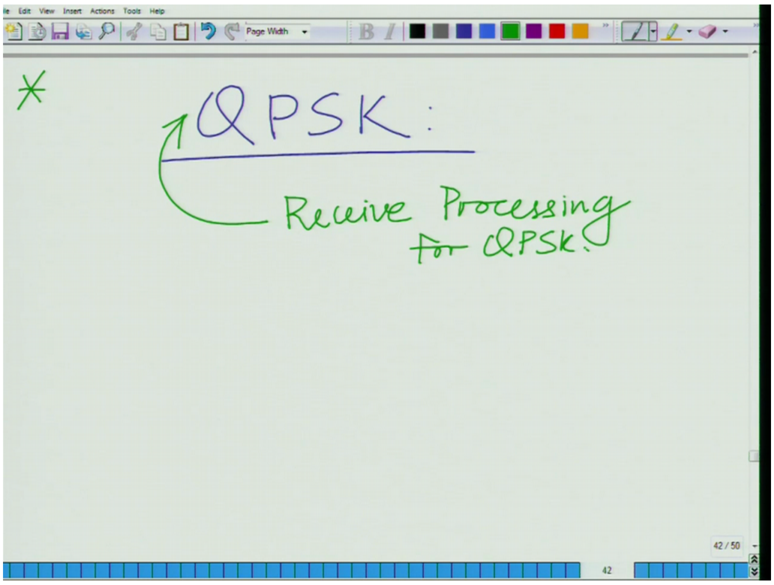
Principles of Communication Systems – Part II
Prof. Aditya K. Jagannatham
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

Lecture - 19
Matched Filtering, Bit-Error Rate and Symbol Error Rate for Quadrature Phase Shift Keying (QPSK)

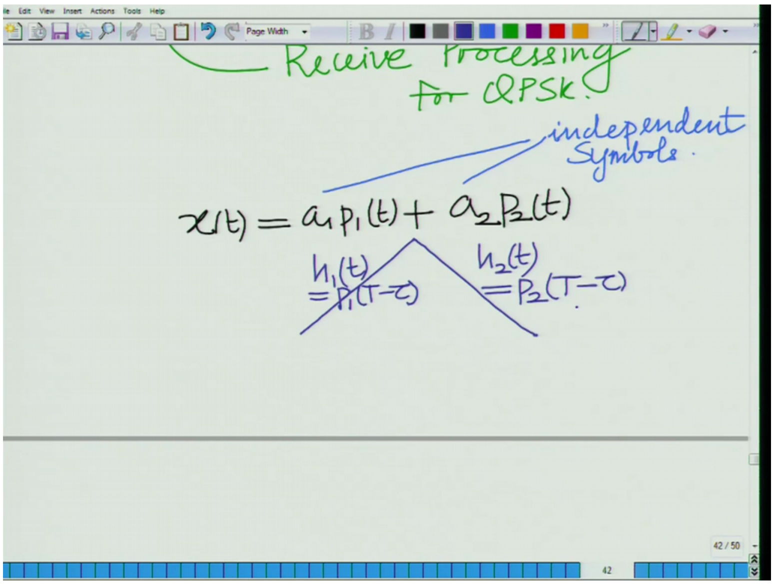
Hello. Welcome to another module in this massive open online course. So, we are looking at quadrature phase shift keying and we have looked at several aspects first how to construct the modulated signal, what are the different pulse wave forms in quadrature shift keying correct, what is the property that they satisfy that is these different waveforms are phase shifted from each other by pi by 2. Now let us look at the detection the processing at the receiver and the resulting performance in terms of it is bit error rate.

(Refer Slide Time: 00:45)



So, we want to look at quadrature we will continue our discussion on QPSK that is quadrature phase shift keying we want to look at the receive processing or the processing at the receiver for QPSK and what we have seen is that for QPSK.

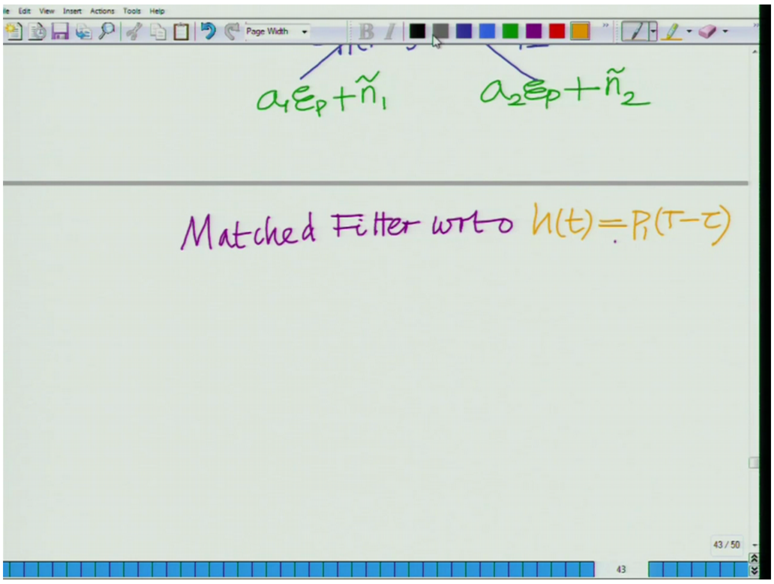
(Refer Slide Time: 01:13)



We have x t equals A 1 P 1 t plus A 2 P 2 t A 1 and A 2 these are independent symbols that is the whole premise of QPSK these are independent symbols. So, what we will do is we can match filter this independently with respect to P 1 and P 2. So, first we will match filter with respect to h 1 t equals P 1 T minus tau and also matched filter with h 2 t equals P 2 t minus tau.

Now, this will give us when we match filter with P 1 the component with respect to P 2 that is pulse P 2 will go away.

(Refer Slide Time: 02:17)

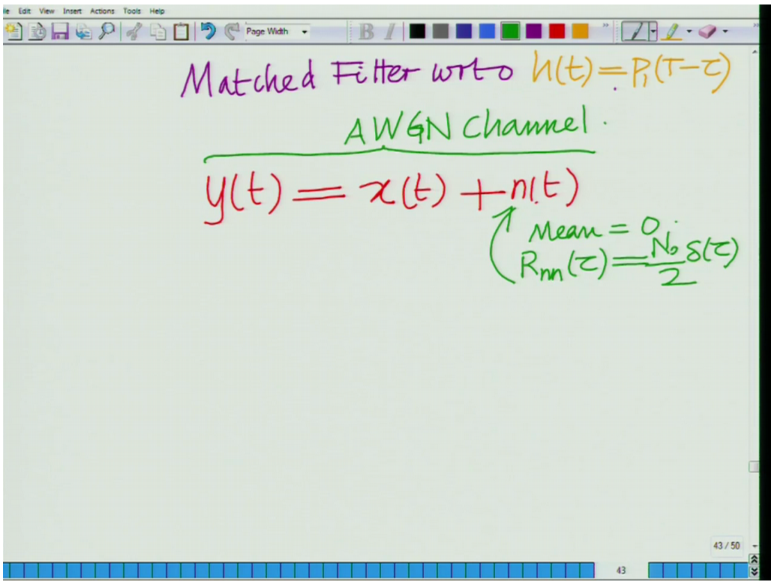


So, this will give us A 1 times E P where E P is the pulse energy which is equal to 1 plus noise and tilde and this will give us A 2 times E P. So, when you match filter with respect to P 2 the component with respect to component along the signal space along the signal space direction P 1 T, that is component along pulse P 1 T will vanish and we get A 2 times E p, A 2 times that is the A 2 is a component along the signal space function P 2 or the signal space pulse P 2 T E P is 1 plus the noise into tilde.

