.

(Refer Slide Time: 05:38)



So, the final set of decision rules will be r T greater than or equal to let us just write it consistently if r T is less than equal to 4 A here just to be consistent we cannot write equal to at both at the places.

(Refer Slide Time: 06:02)

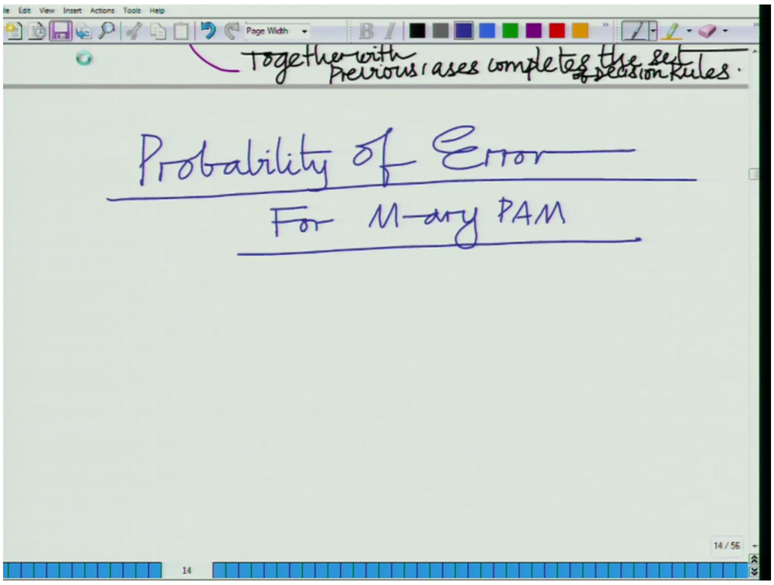


So, just to be consistent let us write this as if r T is if r if or let us write it as this way if r T is less than 2, A 0 less than equal to r T, let us write inequality only on one side because if although it is not a very significant thing because if r T equal to 2 A then we have to decide what do we have to choose if r T equal to 2 A should we choose A naught equal to a or should we choose A naught is equal to 3 what we are saying is that point 2 A, let us choose A naught is equal to 3.

So, 2 A less than equal to r T less than 4 A, so minus 2 A less than equal to r T less than 0 r T equal to 0, we are choosing A r T equal to 2 A we are choosing 3 A r T equal to minus 2 A we are choosing minus A and so on, just to avoid that ambiguity at the boundary now. So, therefore, what we can say is if r T is greater to be consistent in the spirit of consistency if r T is greater than or equal to M minus 1 A your decision r T sorry r T is greater than equal to M minus 2 A. In fact, that is the midpoint r T is greater than equal to M minus 2 A the decision is A naught equals M minus 1 A and similarly if r T is less than minus M minus 2 A then decision A naught equals minus M minus 1 a. So, this together with the previous cases completes.

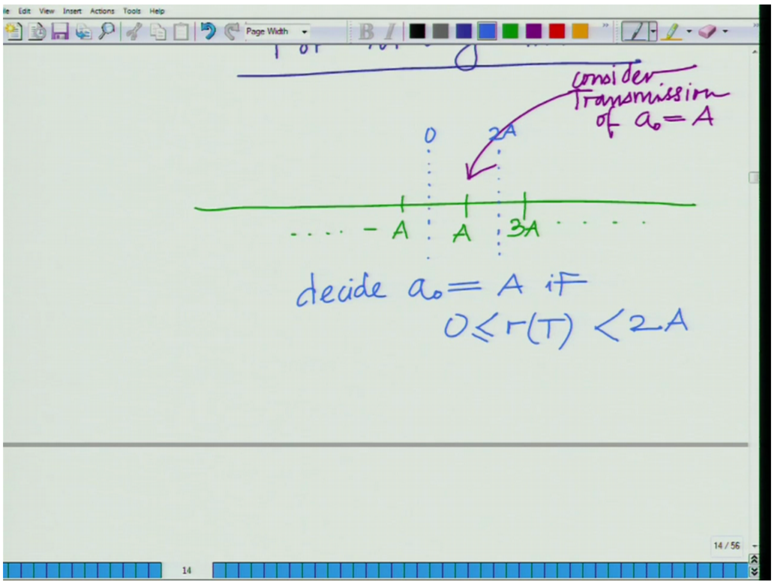
So, together with the previous cases remember it has to be take seen into totality together with your previous cases completes the set of the decision. So, together with the previous cases, but by the previous cases; I mean this for the central points that is for this these are for the let us call these are for the; so the internal points internal points means point which has 2 points that is one point on either side because these points in M-ary PAM these are arranged on the line some points are the internal points which have one point on either side. Then there are the extremities one towards the extreme right which has only points towards the left and no point on the right and one towards the extreme left which has points only on the right and no points on the left these points are respectively M minus 1 A and minus M minus 1 A that is what we have seen and the corresponding decision rules are that is if r T is greater than or equal to M minus 2 A then we decide A naught equal to M minus 1 A if r T is less than minus M minus 2 A decide A naught is equal to minus M minus 1 A.

(Refer Slide Time: 09:29)



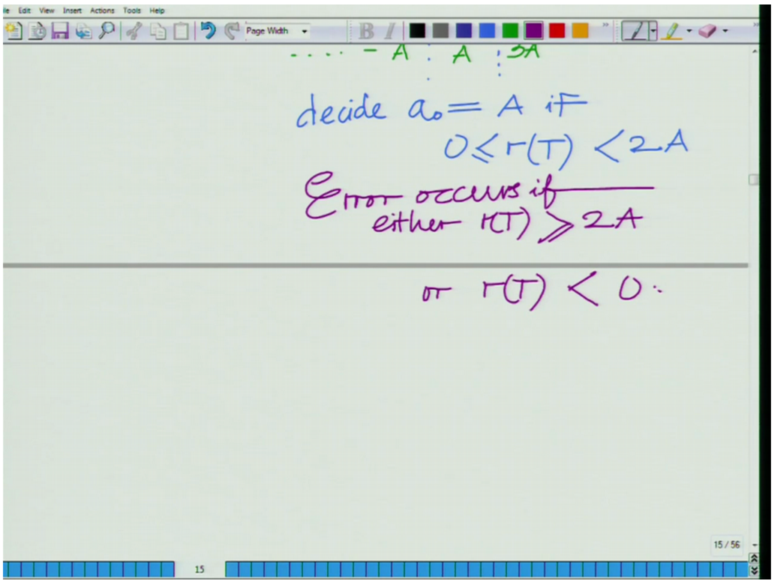
Now, these are the complete now what is the probability of error that is the next question that we wanted to look at we want to ask this question what is the probability of probability of error for this M-ary PAM modulation what is the associated probability of error now consider the transmission of A naught equal to A.

