Principles of Communication Systems – Part II

Prof. Aditya K. Jagannatham
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

Lecture - 20
Introduction to M-ary PAM (Pulse Amplitude Modulation), Average Symbol Power and Decision Rules

Hello. Welcome to another module. So, in this module let us start looking at M-ary another digital modulation scheme based on M levels termed as M-ary pulse amplitude modulation.

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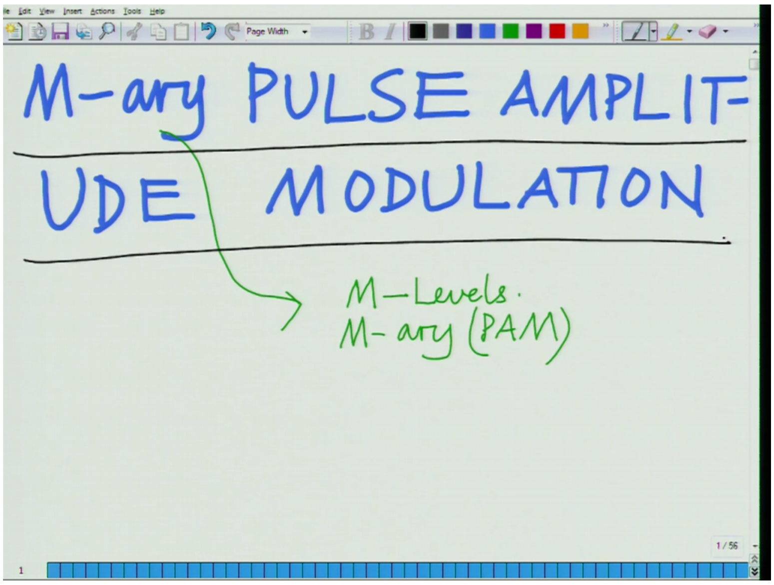

So, we want to start looking at M-ary, pulse, amplitude, M-ary pulse amplitude, modulation, which is based on M levels. Until now we have seen binary modulation schemes. So, this is based on M levels which we are going to describe shortly. And this is abbreviated as M-ary pulse amplitude modulation is abbreviated as PAM M-ary PAM M-ary pulse amplitude modulation; so in this modulation scheme if you can look at the waveform x t.

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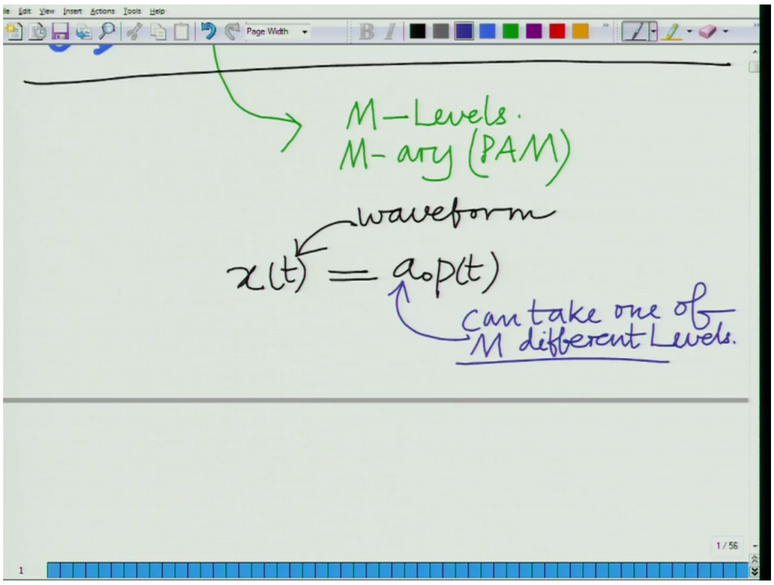

Let us consider the waveform x t equals a naught times p t. This is our waveform the digital communication waveform. Now this a naught can take one of M different levels. So, far we have seen correct; so one of M different levels. So far what we have seen a naught can take at most one or 2 levels that is binary plus a or minus A A or C correct.

Now, we are going to see a general modulation scheme a more general right. In fact, a more capable modulation scheme you can also say as we are going to see shortly which can take not one of 2 levels, but one of M possible levels. So, each symbol a naught is digital symbol a naught can take one of M capital M possible symbols. Now let us represent these M symbols.

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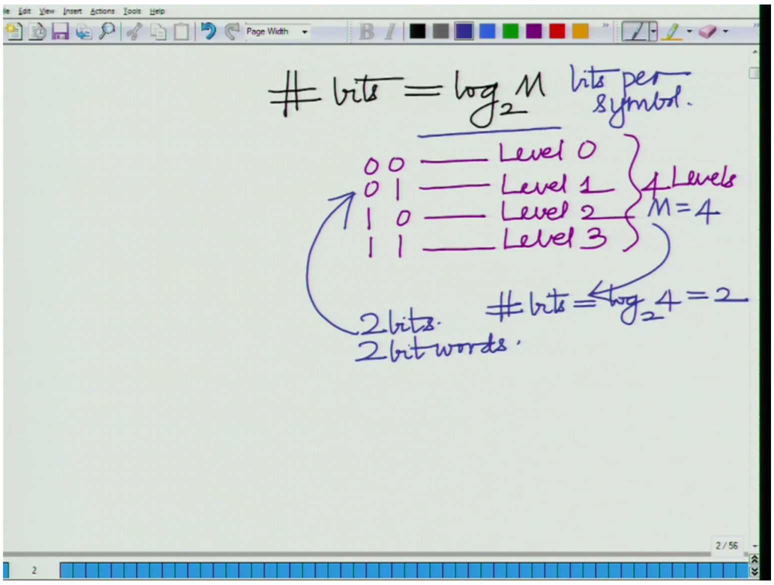

Now, therefore, since there are M symbols the number of bits equals log M to the base 2. For instance, if we have 2 bits 0 0 0 1 1 0 1 1, these can be mapped to 4 different levels. So, this can be your level 0, this can be your level one this can be your level 2 and level 3.