For instance, let us take. So, what we are saying is that basically we match filter with respect to both. So, first we match filter with respect to P 1 that is the signal space basis signal P 1 T. To make a decision with respect to A 1 alright and make a decision with respect to the transmitted signal A 2 we match filter with respect to P 2 that is a pulse P 2 T which is another orthogonal basis function for the signal space.

Now, let us look at how this operates. So, matched filter with respect to. So, when we match filter for instance. So, let us say we match filter with respect to match filter with respect to h t equals P 1 T minus tau.

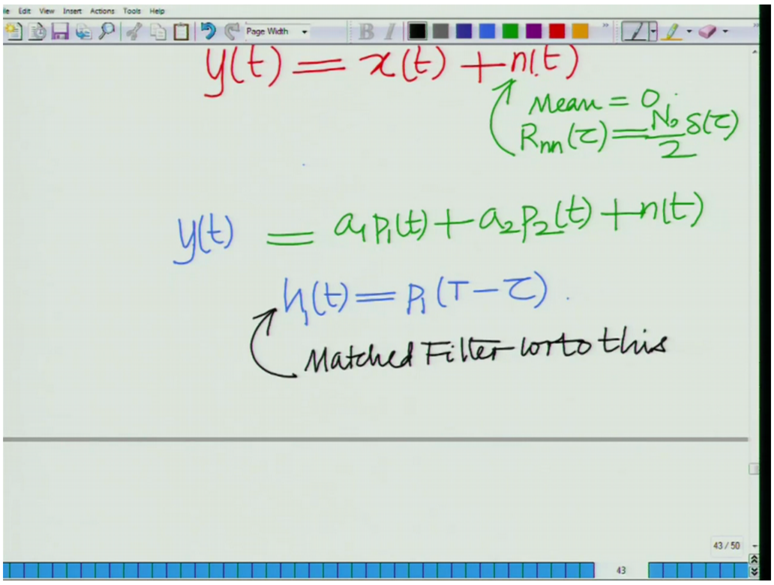
(Refer Slide Time: 04:11)



So, what we perform is we perform convolution that is A 1. So, first of all what we have over here is basically the received signal y t, equals x t plus n t. This is the additive white Gaussian noise channel, this is the model for AWGN channel additive white Gaussian noise channel that goes without saying we have seen this many times before, n t is

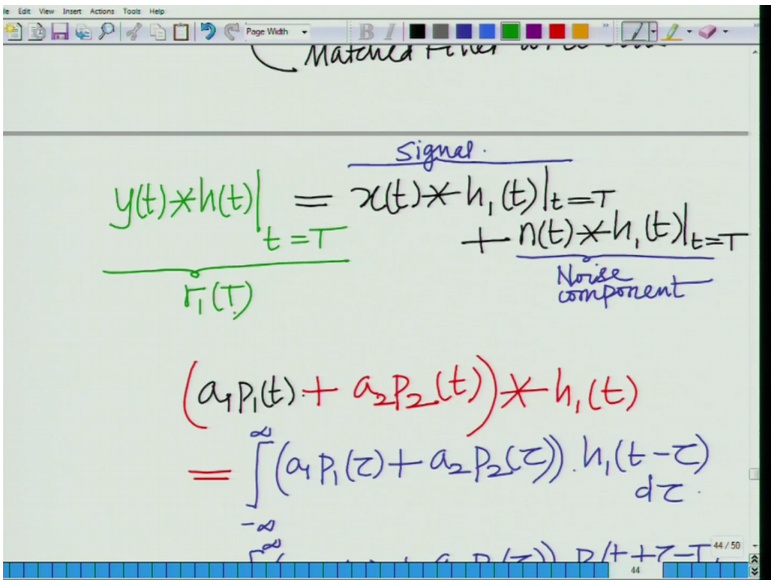
Gaussian noise with mean 0 auto correlation function N naught by 2 delta tau the mean is equal to 0.

(Refer Slide Time: 05:02)



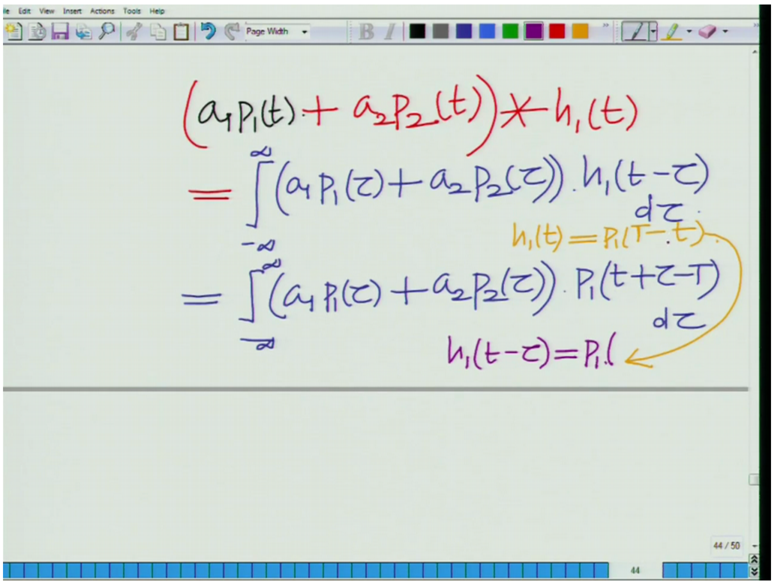
Now, x t, now let us also now expand x t, x t is A 1, P 1 t plus A 2 P 2 t, plus n t. Now when you match filter this with respect to h t, equals P 1 t minus t. So, match filter or let us say match filter with respect to h 1 t, equals P 1 T minus t.

(Refer Slide Time: 05:52)



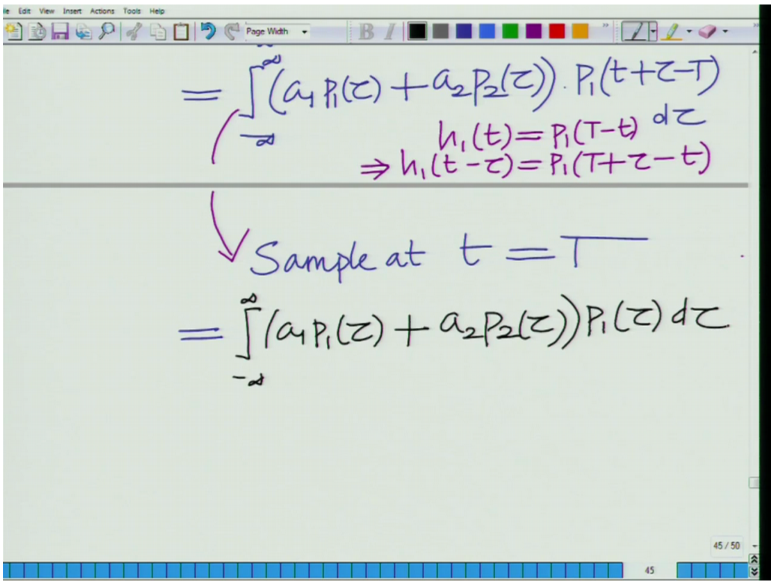
So, we match filter with respect to this. And what we get is basically we have y t equals x t, h 1 t plus n t convolved with h 1 t. Of course, sampled at t equal to t, sampled at t equal to t this is the signal part we have seen this several times before this is the noise component now the signal component is given as follows. So, signal component is well A 1 P 1 t. So, convolution A 1 P 1 t plus A 2 P 2 t, when you convolve with h 1 t that is basically integral minus infinity to infinity A 1 P 1 tau plus A 2 P 2 tau times h 1 t minus tau d tau, but h 1 t is P of t minus tau.