(Refer Slide Time: 10:02)



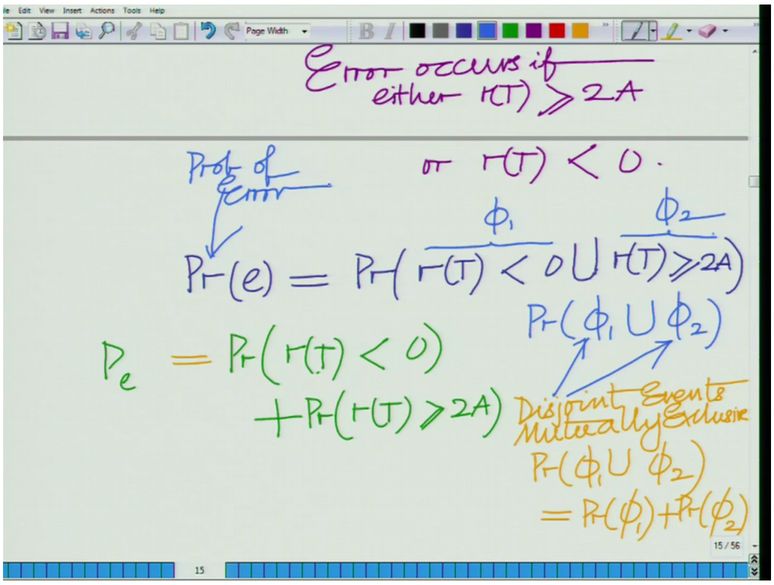
Let us look at this point A naught equal to a here the next point 3 A minus A and so on. So, consider the transmission now error occurs if the received see because if r T is less than if r T. So, if r T is less than; so this side A naught equal to A, if well my; if 0 less than or equal to r T less than 2 A. So, if r T lies in this interval 0 to 2 A, we are deciding that A naught is equal to a therefore, error occurs if either r T is greater than. So, observe that if r T lies in this interval consider the transmission of A naught equal to A; if the received that is the r T that is after match filtering and sampling the statistic r T if it lies in 0 to 2 A then there is no problem because we are deciding A naught equal to a. However, if r T is either greater than 2 A or less than 0 then it will map to some other point and in which case you will have symbol error. So, error occurs if r T is either greater than 2 A or less than 0. So, let us note this.

(Refer Slide Time: 11:47)



So, error occurs if r T is greater than equal to 2 A or r T is less than 0.

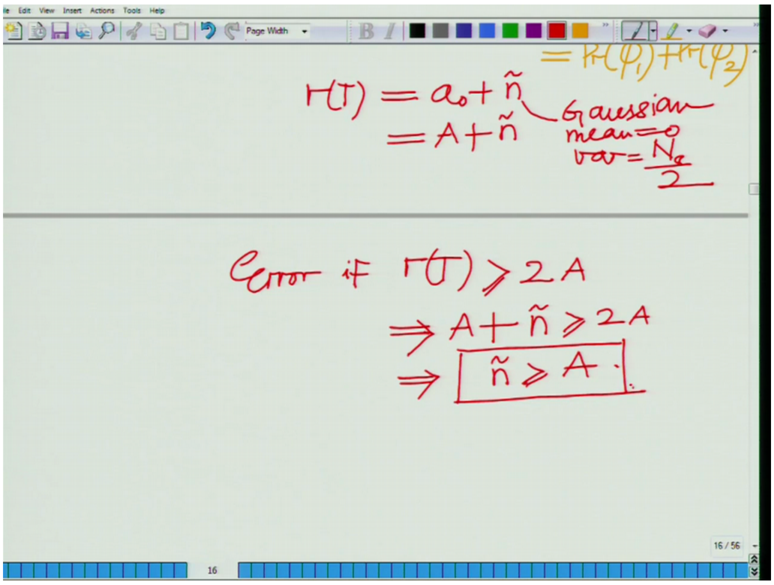
(Refer Slide Time: 12:13)



So, therefore, probability of error; now, probability of error equals simply the probability r T is less than well r T is less than 0, let me write it be a bit more specific because this is the union r T is greater than or equal to 2 A union because there is an error in case of. So, if you call this event as event A; if you call this let us not call it a and b lets call this event as event phi 1 this as event phi 2 then probability of error is basically probability of phi 1 union phi 2 where phi 1 denotes the event that r T is less than 0 phi 2 denotes the event r T is greater than equal to 2 A and remember these are 2 disjoint events because if r T is less than 0 then it cannot be greater than equal to 2 A. Similarly if r T is greater than or equal to 2 A it cannot be less than 0.

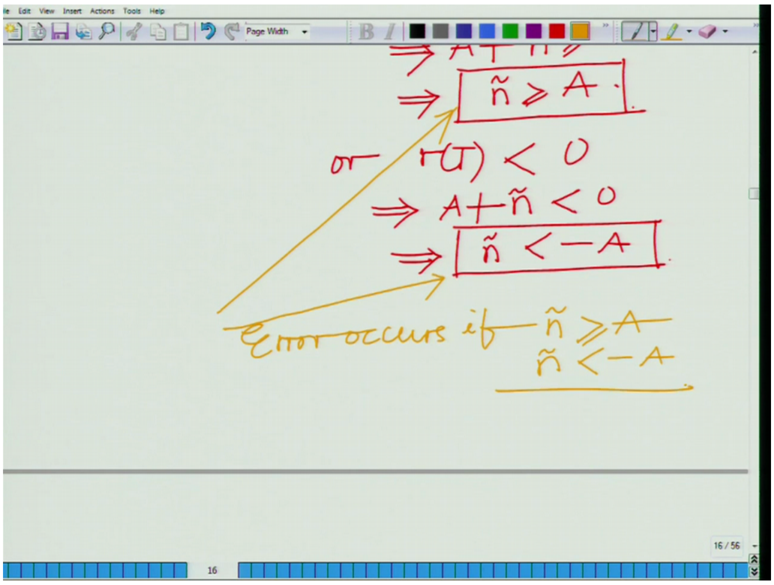
So, these are disjoint events. So, these 2 events these are disjoint events disjoint events means these are mutually exclusive events actually these are the correct term for this is mutually exclusive not disjoint. So, probability of phi 1 union phi 2 from knowledge of probability and random process which is again very important as I have said many times before is equal to probability of phi 1 plus probability of phi 2. Therefore, this is equal to the probability of error p e is equal to probability of phi 1 which is the probability that r T is less than 0 plus probability of r T being greater than or equal to 2 a. So, this is the net probability of error this is the expression for the; remember this is the expression for the probability of error.