So, we have 4 levels implies basically M equal to 4. Therefore, number of bits as we can see is log 4 to the base 2 equals 2. So, we have 2 bits or 2 bit words or 2 bit words. So, in a M-ary constellation, we can we can achieve log M to the base 2 bits per symbol. That is each bit each symbol conveys log M to the base 2 bits log M to the base 2 bits per symbol, for instance example.

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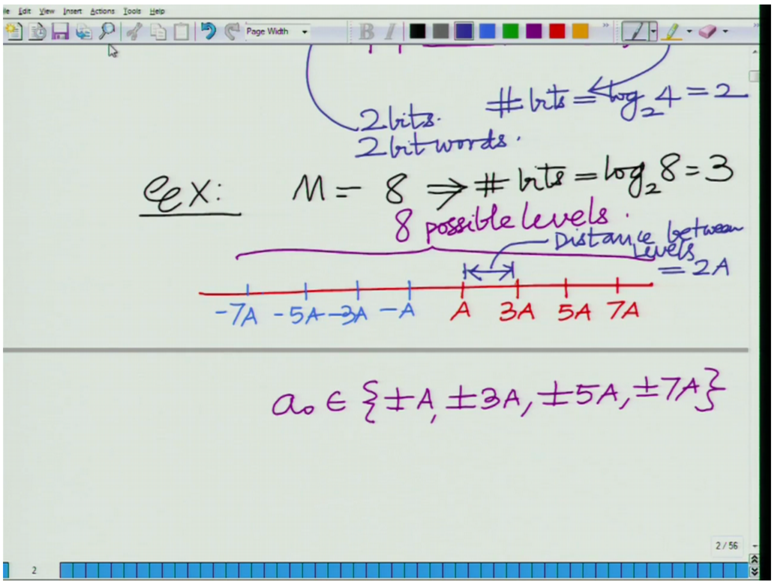

Let us take a simple scenario where M is equal to 8, implies number of bits natural equals log a to the base 2. This is equal to 3. If you look at the M levels you can either be a these are equispaced 3 A 5 A. So, we are generalizing this concept from A and minus A we are allowing M possible levels minus 7 M. So, what we have we have 8 possible levels 8 possible levels. Now a naught can belong to one of these 8 possible levels that is plus or minus A plus or minus 3 A plus or minus 5 A and plus or minus 7. And the distance between these levels. So, we can see the distance we are keeping a constant distance between these levels distance between levels is equal to 2 A. The distance between levels is equal to twice A.

So, we have 8 levels we are considering M equal to 8 constellations which means there are log 8 to the base 2, 3 bits per symbol the a levels are plus or minus A plus or minus 3 A plus or minus 5 A and plus or minus 7 A. So, this is a M-ary that is 8 ary PAM pulse amplitude modulated constellation.

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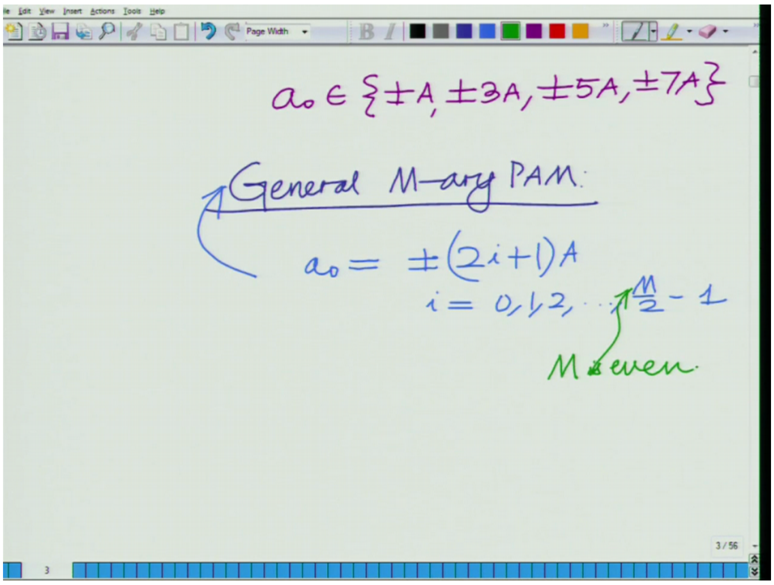

Now, in general for a M-ary constellation in general for general M-ary PAM, we have a naught equals either plus or minus 2 i plus 1 A where I equals either 0 1 2 up to M by 2 minus 1 naturally this is defined only for M is even.

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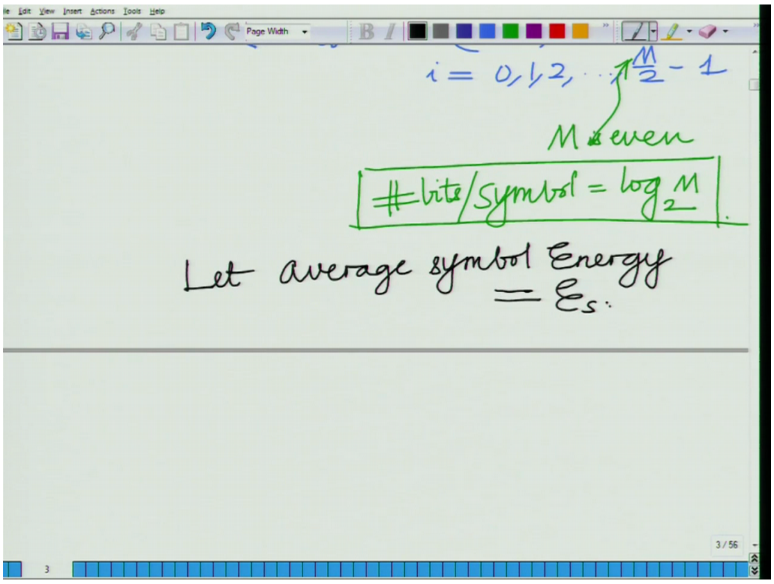