(Refer Slide Time: 07:24)



So, this will be equal to A 1 P 1 tau plus A 2 P 2 tau well times P 1 t plus tau minus t d tau because note here h 1 tau or h 1 t, because we have to note here h 1 t equals P 1 t minus t. So, this implies h 1 t minus h 1 t minus tau equals P 1 of well P 1 of t plus tau minus t.

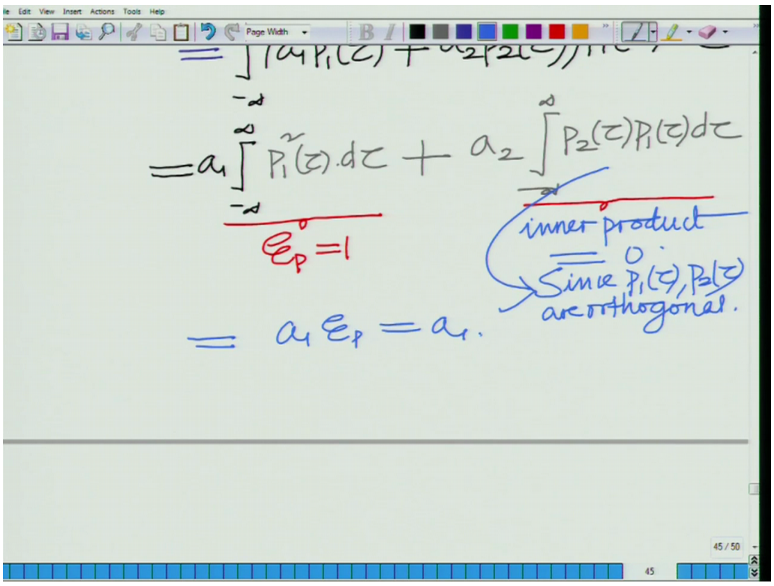
(Refer Slide Time: 08:49)



So, this arises basically because if you look at this, h 1 of t equals P 1 of t minus t implies h 1 of implies which implies h 1 of h 1 of t minus tau equals P 1 of t plus tau minus P of P plus tau minus t. So, this is P of P 1 of t plus tau minus t.

Now, we sample this at t equal to capital T sample at t equal to capital T which means I substitute t equal to capital T and what we have is A 1 P 1 tau plus A 2 P 2 tau into well P 1 tau d tau.

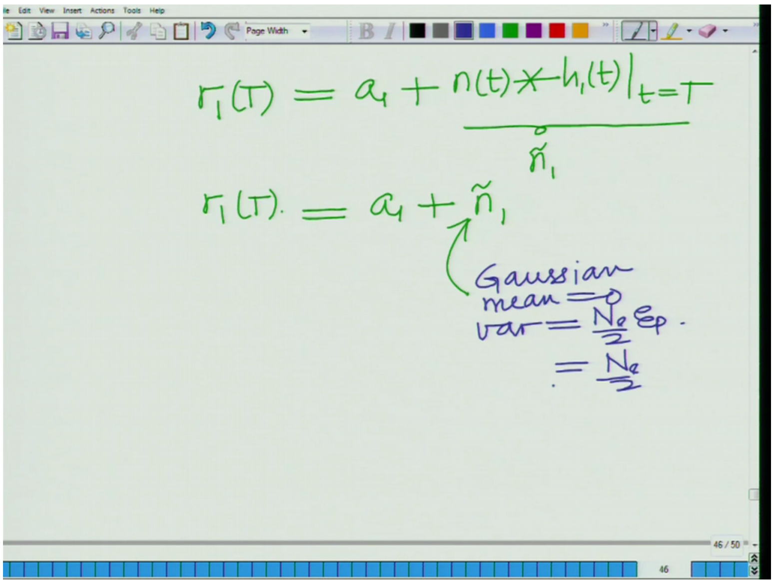
(Refer Slide Time: 10:09)



And now if you look at this I can split this into 2 minus infinity to infinity, A 1 P 1 tau into P 1 tau that is P 1 square tau d tau plus, A 2 minus infinity to infinity P 2 tau d tau. Now, if you look at this integral minus infinity P 1 square d tau this is nothing but E P which is equal to 1. And this you can see is the inner product of P 1, P 2 which is equal to 0 because P 1 is orthogonal to P 2 and we have seen this many times before.

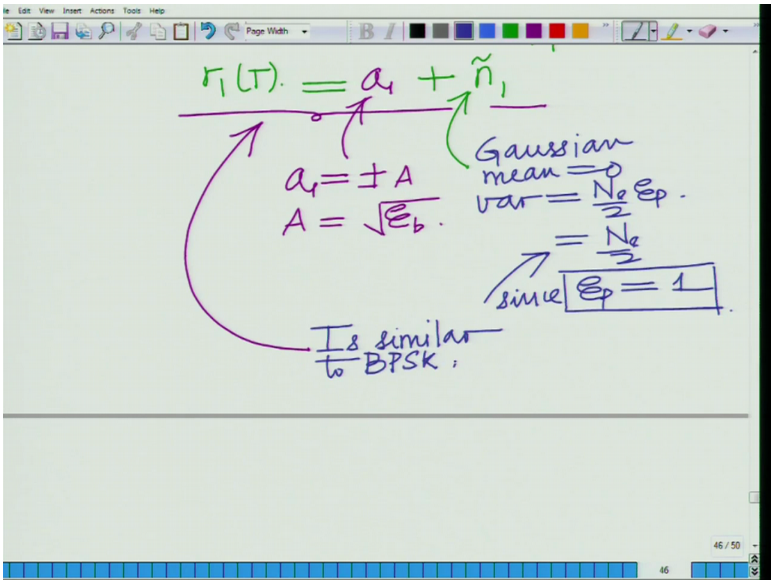
Since P 1 tau comma P 2 tau are orthogonal. So, this is equal to A 1 times E P equals A 1. So, basically what we have is when we match filter with h 1 when we match filter with h 1 t and sample at t equal to t. So, this is basically you are matched filtering which is basically y t with h t and sampled at t equal to t. So, the signal component that is this we can write this as this is nothing but remember this is nothing but we are denoting this part by r 1 t. So, we can write that as output after sampling.

(Refer Slide Time: 12:10)



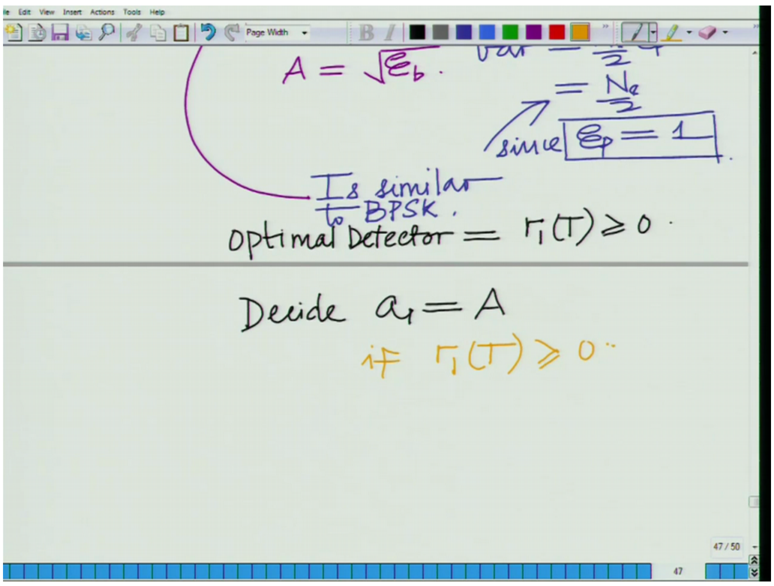
So, r 1 t, A 1 plus of course, n t convolved with n t convolved with h 1 t, at t equal to t, which we are calling as n 1 tilde. And we know after matched filtering we know A 1 plus n 1 tilde. And we know that from matched filtering we know that n 1 tilde is Gaussian. This is the input process n 1 t tilde is Gaussian mean equal to 0 variance N naught by 2 E P which is equal to N naught by 2 since E P is equal to 1.

