An Introduction to Coding Theory
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Module 01
Lecture Number 06
Problem Solving Session-I

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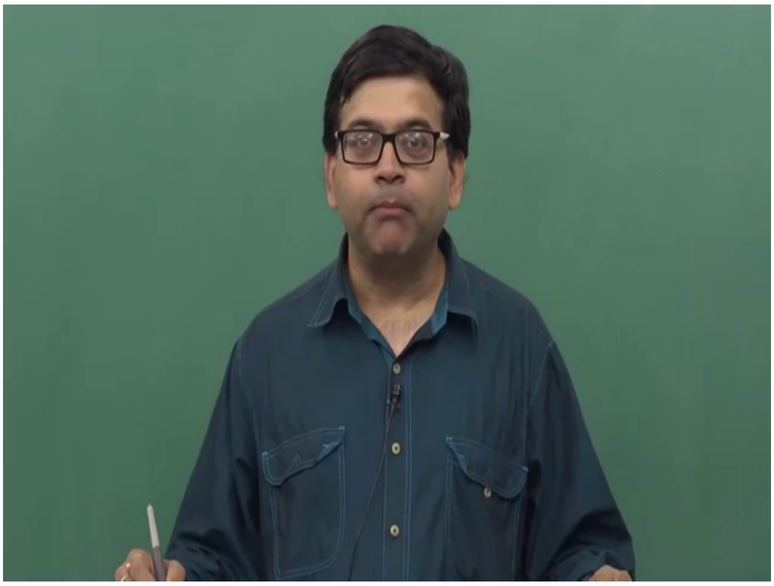

Before we discuss decoding of linear block codes let us solve some problems today.

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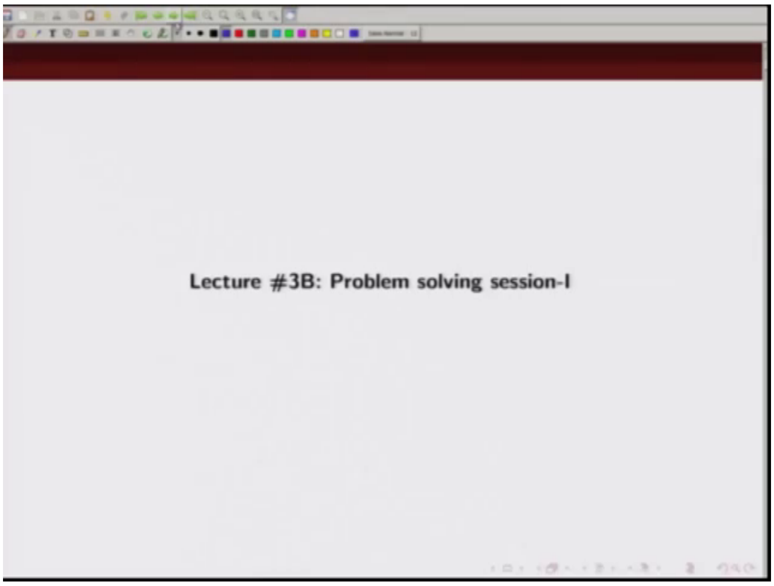

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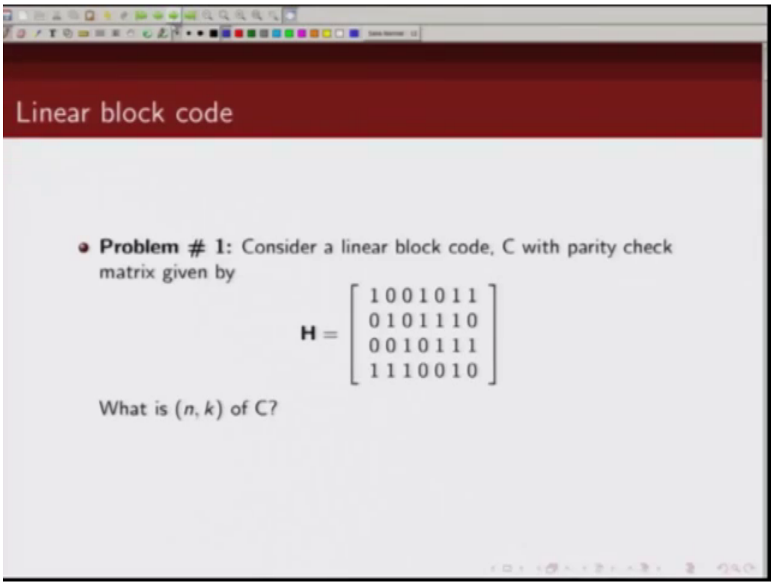