So, M is even and number of bits per symbol is log M to the base 2, which we already said log M to the base 2. That is the number of bits per symbol. Now we have to compute as we have seen several times before we have to normalize things such that the average bit energy or average symbol energy is same. Now since we have more than one bit per symbol. Let us consider start with the average symbol energy. So, let the average symbol energy be denoted by Es k. So, let average symbol energy be equal to Es.

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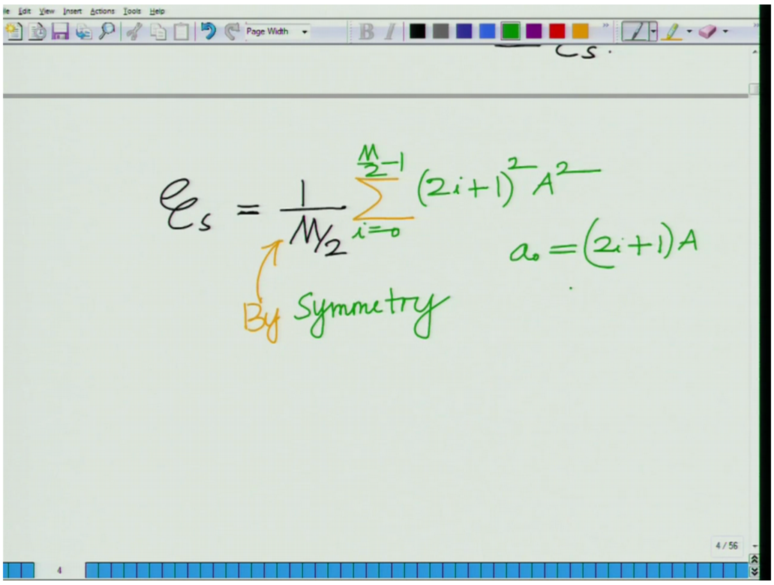

So, we have Es by symmetry Es is equal to if you look at energy of the symbols on the right of 0 greater than 0 is equal to energy of the symbol on the rest of 0. So, I am only going to look at half of that this is my symmetry; I am only going to look at M by 2 symbols because energy of the symbols by symmetry, this is by symmetry energy of the symbols greater than 0 is equal to energy of the symbols less than 0.

So, this is equal to i equal to 0, to M by i equal to 0, to M by 2 minus 1 as we have seen this is plus or minus 2 i plus 2 i plus 1 whole square times A square since we have well a naught equals that is considering symbols greater than 0 a naught equals 2 i plus 1 A. Of course, for less than 0 that is when it is minus 2 i plus 1 A the energy is similar the energy is identical. So, that is why we are only considering half of the symbols that is symbols corresponding to plus 2 i plus 1 into a the rest of the symbols that is for each symbol plus 2 A plus 1 A we have one symbol which is minus 2 i plus 1 times a which has an identical energy.

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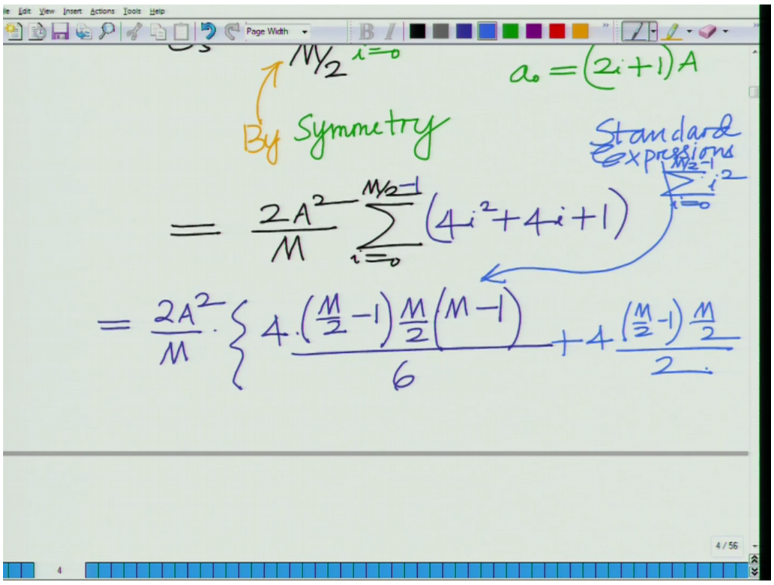

Now, this is equal to let us simplify this average energy, because this is the most important step. So, this is equal to well 2 A square divided by M summation i equal to 0 to M by 2 minus 1 4 i square plus 4 i plus 1. This is 2 A square divided by M. Now here comes the important part M by 2. Now 4 summation i square, now we will remove the summation. So, 4 summation i square would be 4 times from 0 to of course, M by 2 minus 1 that would be 4 times M by 2 minus 1 summation of squares up to n is n into n plus 1 into 2 n plus 1 divided by 6. Here we have summation 0 up to M by 2 minus 1. So, that is M by 2 minus 1 into M by 2 minus 1 plus 1. So, that is M by 2 times 2 M by 2 minus 1 plus 1. So, that is M minus 2 plus 1 that is M minus 1 divided by 6.