(Refer Slide Time: 13:11)



Since, the normalized pulse energy since the pulses are unit norm since E P equals 1. And therefore, again now remember A 1 equals plus or minus A where A equals square root of E b. So, A 1 equals A. So, this is similar to BPSK. So, after matched filtering by h 1 this becomes similar to BPSK not saying that QPSK is similar to BPSK, but after matched filtering by h 1 t which is equal to P 1 capital T minus tau capital T minus t where capital T is the duration of the sampling instant this is the output sample r 1 r 1 t is basically it is basically similar to what we have for binary phase shift keying that is what on this branch after match filtering with h 1 t is similar to BPSK.

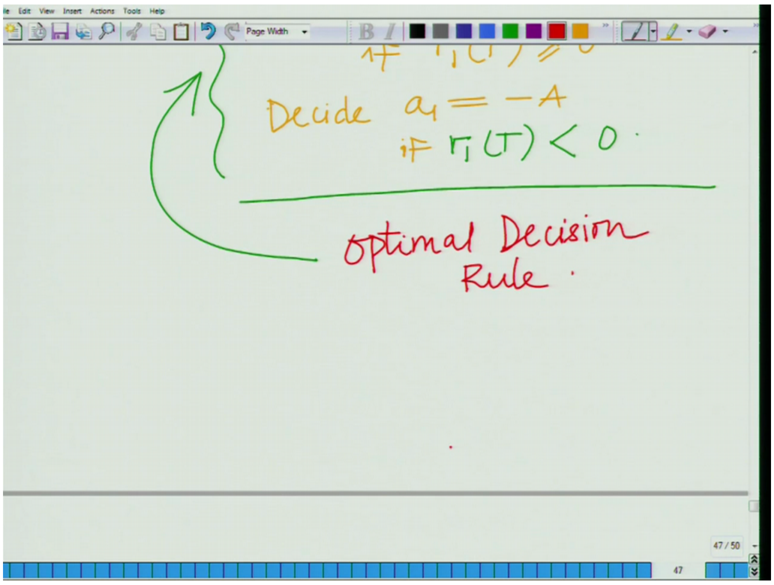
(Refer Slide Time: 14:36)



And therefore, the optimal detector is again the simple threshold based detector is basically r 1 t greater than equal to 0.

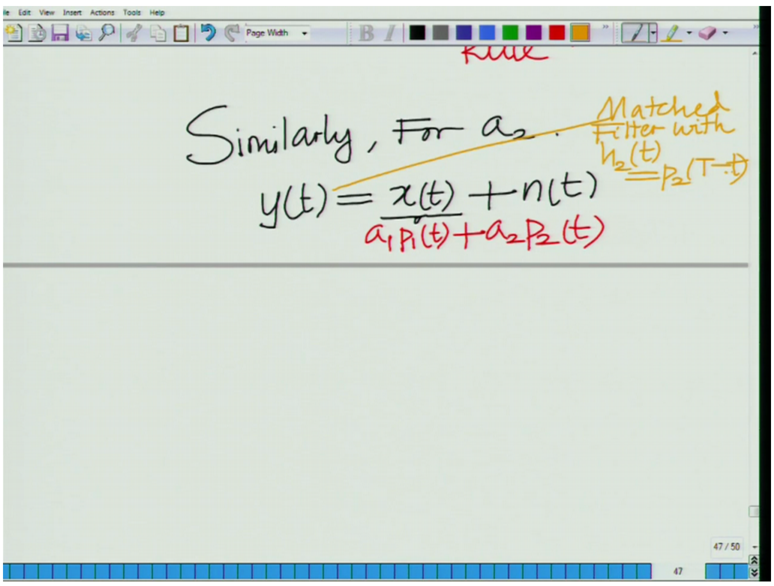
So, decide A 1 equals A 1 equals 0 or A 1 equals decide A 1 equals well basically A 1 equals the transmitted symbol A 1 equals A if r 1 t greater than or equal to 0.

(Refer Slide Time: 15:15)



And decide A 1 equals minus A, corresponding to bit 0, if r 1 t less than 0. So, this is your optimal decision rule optimal detector or optimal decision rule optimal detector or optimal decision rule.

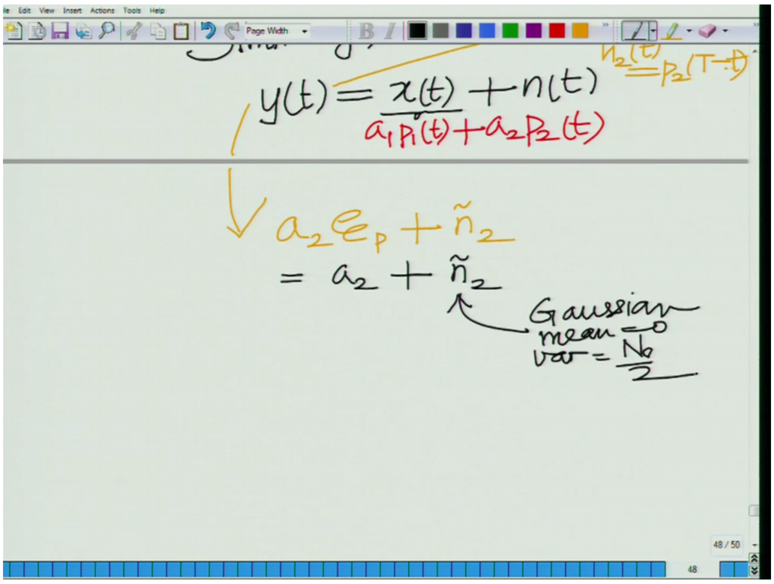
(Refer Slide Time: 15:57)



And therefore, now similarly for A 2 correct we have y t equals x t plus n t, where this x t is basically your A 1 P 1 t plus A 2 P 2 t.

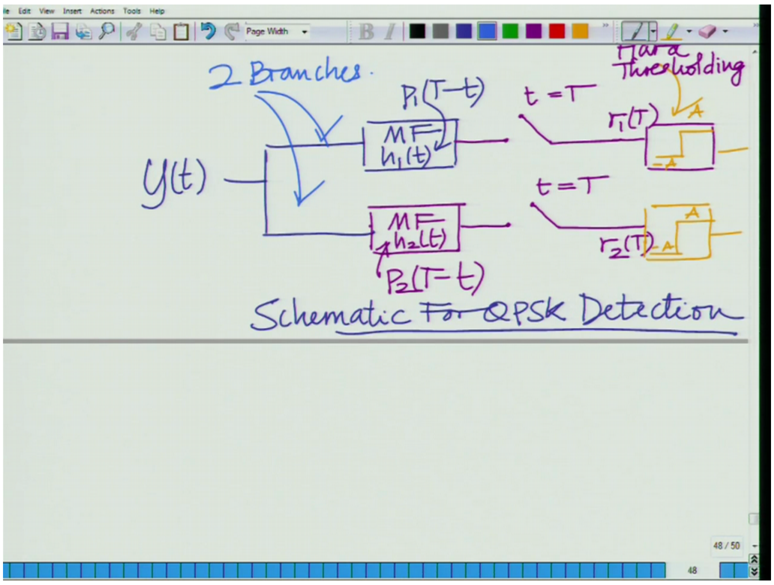
Now, this you matched filter with h 2, t equals P 2 t minus t you match filter with this.