So first question we are going to look at is consider a linear block code C whose

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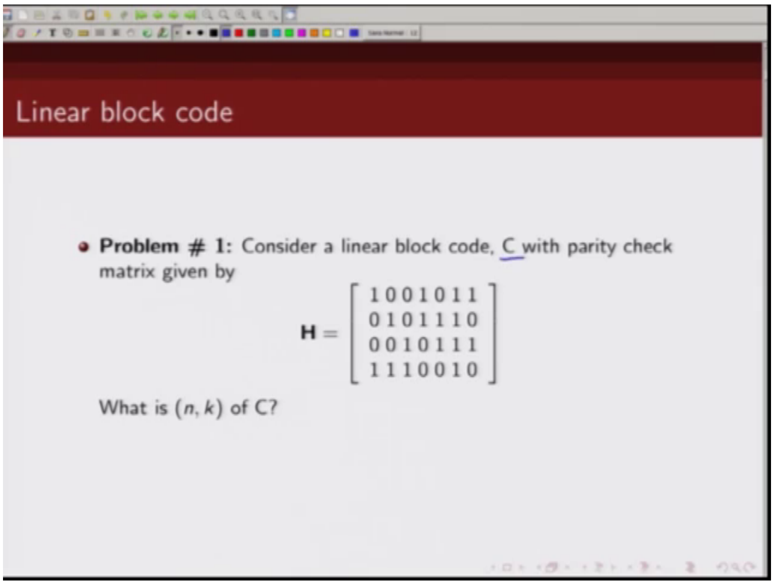

parity check matrix is given by this.

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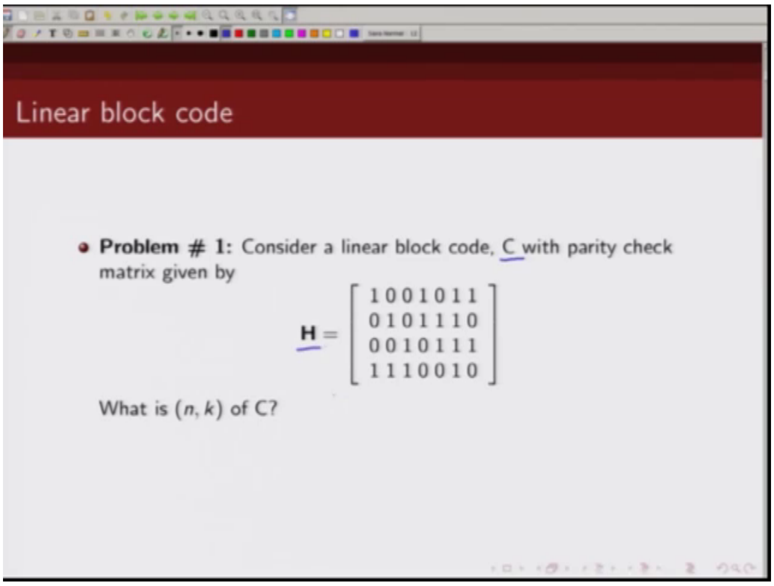

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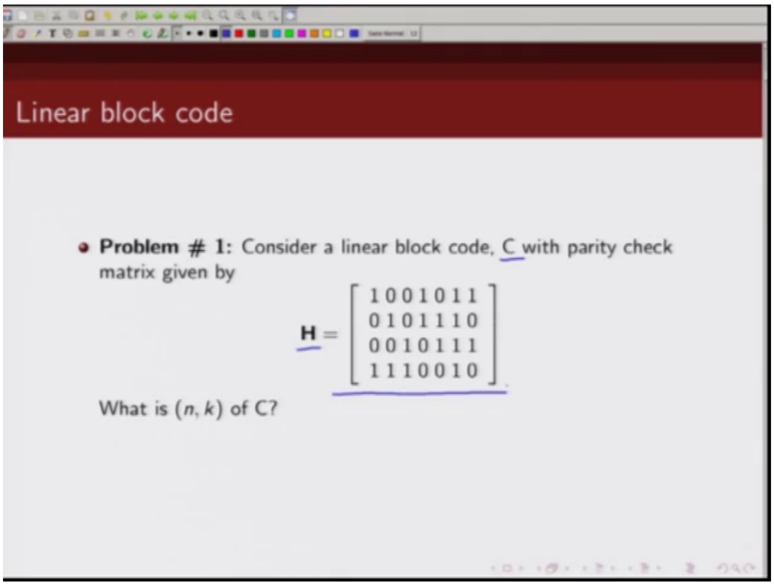