So, these you should be able to get these you should be able to evaluate from standard expression. So, this I am not going to repeat this. Because you are expected to know this evaluating from standard expressions for summation of i square i equal to 0 to M minus 1. These expressions are standard, plus well 4 times summation i. So, summation i, i 0 to n is n into n plus 1 divided by 2.

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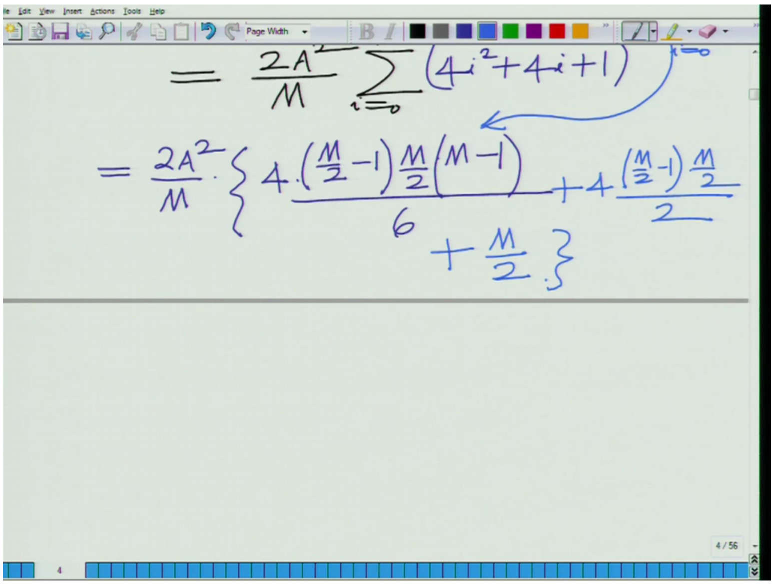

So, from 0 to M by 2 minus 1 is M by 2 minus 1 into M by 2 minus 1 plus 1 that is M by 2 divided by 2 plus summation of i, i equal to summation of one I equal to I to M minus 1 is simply M by 2 that is 1 added M by 2 times so that component is simply component is simply M by 2.

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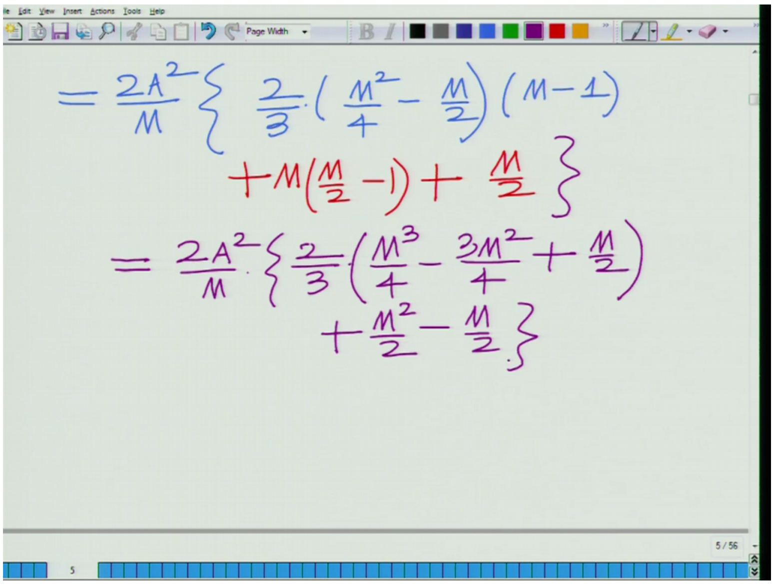

Now, we are going to simplify this expression to compute the average energy. So, this can be simplified as well 2 A square divided by M. You can check this this is 2 by 3 times M square by 4 minus M by 2 times M minus 1 plus M into M by 2 minus 1, plus M by 2 which is equal to 2 A square, divided by M into 2 by 3, M cube by 4, minus 3 M square divided by 4, plus M divided by 2, plus M square by 2, minus M plus M by 2 that is minus M by 2.

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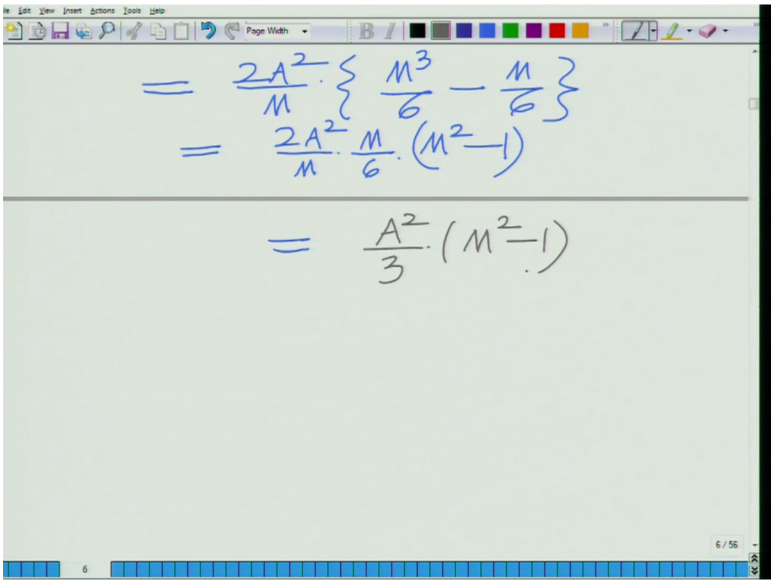

And if you now simplify this further what you will have is 2 A square divided by M into M, cube by 6 minus M by 6 which is equal to 2 A square divided by M into M by 6 into M square minus 1, which is equal to well A square divided by 3, sorry A square divided by 3 A square divided by 3 times M square minus 1.