(Refer Slide Time: 14:45)



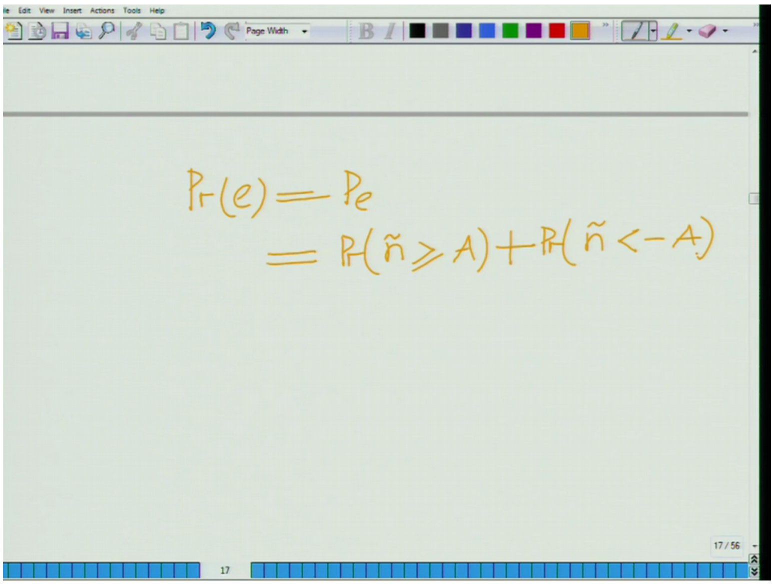
Now, what is r T remember we have derived the expression for r T; r T equals A naught; A naught plus n tilde A naught plus n tilde; n tilde is Gaussian, similar to BPSK mean equal to 0 variance equals n naught by 2. Now, therefore, error occurs if error if either r T greater than 2 A, r T greater than equal to 2 A this implies A naught remember A naught is a because we are considering with transmission of A which means A plus n tilde is greater than equal to 2 A which implies n tilde is greater than equal to A.

(Refer Slide Time: 15:52)



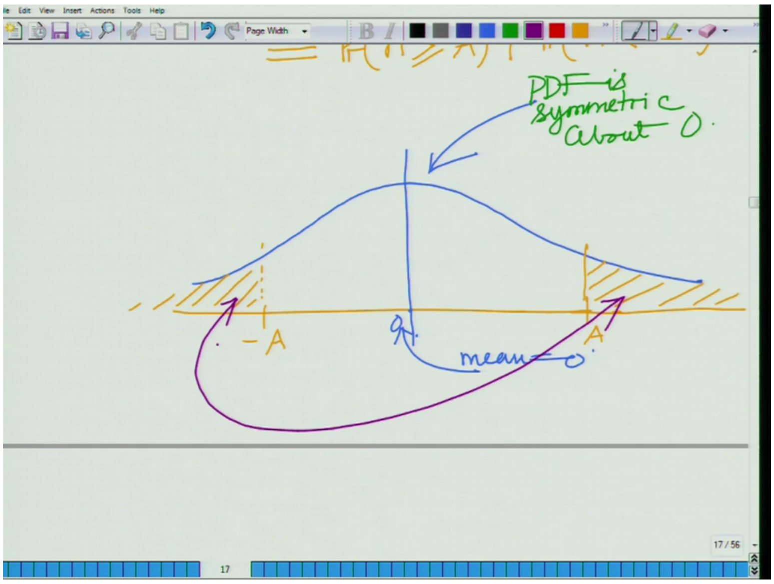
So, first scenario is error occurs if n tilde is greater than equal to A or r tilde or r T is less than 0 which means a plus n tilde is less than 0 which implies n tilde is less than minus A. So, error occurs if either n tilde is greater than A or n tilde is less than minus A. So, error occurs considering transmission of A naught equal to A if n tilde is greater than equal to a or n tilde is less than minus A now what is the probability that n tilde is.

(Refer Slide Time: 16:46).



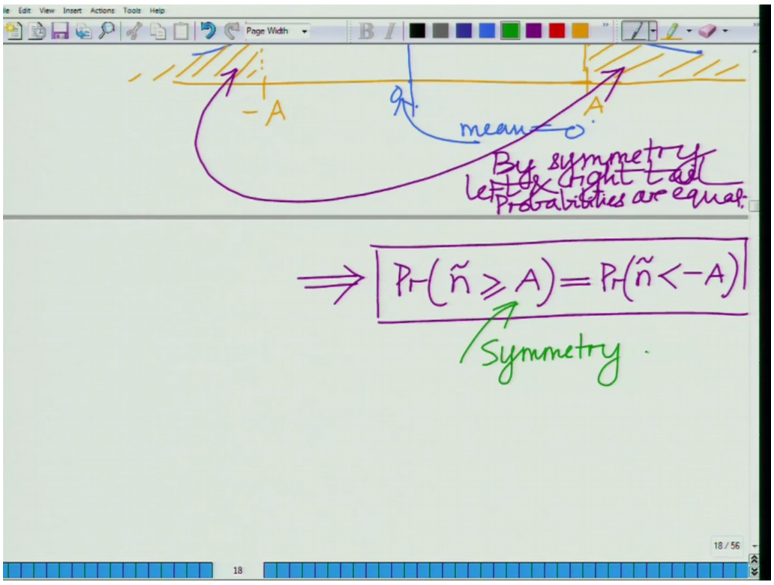
So, probability of error equals P e as we have seen these are disjoint events equals P of n tilde greater than equal to A plus probability n tilde less than minus A.