And you are asked what are the code parameters, n and k;

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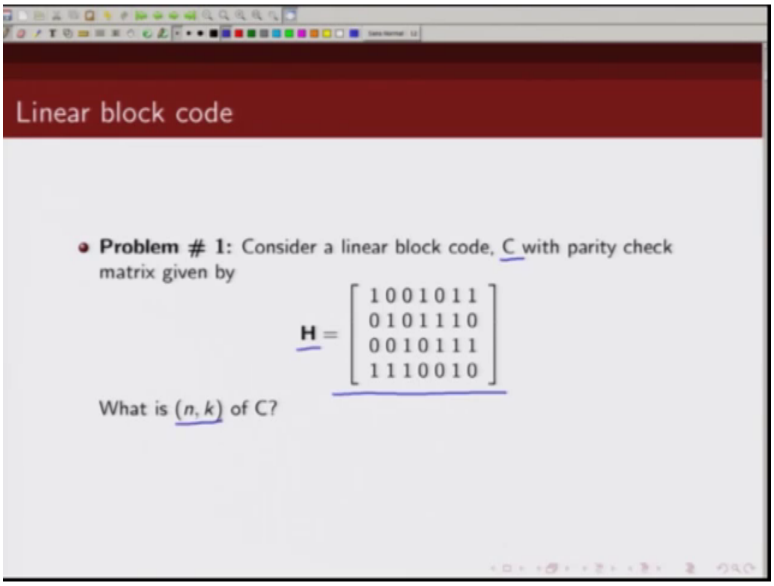

n which is a block length, codeword length and k is the size, it's the dimension of the, basically information sequence length s k. Now how do we solve it? We know, we will first find out what is the rank of this matrix H. Now you can see this is a 4 cross 7 matrix right? So the maximum rank

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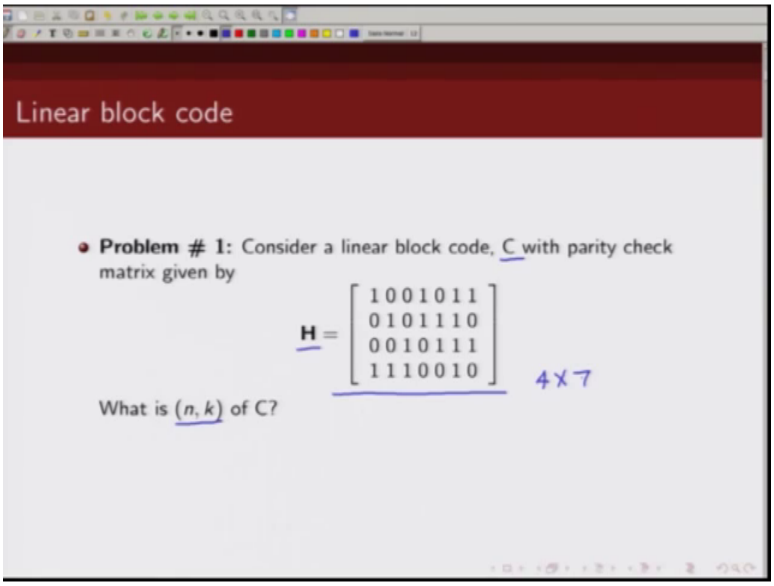

possible is 4. Let's see whether it has rank 4. Now if you add row 1, 2 and 3 what do you get? 1 1 1 1 0 1 0, sorry 1 1 1 0