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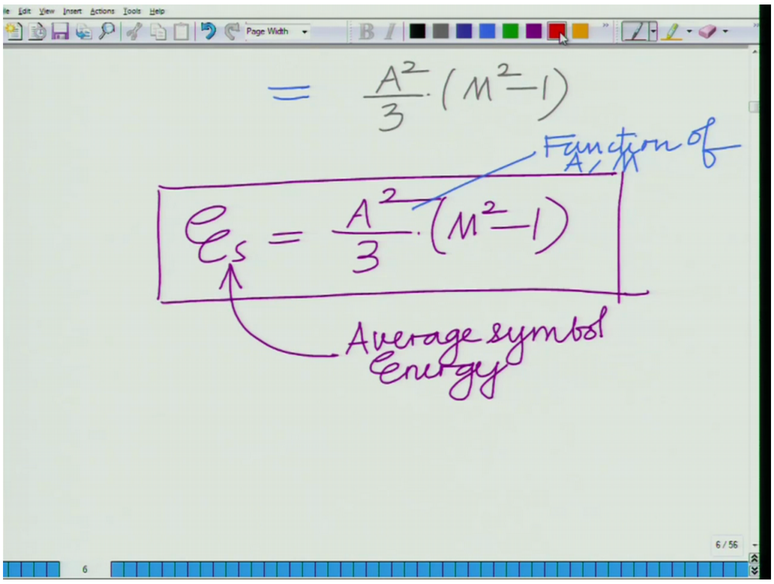

So, we have the final result that Es average symbol energy as a function of A and M is A square by 3 minus M. So, this the average symbol energy or you can also say average energy per symbol. And this is equal to this is a function of A and M. Naturally, it depends on A where 2 A is the spacing, now we cannot say a is the amplitude because there are several amplitudes A 3 A 5 A. So, we can say we can only characterize it by the spacing between the successive levels and the spacing is basically given by 2 A. So, this is a function of a which is the spacing and also M which is the number of levels that is the point.

So, function of a comma M, where spacing is spacing between levels is equal to spacing between M-ary PAM levels is equal to 2 A.

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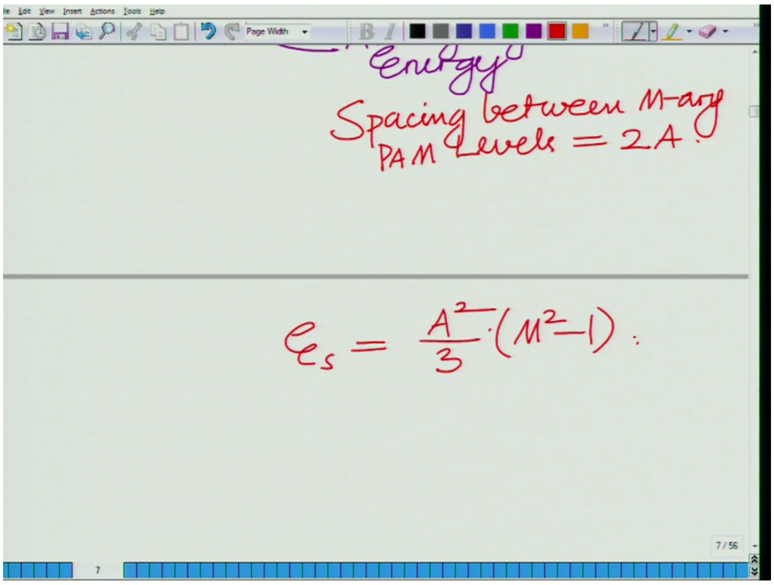

Spacing between M-ary PAM levels is equal to 2 A, and therefore, now look at this we have Es is equal to well A square by 3 into M square minus 1 if we are given Es. Therefore, if we set A is equal to 3 Es, remember that is what we want to going to choose an appropriate a where 2 A is this thing 3 Es divided by M square minus 1 square root.

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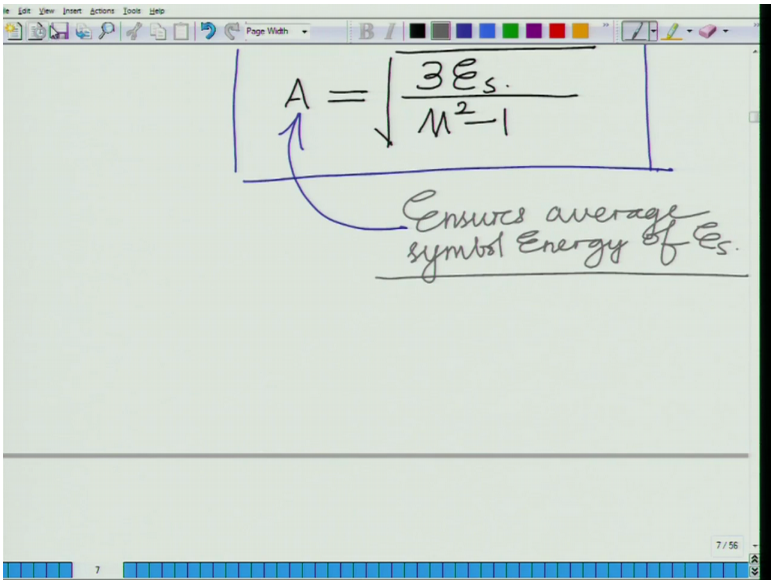

So, setting A equals square root of 3 Es by M square minus 1 this ensures average symbol energy of Es. So, setting this value of A this ensures average symbol energy of this ensures an average symbol energy of Es, that is the whole point. And therefore, now we have got what is the A what is the parameter A where 2 A is the spacing between these 2 levels.