(Refer Slide Time: 16:59)



And that will give you naturally that gives you now when you match filter with h 2 which is equal to P 2 which is matched to pulse P 2 the component corresponding to P 1 will be 0. So, this will give you A 2 times E P plus n 2 tilde E P is one which is equal to A 2 plus n 2 tilde, n 2 tilde is Gaussian mean equal to 0 variance equals N naught by 2 E p, but E P is 1.

(Refer Slide Time: 17:40)

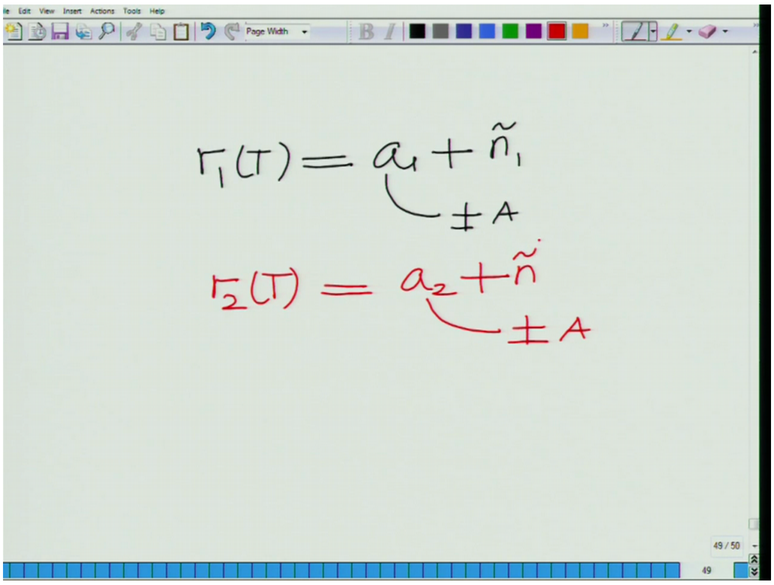


So, basically if you have to represent this schematically, what we have? We have this incoming signal not x t, we have this we have this y t correct we have y t. So, basically we can now represent this as 2 separate branches, 1 for each matched filter. So, we can have 2 branches 1 for each matched filter. So, matched filter with. So, you match filter with h 1 t h 1 t equals P 1 t minus t, the second branch you match filter with h 2 t correct this is equal to P 2 t minus t, followed by sample at t equal t, you sample this second branch also at t equal to t, and therefore, you get here you get well you get r 1, t you get r 2 t and this you basically apply to a thresholding device.

So, basically what this does is if it is greater than 0 it decides plus a less than 0 decides minus A. So, this is basically a hard thresholding operation. So, this is basically hard thresholding this is optimal decision rule. So, this is basically for the schematic of QPSK detection. So, let me just write this appropriately. This is your schematic for QPSK and these are 2 parallel branches one for h 1 t. So, this has 2 branches unlike previous modulation schemes like for instance amplitude shift keying or binary phase shift keying where your are match filtering with only one matched filter, here because you are transmitting 2 different bit is on 2 different basis functions orthogonal basis functions orthonormal basis functions in of the signal space in right.

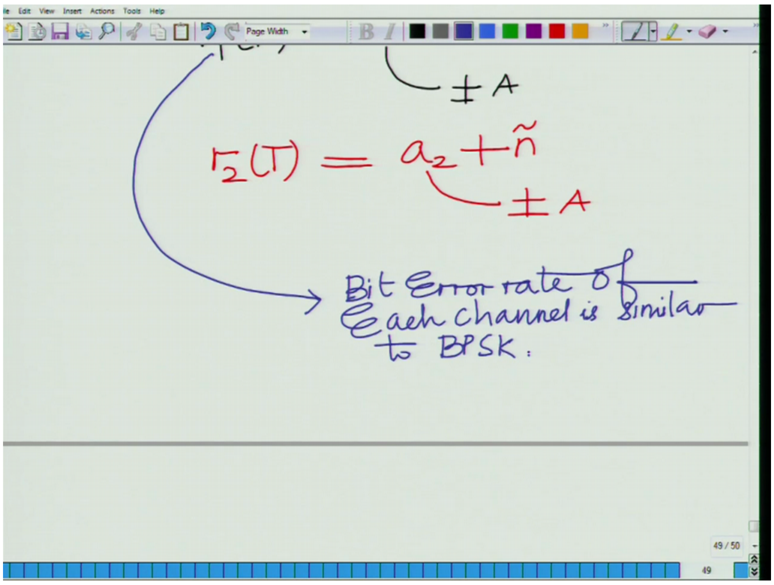
So, we match filter separately one branch performing match filter with respect to h 1 t another branch performing match filter with respect to h 2 t. So, h 1 is matched to P 1 t the pulse shift P 1 h 2 t is matched to the pulse shift P 2. And naturally what we get is we have after match filtering every channel is similar each channel is similar to BPSK.

(Refer Slide Time: 21:00)



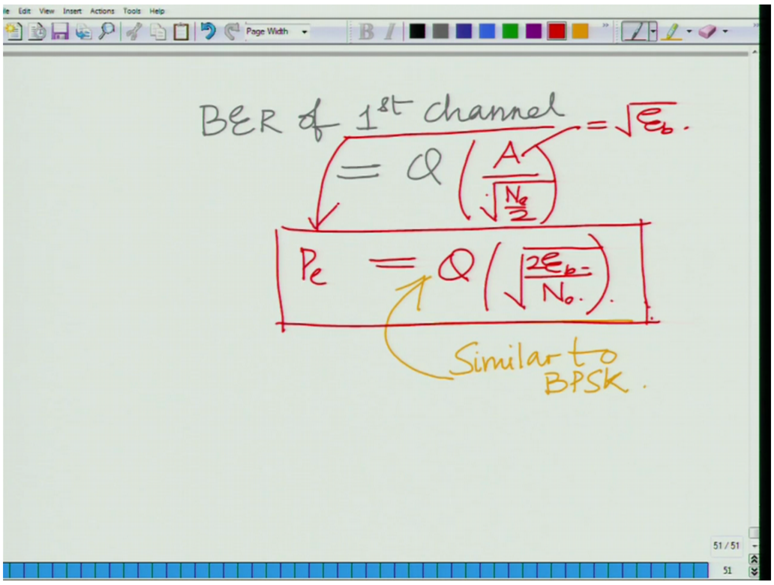
So, this is A 1 plus n tilde where A 1 is plus or minus A r 2 t equals A 2 plus n tilde where this is equal to plus or minus A.

(Refer Slide Time: 21:34)



And therefore, the bit error rate will also be similar to BPSK, bit error rate for this bit error rate on each channel, of each channel note that this is not the overall bit error rate bit error rate of each channel.