(Refer Slide Time: 17:13)



Now, if you look at n tilde n tilde is a Gaussian correct n tilde is a Gaussian with mean 0 which means it has a symmetric PDF about 0. So, mean equal to 0 implies PDF is symmetric about 0 Gaussian PDF, Gaussian PDF is symmetric about the mean which is 0 therefore, the probability. Now if you can look at any threshold A; the probability that n tilde is greater than equal to a is the Gaussian tail probability greater than equal to a. Similarly if you look at the probability that n tilde is less than equal to minus A that is the left tail probability correct and you can see by symmetry these 2 probabilities are equal.

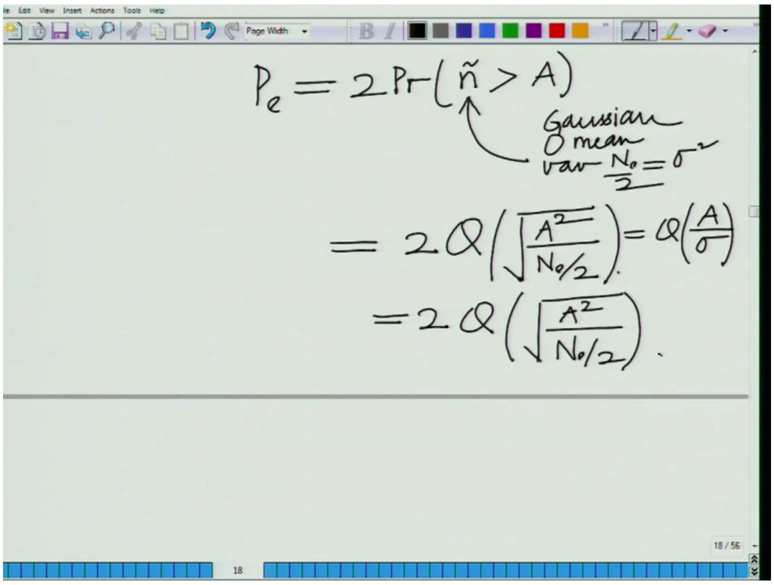
(Refer Slide Time: 18:30)



So, by symmetry or you can also say you can also say the same thing by symmetry you can see by symmetry these 2 tail probabilities are equal by symmetry left and right tail probabilities left and light tail probabilities the left and light tail probabilities are equal therefore, this implies that probability n tilde greater than equal to a well of course, I should not say they are exactly equal because probability because you have that one point a, but that probability is 0.

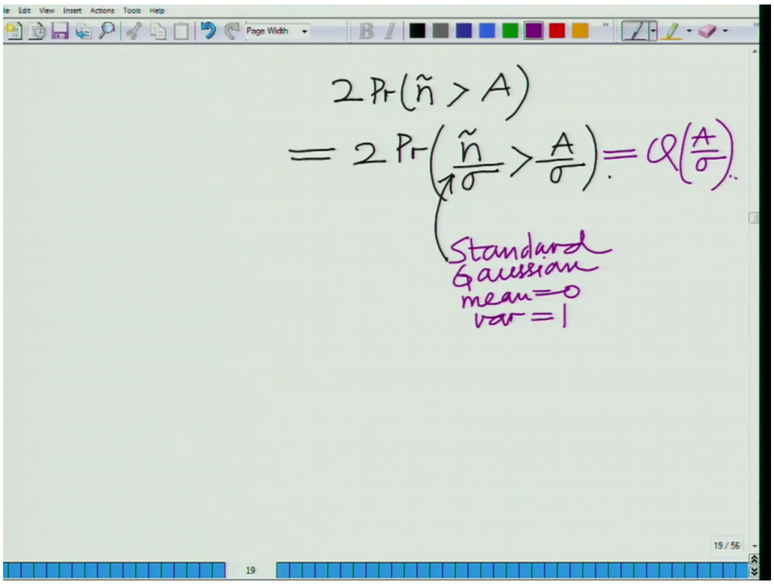
So, probability; so I will write this is equal to probability n tilde less than minus A and this is actually correct because that probability of that one point A is 0. So, this is basically follows by symmetry and therefore, what we have is probability of error if you are following the argument the sequence of arguments that we have made probability of error is probability n tilde is probability r T greater than equal to 2 A plus probability r T less than 0 which is equivalent to which is equal to probability n tilde greater than or equal to a plus probability n tilde less than minus A. And we are saying probability of n tilde greater than equal to 2 greater than equal to A is same as the probability n tilde less than minus A by symmetry therefore, probability of error is simply twice the probability that n tilde is n tilde is greater than or equal to A by symmetry.

(Refer Slide Time: 20:27)



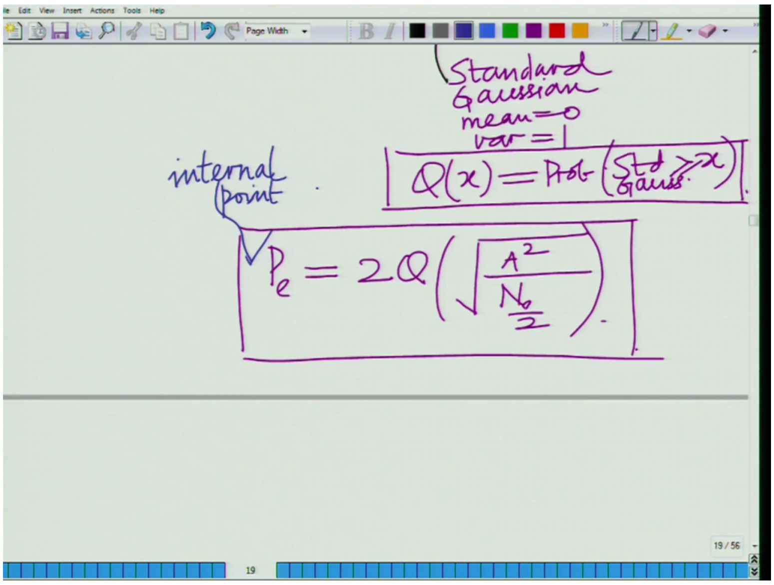
So, probability of error is simply twice the probability that n tilde is greater than A where n tilde is 0 mean Gaussian variance eta naught by 2. And now we are done from BPSK we know that probability n tilde greater than a is Q square root of A square divided by n naught by 2 this is Q of which is equal to basically Q of n naught by 2 equals variance is sigma square. So, this is Q of A divided by sigma.