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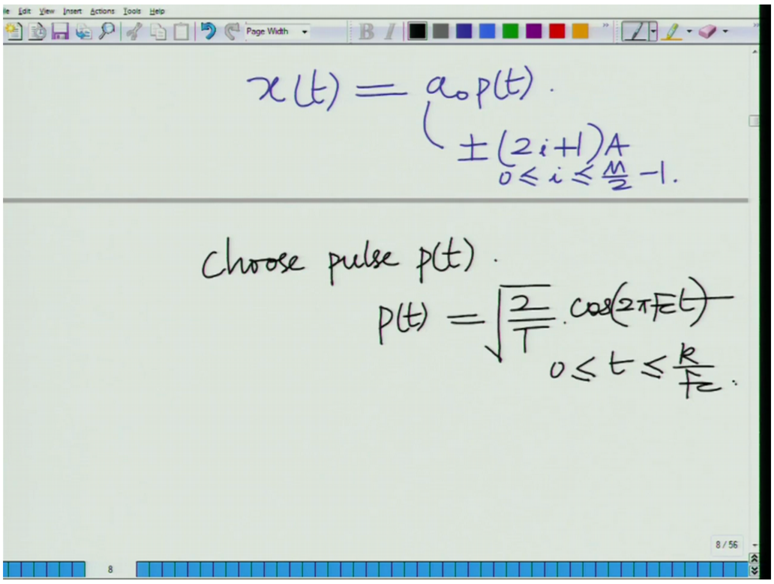

We also know that our waveform x t equals a naught times p t a naught is plus or minus 2 i plus 1 times a, 0 less than equal to I can take the values 0 less than equal to i less than equal to M minus 1. We can choose any pulse p t. In particular, one can choose the pulse p t which is square root of 2 over t cosine 2 pi f c t. We have seen that this is basically seen several times before. In fact, this is basically where 0 less than equal to t less than equal to k over f c that it contains an integer number of cycles.

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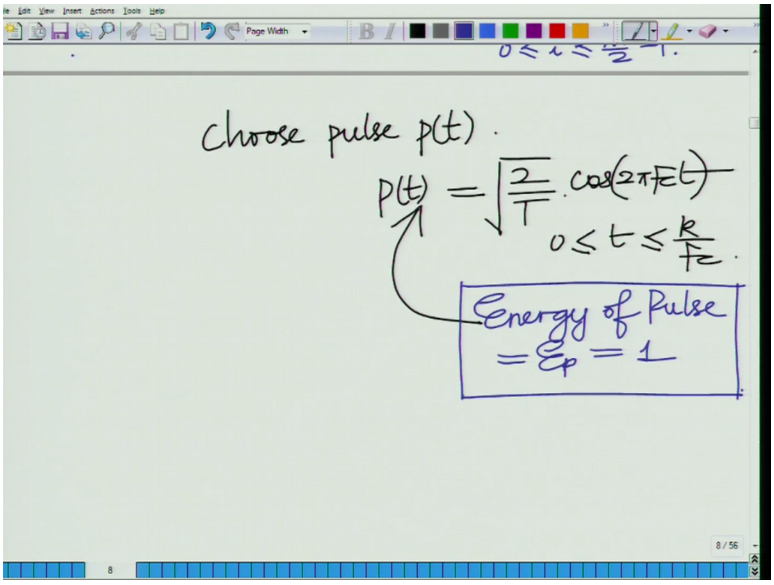

And we have seen that energy of this pulse when you set the pulse like this energy of the pulse equals E p equals the energy of the pulse is unity, when you set it like this the energy of the pulse is unity. And; obviously, now we can again we have a pulse p t. Therefore, I can use the matched filter receiver. That is also similar to what you did before which is the matched filter receiver at the receiver.

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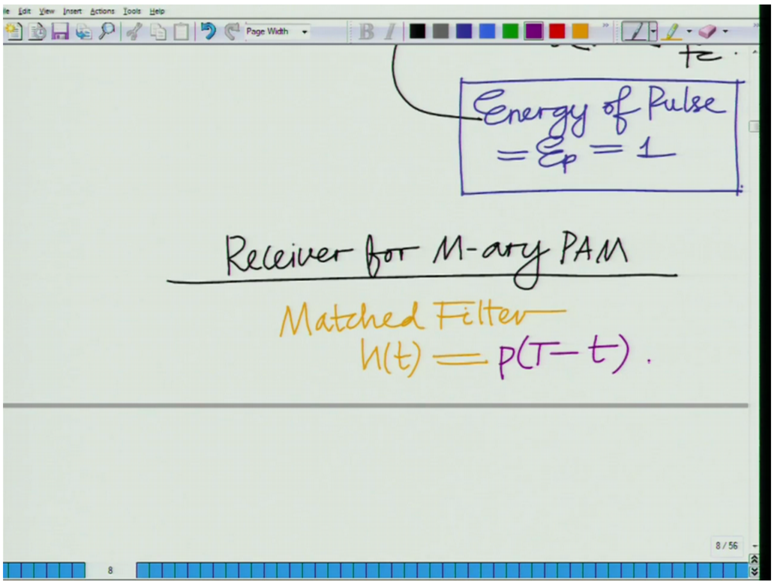

So, receiver operation receiver for M-ary PAM; so obviously, we are again going to use the matched filter. Matched filter h t matched to p t which is equal to that is again the same matched filter maximizes SNR because for the if you go back and look at the derivation of the matched filter, we did not assume any particular modulation we have only assumed a pulse shape, and a symbol a naught a naught need not belong to any particular modulation.

So, for any given pulse shape p t irrespective of the modulation scheme a naught the matched filter maximizes the SNR. Therefore, we are choosing matched filter h t which is p capital T minus t where capital T is the symbol duration. So, that remains unchanged.