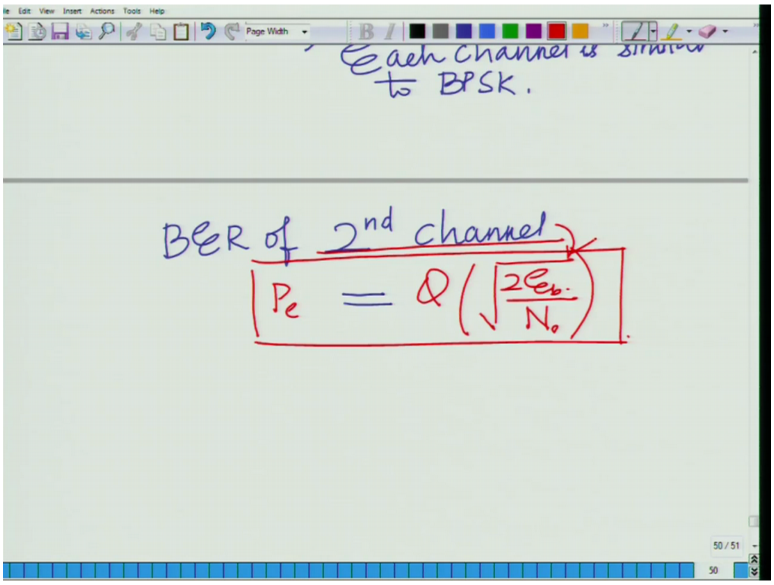
(Refer Slide Time: 22:08)



So, b E r of channel 1, Q well A divided by sigma A divided by sigma or A divided by well sigma is square root of eta naught divided by 2 and remember A equals square root of E b, which is equal to Q square root of E b divided by 2 E b divided by n. So, this is similar to corresponding to bit energy of E b, average bit energy of E b on transmitted along pulse P 1 correct and after match filtering with respect to h 1 matched to P 1 alright the output of this branch the first branch is similar to basically binary phase shift keying. And therefore, bit error rate will also be similar to that of binary phase shift keying with average bit energy E b.

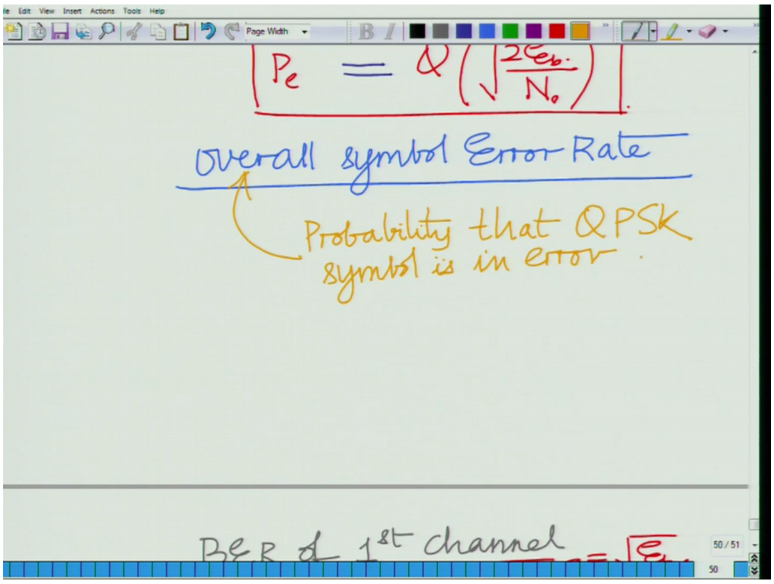
So, this is also similar to that of BPSK, now similarly BER of second channel. Now if you look at this.

(Refer Slide Time: 23:41)



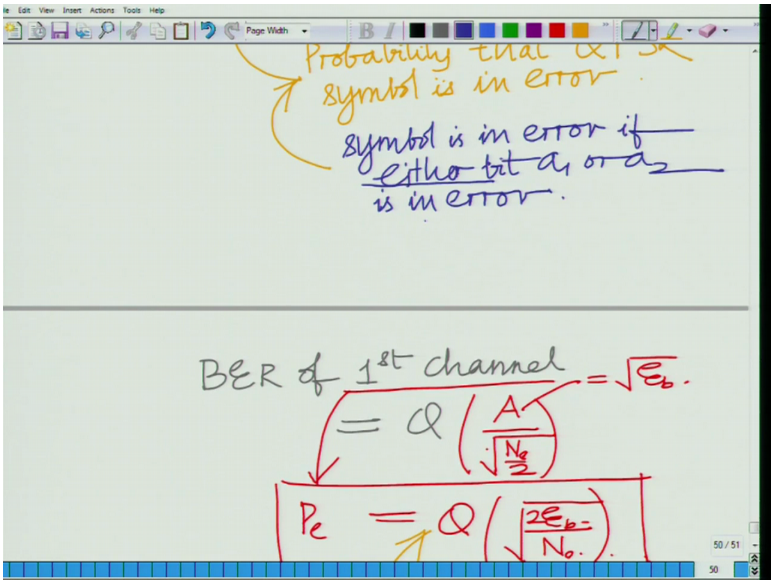
Similarly, if you look at the BER of the second channel, this is also equal to Q well this is also equal to Q square root of 2 E b over N naught. So, this is also the probability of error on the second channel this probability of error, we can say this is the probability of error on the first channel. This is for the first channel bit error rate of the first channel. This is the bit error rate of the second channel, now we come to the overall bit error rate.

(Refer Slide Time: 24:33)



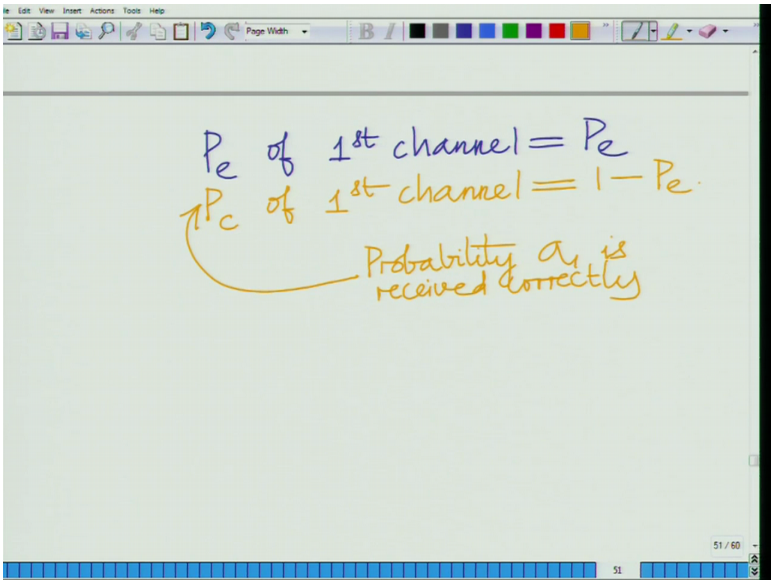
Now, we come to the overall bit error rate or let us say we come to the overall symbol error rate, that is the probability that the symbol is in error probability that the QPSK symbol is in error. Now realize that the QPSK symbol the overall QPSK symbol has 2 bit is A 1 and A 2. So therefore, the symbol is in error if either of the bit is A 1 comma A 2 are in error.

(Refer Slide Time: 25:35)



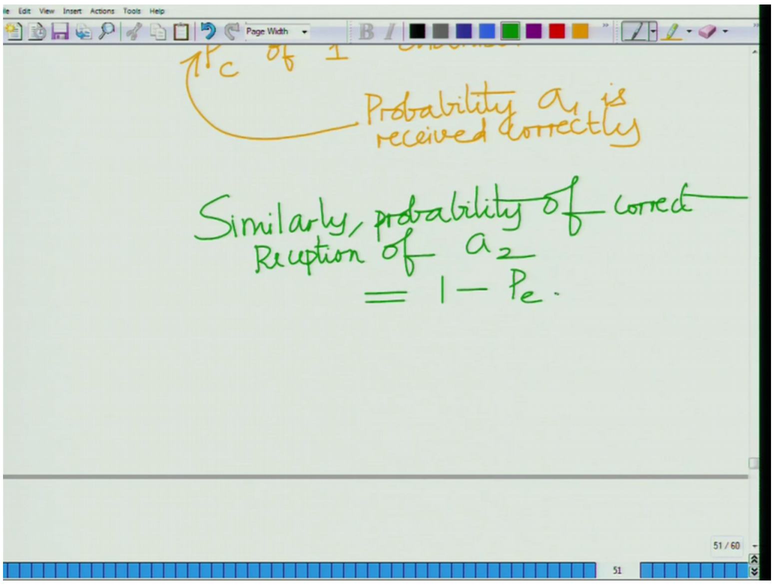
So, let us note that symbol is in error symbol, is in error, if either bit A 1 or A 2 because even if a single bit A 1 is in error then the symbol is in error, if A 2 is in error symbol is in error if A 1 and A 2 both are in error then of course, the symbol is in error. So, the symbol in error corresponds to basically the union of 2 these 3 events A 1 in error, A 2 correct A 2 error, A 1 correct A 1 and A 2 both in error that is the point.