(Refer Slide Time: 21:32)



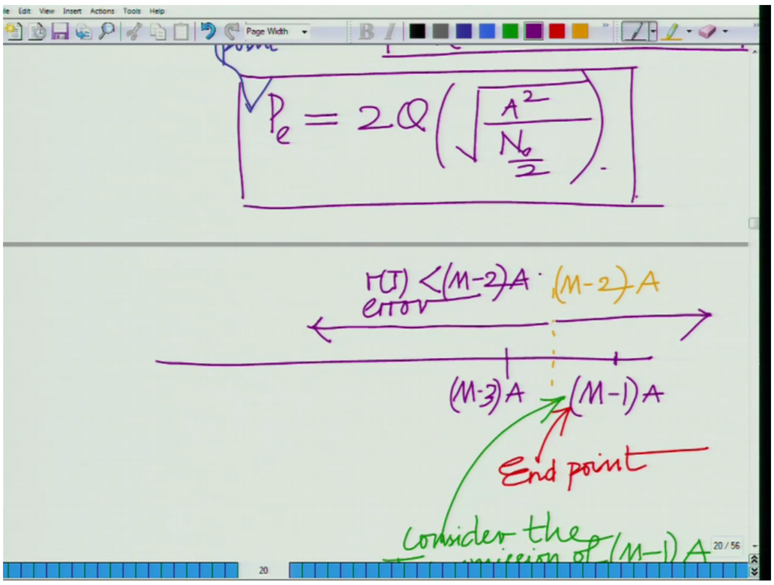
So, this is Q of twice Q of square root of A square divided by n naught there is also another way to see this because if you look at probability of n tilde twice probability of n tilde greater than A or rather greater than equal to A its one and the same thing greater than a or greater than equal to A. This is equal to I can write this as twice probability of n tilde over sigma greater than n tilde over sigma greater than a over sigma and now once I divide n tilde divided by sigma n tilde is 0 mean variance sigma you divide by sigma. You see n tilde by sigma is a standard Gaussian that is with mean equal to 0 mean equal to 0 variance equal to 1 there is a probability that standard Gaussian is greater than A or sigma this is equal to Q of A over sigma because Q of x.

(Refer Slide Time: 22:34)



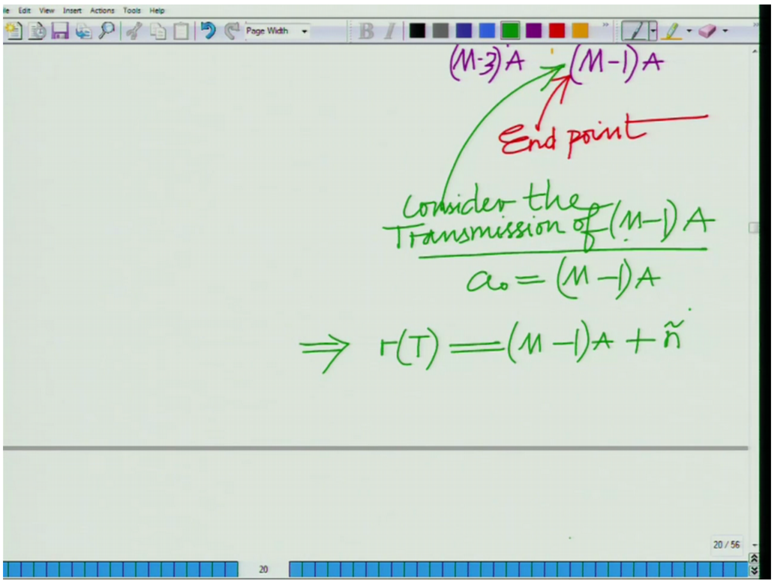
Remember Q of x equals probability standard Gaussian random variable mean of 0 variance one is greater than equal to x that is the definition from the very definition of the Q function. So, anyhow for the inter for the internal point it follows that the probability of error equals twice Q square root of A square divided by n naught by 2 remember, but remember this is only for an internal point any internal point.

(Refer Slide Time: 23:22)



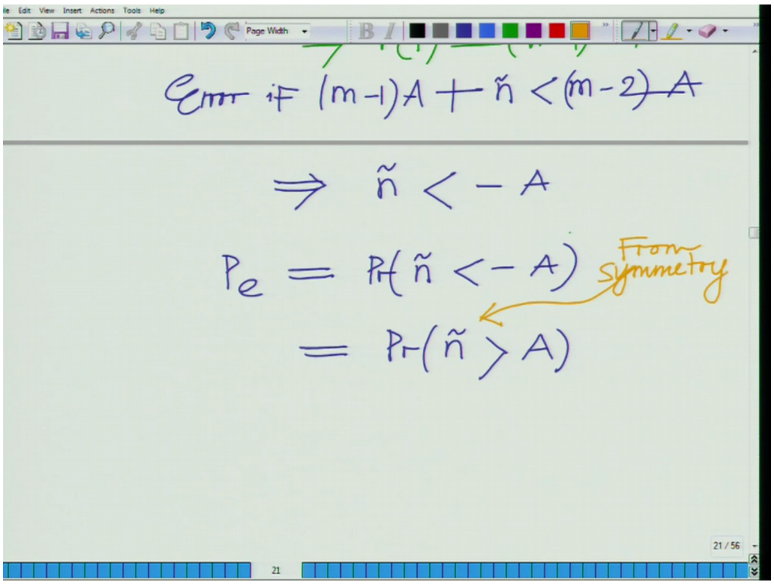
Now, consider an external point or consider a point points endpoints now consider for instance endpoints M minus 1 A M minus 3 A the threshold is M minus 2 A. So, this is one of the endpoints. So, consider the transmission of M minus 1 A.