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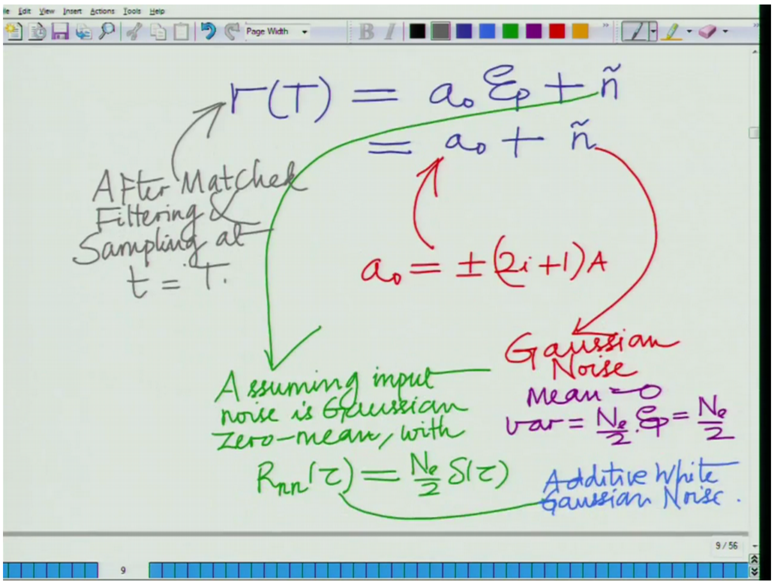

So, matched filter still maximizes the output SNR and in fact, once you matched filter. So, we have E p equals 1. In fact, once you matched filter and sample you have again R t equals a naught plus n tilde E p equal to 1, which is equal to a naught plus n tilde. Which is very similar to what you have obtained for binary phase shift keying only thing is a naught changes only thing is now a naught equals plus or minus 2 i plus 1 a. So, that is the only difference. So, we still have a naught times E p, E p equals 1. This is Gaussian noise. Again the noise part is exactly the same as we have BPSK Gaussian noise mean equal to 0 variance equals a naught by 2 E p, where E p equals 1 this is a naught by 2.

And of course, we are assuming Gaussian noise assuming the input noise process, we are assuming or let me assuming input noise process, assuming that input noise process is Gaussian. White sense stationary 0 mean of course, in the Gaussian noise process white and stationary, it automatically means it is strict and stationary also we have seen that or follows from the properties of random process with R n and tau that is 0 R n and tau equals n naught by 2 delta tau basically this is white noise additive white Gaussian noise the same thing.

Additive white Gaussian noise that is the same thing, so this is basically let me clarify this is after matched filtering and sampling at t equal to tau t equal to capital T, after match filtering and sampling at t equal to t.

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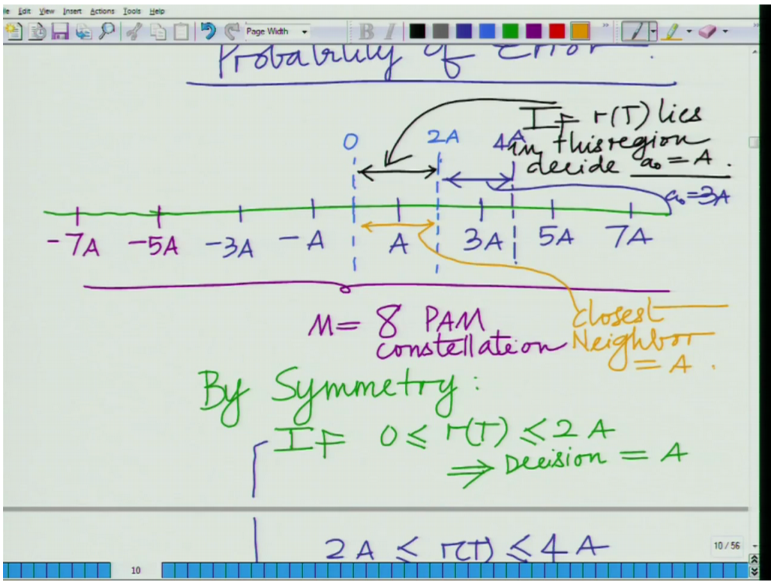

Now, let us come to the probability of error the discussion about probability of, now this is going to be slightly interesting it is going to be different from what we have seen because the modulation scheme is complicated it is going to be slightly different than what we have seen for BPSK and other modulation schemes.

So, the probability of error discussion as I was just saying is going to be slightly more different. And of course, more complex because the modulation scheme is complex naturally the analysis also is going to be complex. It is going to be different from what we have seen earlier for BPSK. BPSK it was still fairly simple. So, this is going to be slightly more complicated because the nature of different points in this constellation is different.

Because if you look at these points on the line some of the points are in the middle surrounded by or basically flanked by point on each side some of the points 2 of the points precisely are on the end. So, the probability of error for each of these things or the average probability of error for each of these considering the transmission of each of these symbols is going to be different that is the point I want to emphasize.

So, what I am saying is basically first of course, you have to come to the decision rule let us talk a little bit about the decision rule. Now we have the levels A for instance let us again consider our M equal to A case because that is simple to handle or let me just be minus A minus 5 A and minus 7 A. This is your M equal to 8 PAM now again once again the detection is very simple if you look at this point A interior point A by symmetry you can look at the midpoint between A and 3 A, and you can look at the midpoint between minus A and so this midpoint is 0 this midpoint is 2 A. So, if R t lies in this region that is we are using something very similar to BPSK again if you think about it we are looking because it has 2 constellation points.