(Refer Slide Time: 26:37)



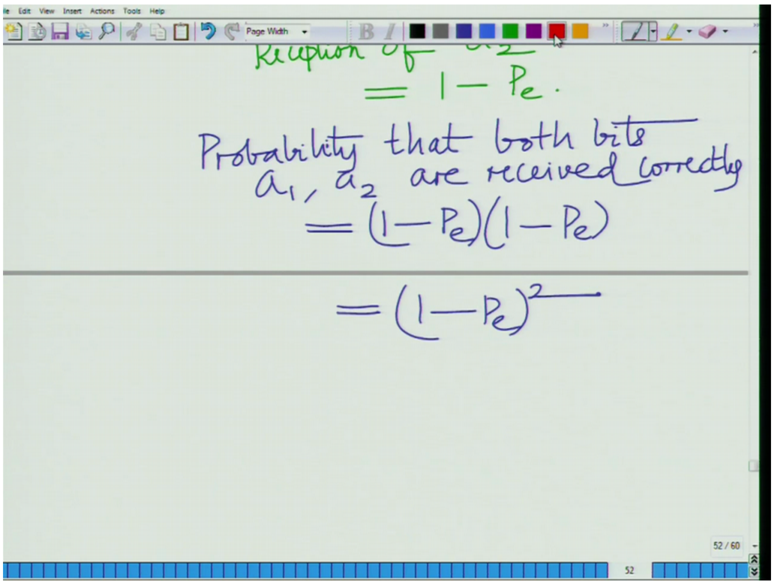
Now, therefore, if you look at this the overall symbol error can be calculated as follows. The overall symbol error can be calculated as follows. Probability that probability of error, of first channel equals P e. Therefore, probability correct of first channel first channel means probability that A 1 that is probability that is received correctly that is without error this is equal to 1 minus P because probability of error is P probability of correct reception is 1 minus P.

(Refer Slide Time: 27:25)



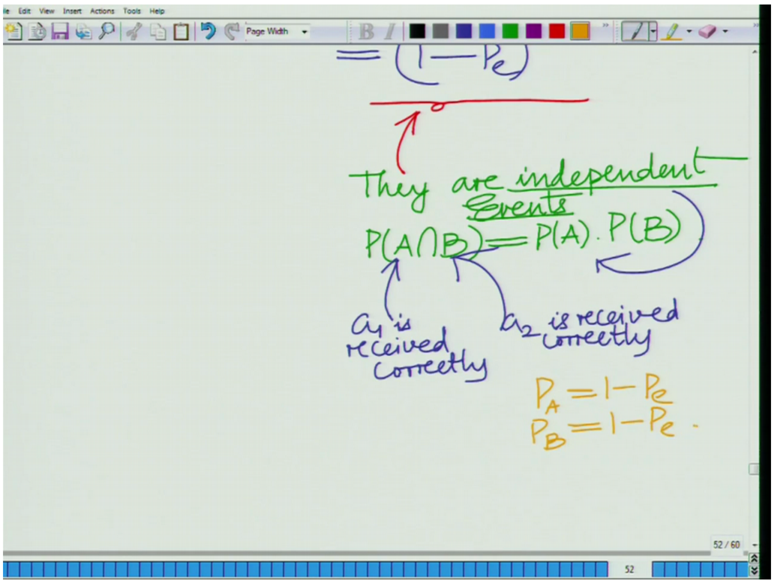
Similarly, probability of correct reception equals 1 minus P e, now probability that both bit is are received correctly.

(Refer Slide Time: 27:54)



Here we are going to use the results from probability theory probability bit is A 1 comma A 2 are received correctly equals 1 minus P e, into 1 minus P e this equal to 1 minus P e square.

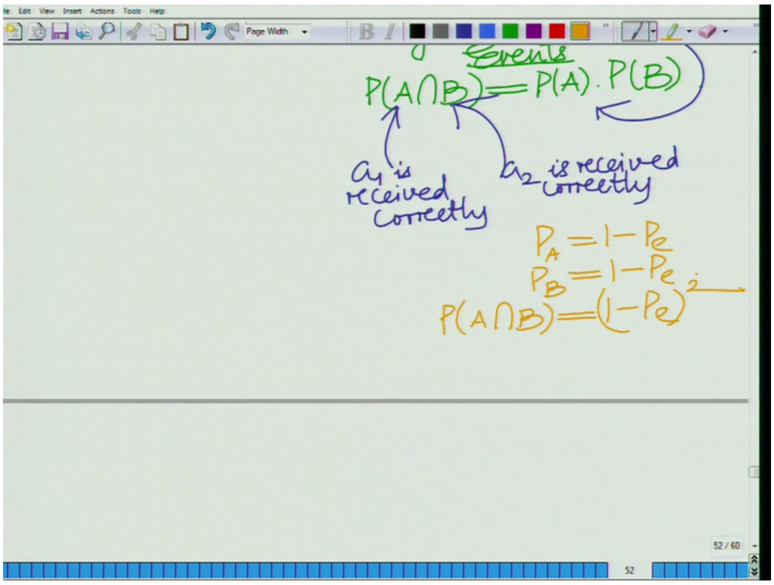
(Refer Slide Time: 28:41)



Where, we are using the fact that this is based on the fact that they are independent events. They are independent events and for independent events, A comma B probability of A intersection B is probability of A times probability of B for only for independent events we are assuming that these 2 events error events are independent.

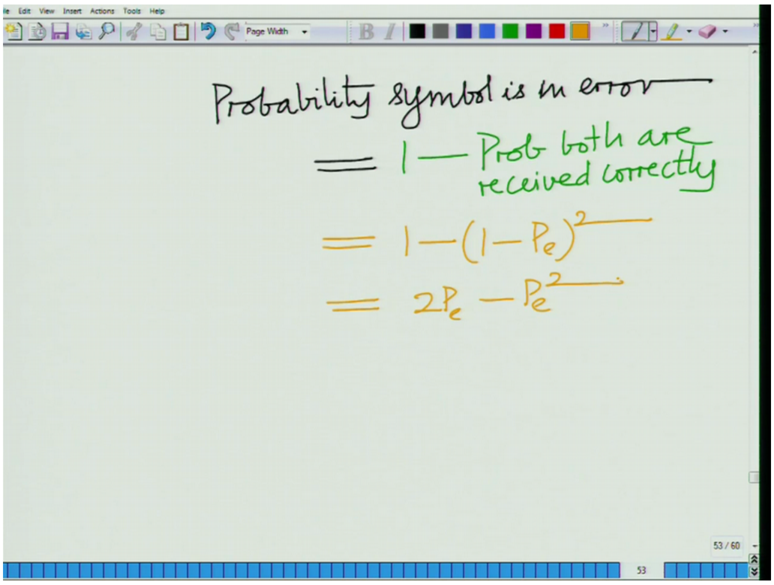
Therefore, this is let us say - this is a event A that A 1 is received correctly and this is event b that A 2 is received correctly. So, P A equals. So, we have P A equals 1 minus P e P b equals 1 minus P e.