(Refer Slide Time: 23:55)



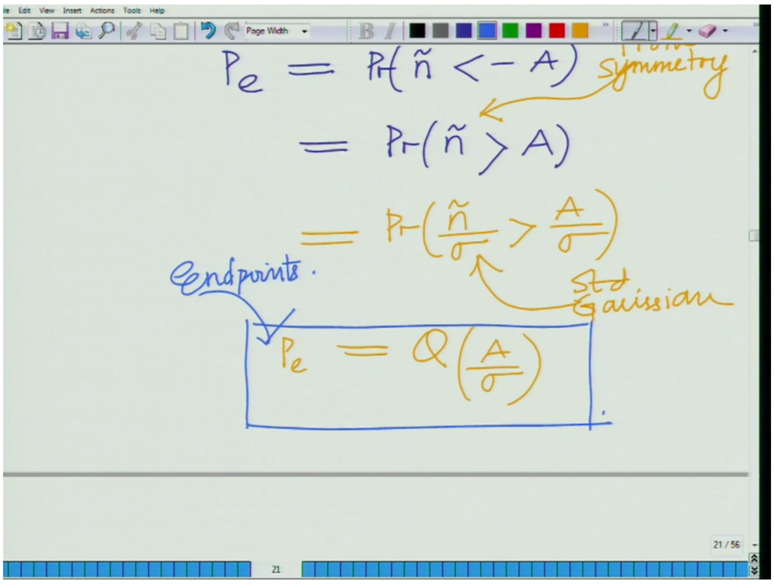
So, let us say we consider transmission of M minus 1 A consider the transmission of M minus 1 A, alright, we are considering the transmission of one of the end points of the constellation that is M minus 1 A you can similarly also consider minus M minus 1 A does not make any difference. So, we are saying A naught equals M minus 1 A implies r T equals A naught plus n tilde that is equal to M minus 1 A plus n tilde. Remember we decide r T equal to M minus 1 A if r T is greater than M minus 2 A. So, the error occurs if r T. So, error if r T is less than M minus 2 A that is M minus 1 A plus n tilde error.

(Refer Slide Time: 25:05)



If M minus 1 A plus n tilde is less than M minus 2 A, we decide implies n tilde implies n tilde is less than error if n tilde is less than n tilde is less than M minus 2 A minus M minus 1 A n tilde is less than minus A and by sym. And therefore, probability of error for the end point for this end point is the probability that n tilde less than minus A which is equal to probability again probability n tilde because n tilde is symmetric Gaussian noise which is 0 mean this is symmetric. So, by symmetry this is equal to probability n tilde is greater than A this follows from symmetry.

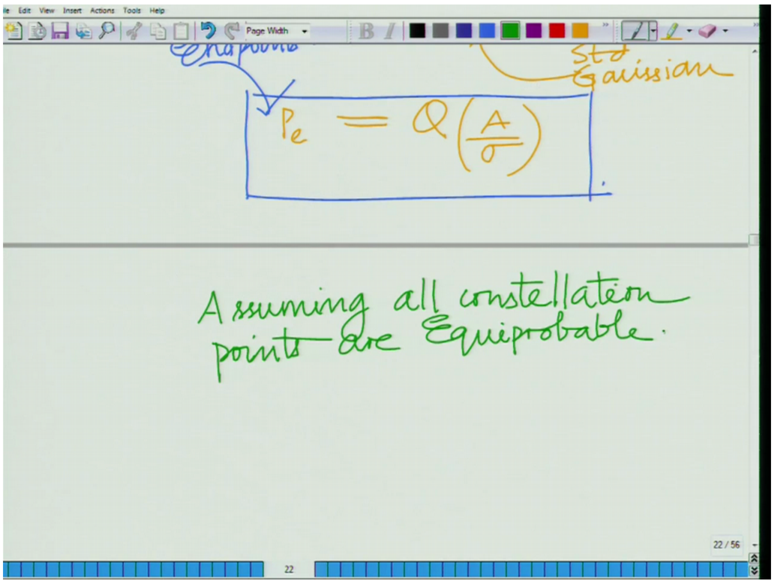
(Refer Slide Time: 26:26)



Which is equal to probability n tilde by sigma is greater than a over sigma and now observe that n tilde over sigma this is a standard Gaussian probability that standard Gaussian is greater than A over sigma this is equal to Q of A over sigma.

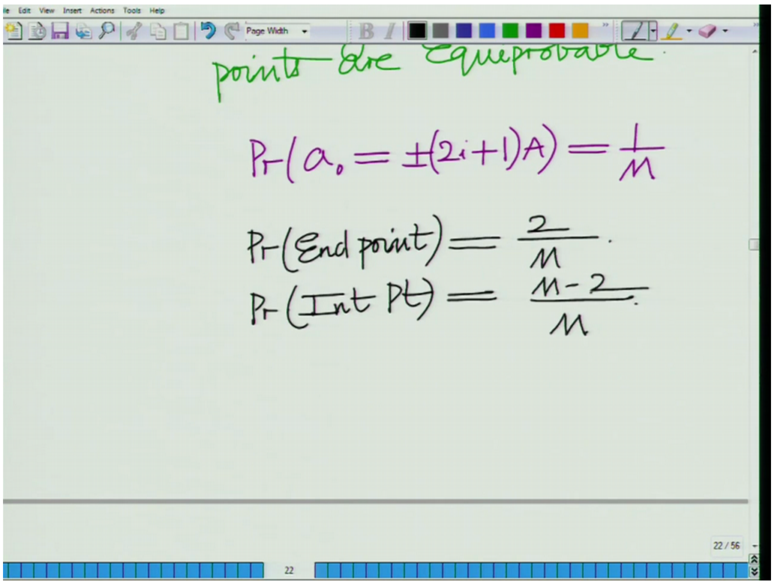
So, probability of error equals Q of A over sigma this is for the endpoints. So, we have probability of error for an internal point which is twice Q of A over sigma probability of error for an endpoint is only Q of a over sigma that is the difference. Now assuming all the points are equiprobable that is how many points do we have remember in M-ary PAM.

(Refer Slide Time: 27:28)



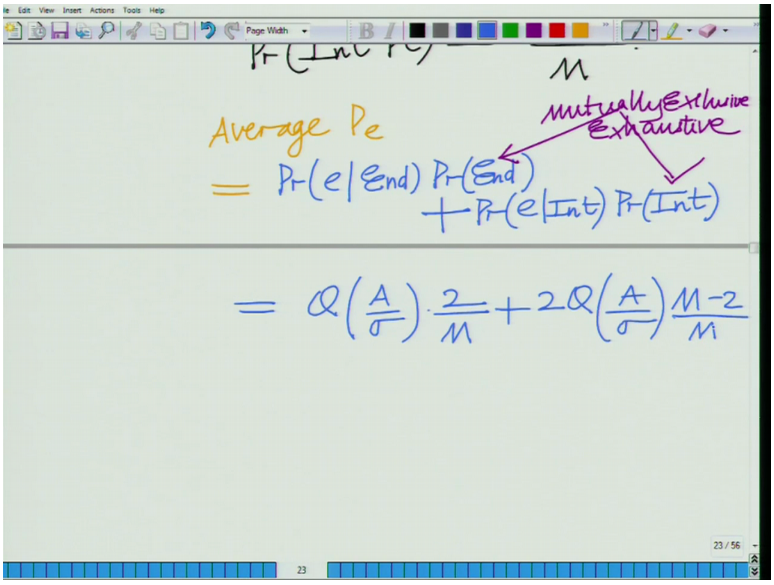
We have M levels assuming all the levels are equiprobable all of them each has probability one over M assuming all M all constellation points are equiprobable assuming all constellation points are equiprobable.