So, if R t lies in this region, decide a naught is equal to A. Now the point is in BPSK you have only one point. So, you look at the midpoint between the 2 points. Now here you have a point there is a point on one side there is a another constellation point on the other side. So, we have to basically look at 2 midpoints one on each side. So, the midpoint with respect to 3 A is 2 A midpoint with respect to minus A is 0. So, if R t lies in this range 0 to 2 A correct then one can decide A and similarly for all the interior points.

So, again it is fairly simple, by symmetry again one can again say by symmetry, if 0 less than equal to R t less than equal to 2 A, implies decision equals A. 0 less than equal to or not zero, but now if you look at the midpoint this other set of the midpoints 2 A and midpoint between 3 A and 5 A, that is going to be that midpoint is going to be 4 A. So, in this region we; obviously, have to decide a naught equals 3 A.

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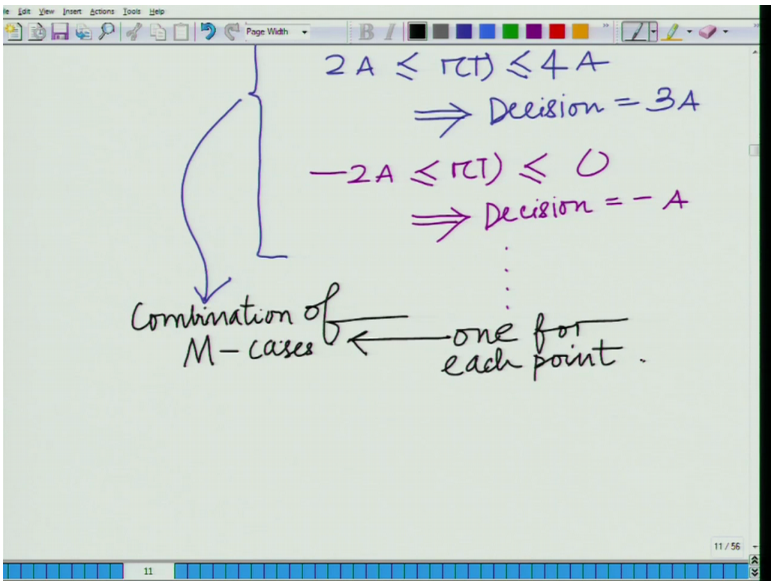

So, if R t 0 less than equal to R t less than equal to I am sorry again not 0, this is if 0 less if 2 A less than equal to R t less than equal to 4 a implies decision equals 3 A.

Similarly, for the negative side of course, you have this positive side similarly for the negative side again it is symmetric on the negative side. So, minus 2 A less than equal to R t less than equal to 0 implies decision equal to minus A and so on and so forth. So, one can proceed by symmetry I think this is the complete. So, there are going to be M decision rules naturally for the M constellation points it is going to be a combination of M cases let us say it is not say there is only one decision rule, it is going to be a combination of M cases, combination of M cases, one for each point similar to what you have in programming any programming language, where you have a case and depending on a case there is an outcome. So, you have different cases that are depending on the boundaries of the box in which R t lies one has to choose the appropriate value for a naught.

Now obviously, another interesting aspect you can see here also here is that you can also call this as a nearest neighbor criterion. Because if R lies in 0 to 2 A then the nearest neighbor is A that is it is closer to A than it is closer to. So, this can also be called. So, if R lies. So, one thing you can observe here is that in this region closest neighbor is A. So, this decision rule is basically you are taking R t looking at what is the closest neighbor and assigning it to the closest neighbor.

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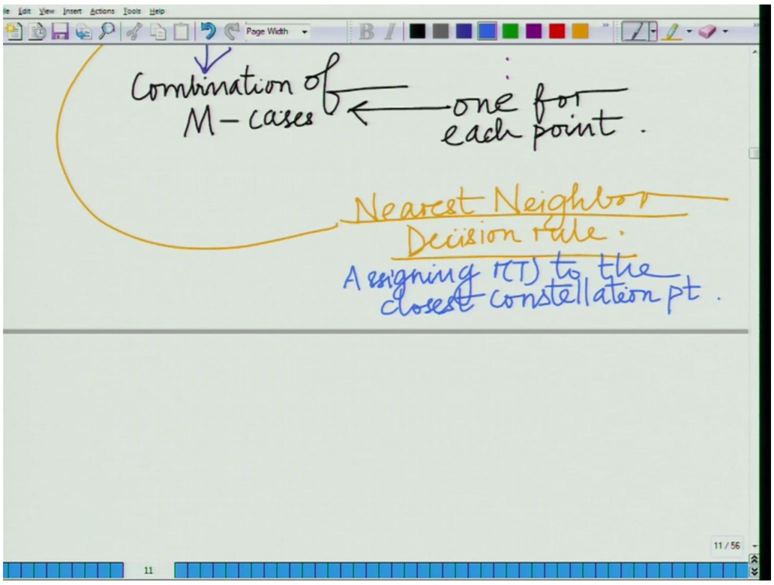

So, this can also be called as the nearest neighbor decision rule. This is a very important criteria nearest neighbor decision that is we are assigning R t to the closest constellation point you can check this. So, you are assigning R t to the closest constellation point.

So, that takes care of the interior points. Now what happens to the other points the n points because they are special, because see each interior point has 2 points one on each side the n points that is if we can look at the corresponding to i equals M by 2 minus 1 and i equals minus M by 2 or that is corresponding to i equal to M by 2 minus 1 these 2 points that is 2 i plus 1 minus of 2 i plus 1.

So, these 2 points have only one neighbor of course, the point on the rightmost has one neighbor on the left point and leftmost has one neighbor on the right so the analysis. So, the decision rule with respect to these 2 points is going to be slightly different, which we look at the next module and also the corresponding analysis of the probability of error.

Thank you.