(Refer Slide Time: 29:54)



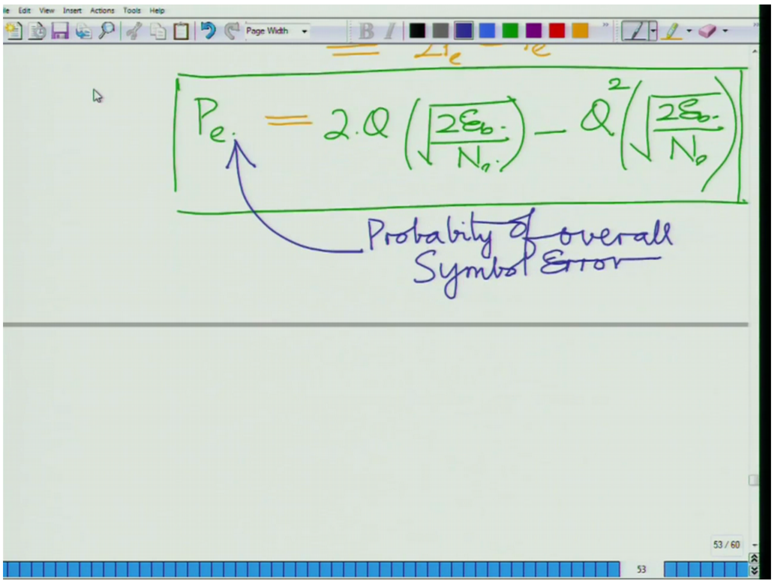
So, probability of both received correctly is P A intersection B equals 1 minus P e square.

(Refer Slide Time: 30:05)



And therefore, probability that symbol is in error equals 1 minus the probability 1 minus the probability, both are received correctly that is 1 minus 1 minus P square which is equal to basically 2 P e minus P e square.

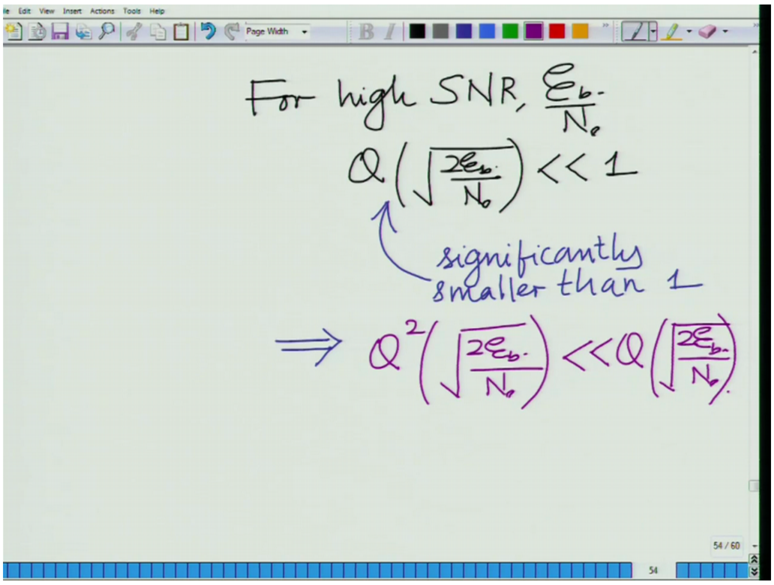
(Refer Slide Time: 30:53)



Which is equal to we know what is P e which is equal to 2 Q square root of 2 E b divided by N o minus Q square root of 2 E b divided by N o this is the probability of the error for the probability of overall this is important. This is probability of overall symbol error this is the probability of overall symbol error.

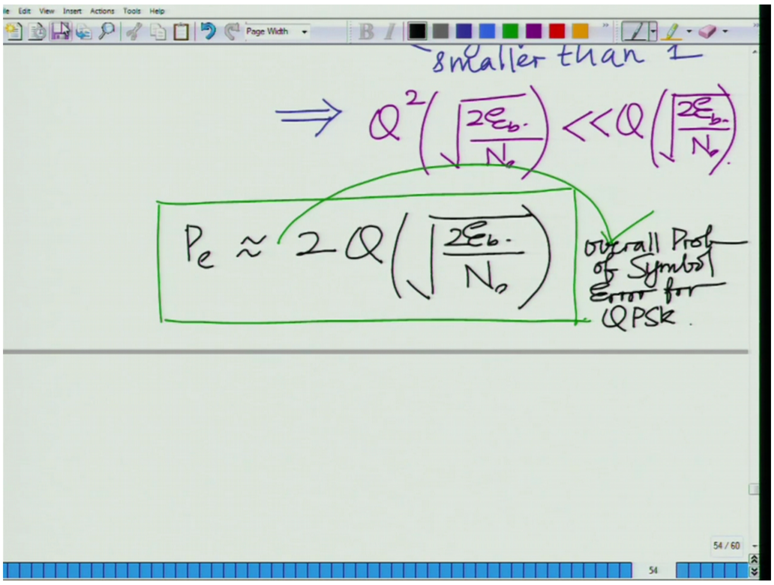
So, the probability of overall symbol error is 1 minus 1 minus P e square that is 2 P e minus P e square where P e is Q square root of 2 E b over N naught.

(Refer Slide Time: 31:54)



And now you can see for high SNR SNR that is E b over N o Q square root of 2 E b over N o is much smaller than 1. That is this is significantly smaller than smaller significantly smaller than, this implies that your Q square root of 2 E b over N o is significantly smaller than q, because this quantity is less than 1. So, square Q square of 2 E b over N o is significantly smaller than Q of 2 E b or Q of square root of Q square of square root of N o significantly smaller than Q of square root of 2 E b over N o.

(Refer Slide Time: 33:10)



Which means the probability overall probability of error can be approximated by ignoring the square term the overall probability of error can be approximated as 2 square root of 2 E b over No. This is the overall probability of symbol, a valid approximation for overall probability of symbol error. So, this is an approximation for the overall probability of symbol error QPSK.

So, basically what we have seen in this module and in continuation of the previous modules is we have thoroughly examined this different new modulation scheme which is QPSK. And it is very interesting because remember now we have used the concepts of signal space. So, we are using 2 orthonormal basis functions that is we are using A 2 dimensional signal space with orthonormal basis functions P 1 t and P 2 t. And we have seen that we can transmit 1 bit along P 1 t that is by using pulse P 1 t and another bit independent of that along P 2 t that is along the pulse P 2 t.

And both these bit is can be recovered at the receiver one first one by matched filtering using the filtered match pulse shift P 1 t other filter matched to the pulse shift P 2 t alright. The decision rules on both these branches are independent alright the probability of bit error on each of these branches is Q square root of 2 E b over N naught similar to that of BPSK binary phase shift keying and the overall probability of the symbol error is 1 minus 1 minus P e square which we have well approximated as twice Q of square root of 2 E b over N naught which is basically twice that of the bit error rate of a BPSK.

So, basically roughly this means there are 2 bit is each bit can be in error, bit error rate is twice roughly twice that of BPSK although it is not exactly true because P e of because using the prob using the results from probability theory, we know that probability of A union B is probability of A plus probability of B minus probability of A intersection B. In fact, that is what the square term represents, but since the square term is much smaller we can neglect it at high SNR, and we can simply write it as twice Q of square root of 2 E b over N naught. That is the overall probability of symbol error for this QPSK modulation scheme with average bit energy E b alright.

So, we will stop here and explore other schemes in subsequent modules.

Thank you very much.