(Refer Slide Time: 27:53)



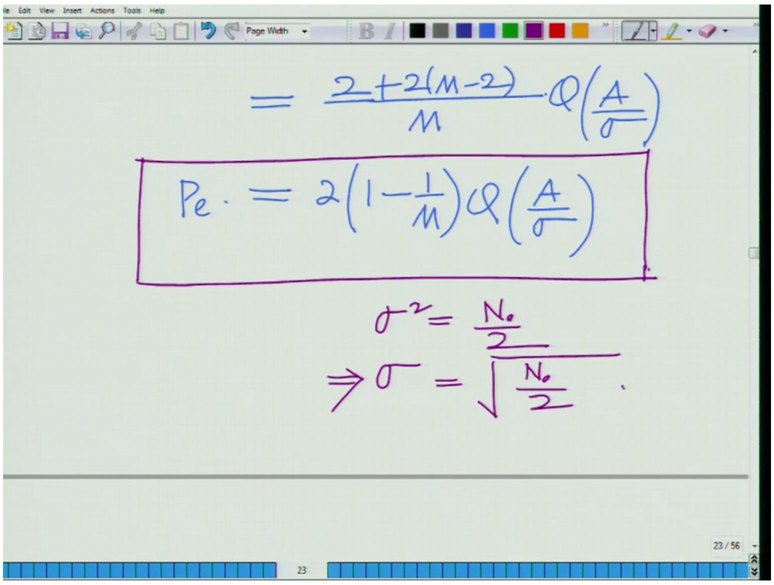
Probability of that is each point A naught is equal to any plus or minus 2 i plus 1 A is equal to 1, by M all points there are M points all points are equiprobable implies that each point must have probability 1 over M. So, now, average probability of error. So, because probability of error for end points and probability of error for internal points is different, now how many internal points are there how many end points are there; there are 2 end points and M minus 2 internal points. So, probability of an end point is 2 divided by M 2 end points 1 by M plus 1 by M probability that it is an internal point is M minus 2 divided by M. So, probability of end point equals 2 divided by M probability of an internal point is equal to M minus 2 divided by M.

(Refer Slide Time: 29:10)



Therefore average probability of error now you can say average probability of error equals probability that it is an end point 2 over M let me write this little bit more explicitly. So, that you get it clearly probability of error probability of error given end point into probability of endpoint this is total probability rule plus probability of error given internal point into probability of an internal point this is the rule of probability. So, that is probability that is because end point and internal point are mutually exclusive and exhaustive events this is something that you have to remember again. I urge you to look at the lectures on probability these are mutually exclusive and also exhaustive anyway. So, this is equal to probability of error given n we have already calculated that is Q of A divided by sigma times probability of end points that is 2 divided by M plus probability of error given interior point twice Q of A divided by sigma into probability of interior point that is M minus 2 divided by divided by M.

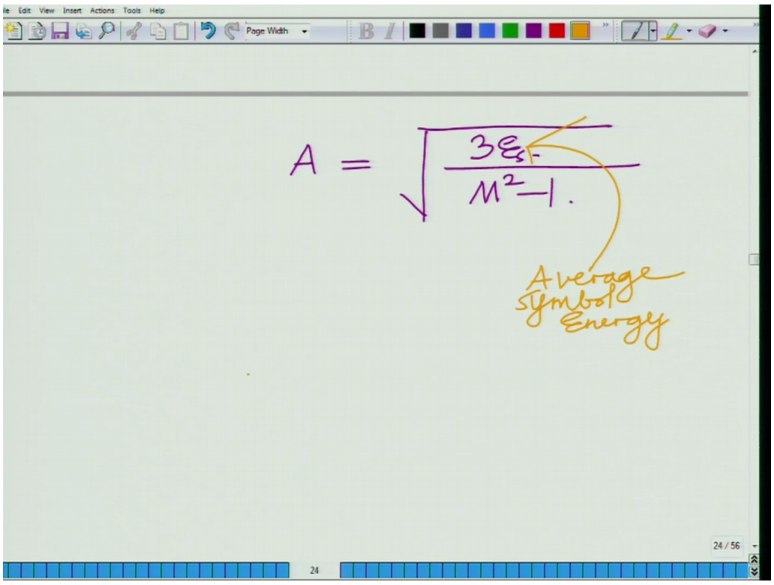
(Refer Slide Time: 30:50)



Now, this is equal to twice 2 plus 2 M minus 2 divided by M Q times A Q of A over sigma 2 plus. So, this is twice 1 plus M minus 2 that is 1 minus. So, twice 1 plus M minus 2 that is M minus 1 over M that is 1 minus 1 over M Q of A over sigma this is the average probability of error averaged over both the end points and internal points this is the average probability of error.

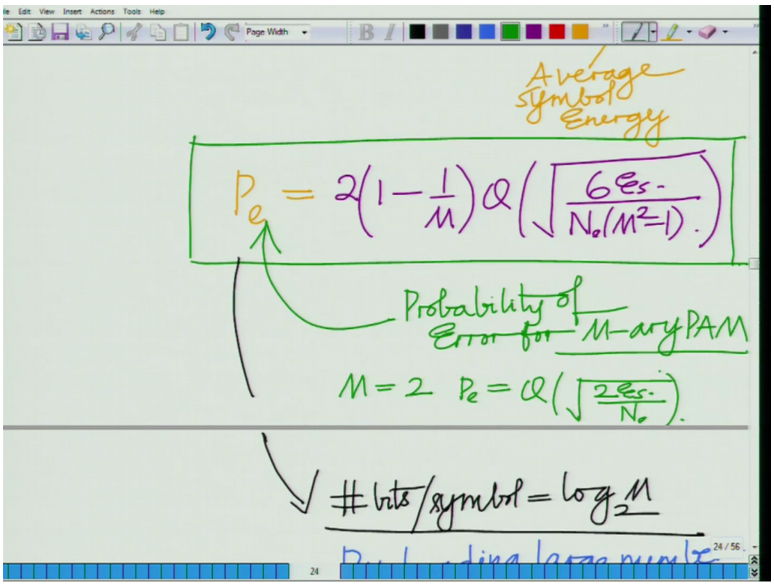
Now, all we have to do is we have to substitute a expression for A and expression for sigma; sigma is of course, very simple sigma square equals noise variance is n naught by 2 E P; E P is 1. So, its n naught by 2 which implies sigma equals square root of n naught by 2 and A remember if you remember A; it is if symbol energy equal to E s.

(Refer Slide Time: 32:01)



A; we have already derived very elaborately for the M-ary PAM. In fact, that was 1 of the first things that we derived a is equal to square root of 6 e s by no I am sorry, A equal to square root of 3 E s divided by M square minus 1 where E s remember E s is average symbol energy.

(Refer Slide Time: 32:38)



So, average probability of error substituting this expression for both A and n naught by 2 that is twice 1 minus 1 by M Q square root of A square which is 3 E s divided by n square minus 1 divided by n naught by 2. So, that will be 6, E s divided by n naught into

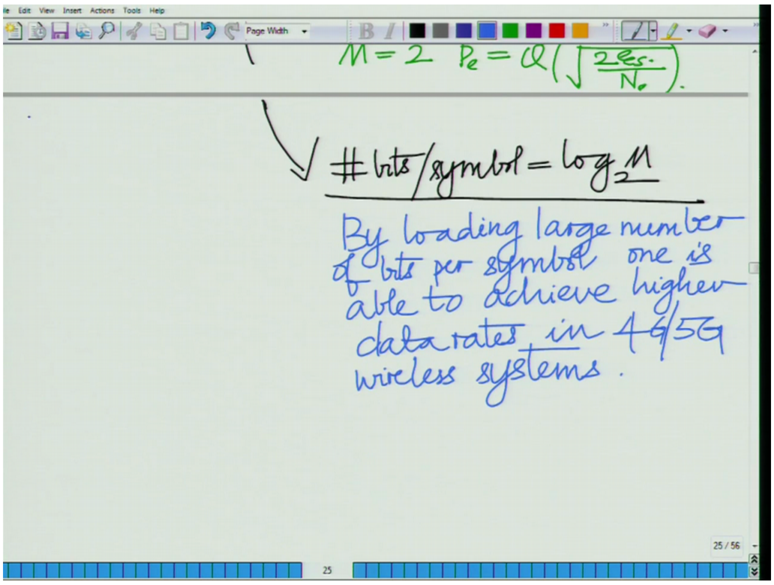
M square minus 1. So, this is basically the average probability of error for M-ary PAM it is slightly elaborate, but it is a very interesting derivation it is one of the fundamental expressions because M-ary PAM is a very general digital modulation scheme.

So, this is let me describe a little bit more about it, now if I substitute M equal to 2 if you substitute M equal to 2, you can see that probability of error will be twice 1 minus 1 over twice 1 minus half is basically half twice into 2 into half is 1. So, this reduces to Q square root of well n square minus 1 is M equal to 2. So, M square minus 1 is 3. So, six divided by 3 is 2 twice E s by n naught which is same as what we had before what we had before is E b Q of square root of 2 E b over n naught, but for 2 PAM that is when M equal to 2 the average bit energy is the same as average symbol energy because number of bits per symbol is 1, alright.

So, average bit energy is the same as average symbol energy and therefore, what you have seen is we have derived the average bit energy for M-ary PAM as I have said it is a very general modulation because M-ary PAM, you can transmit log M to the base 2 bits per symbol. So, unlike the previous modulation schemes such as binary phase shift keying etcetera you are not limited to 1 bit per symbol you can load a larger number of bits on each symbol and that is helpful especially in modern high speed wireless communication systems because if you have a good channel where the SNR is good you can keep loading a large number of bits on each symbol such as 4 PAM, 8 PAM, 128 PAM.

In fact, we are going to see a much more general modulation scheme next that is known as quam which is similar to QPSK that is we have 2 basis functions and you can load any number of bits on each on each basis function correct. Each orthogonal basis function or each orthogonal basis symbol thereby by increasing the number of bits per symbol for the same symbol rate you can significantly increase the bit rate and that is how one achieves higher and higher data rates for instance in 4 G wireless communication system or now people are talking about somewhat five g wireless communication system. So, by loading by using higher these are known as higher order modulations by using higher values of M by loading large number of bits on each symbol one is able to achieve higher data rate that is why the performance of this general modulation scheme is important.

(Refer Slide Time: 36:03)



So, by loading; so the final point, I want to mention is number of bits again emphasize number of bits per symbol equals log M to the base 2 by loading large number of bits per symbol rather than simply one bit per symbol using large of number of bits per symbol one is able to achieve higher data rates in for instance 4 G and 5 G. So, these are very important 4 G and 5 G that is the whole point. So, this is something. So, this is basically summarizes the probability of error for M-ary PAM, let me just write it over here. So, write its clear probability of error for M-ary PAM.

So, basically that wraps up this module where we had the previous module we have introduced M-ary PAM partly discussed the decision rules for the internal points. Now we have completed the discussion regarding the decision rules for the end points there are 2 end points and what is the probability of error for each point if it is an internal point probability of error for an end point and therefore, what is the average probability of error and we have derived an expression for that all right.

So, please go through this derivation again because it involves several steps a series of logical arguments that we have to make and the series of computations that we have to make at every step and we have used several different properties such as properties of random variables such as properties of mutually exclusive events properties of mutually exclusive and exhaustive events the total probability rule etcetera.

So, this is a very interesting derivation that also that of course, tells you what is the average probability of error what is optimal decision rule what is an M-ary PAM constellation, what is optimal decision rule, what is a resulting probability of error and also explains or gives or a also this practical example helps you better understand several principles of probability and random processes alright.

So, we will stop here. We will look at other aspects in the subsequent modules.

Thank you very much.