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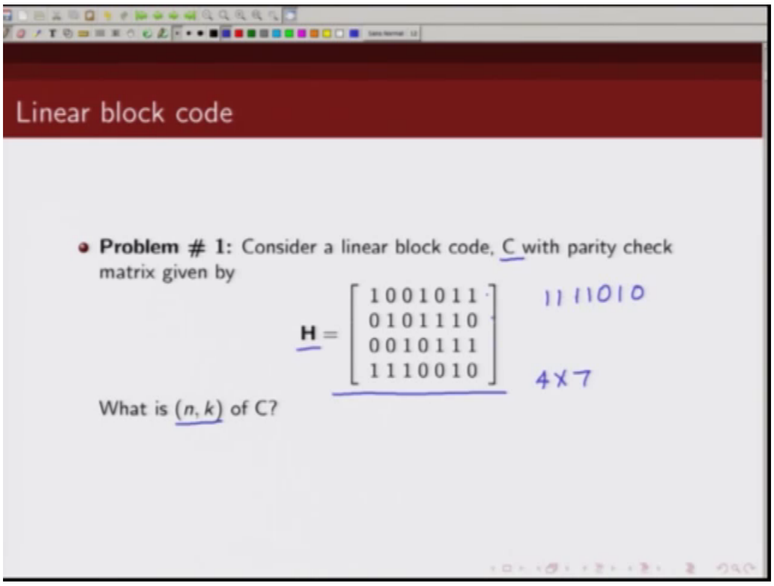

this is 0, 0 1 0. This is what you get, you can see this is 1, this is 1, this is 1, this is 0, this is 0 this is 1and this is 0.

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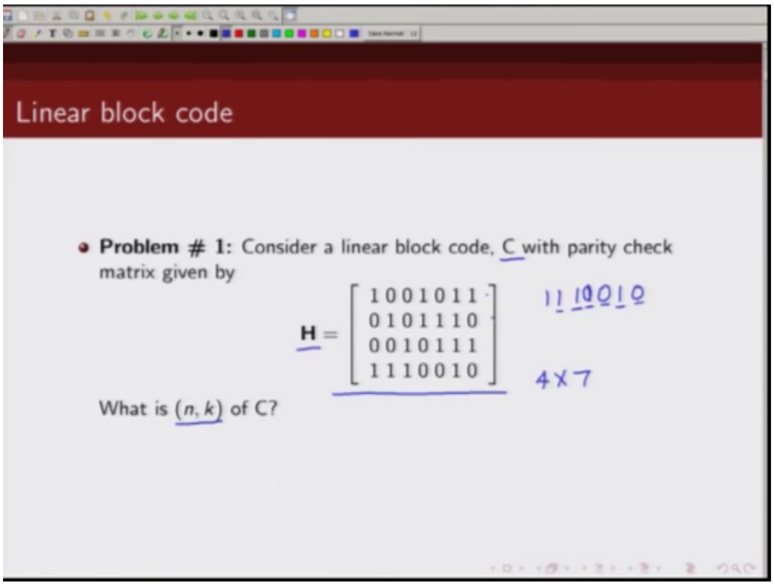

And what is row number 4? It's exactly same

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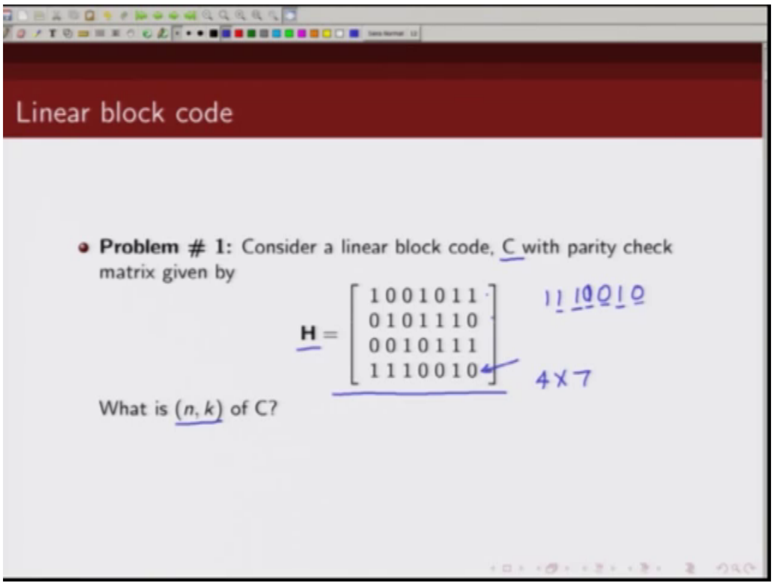

same as this. So you can see row 1, row 2, row 3 and row 4 add up to 0. That means it does not have

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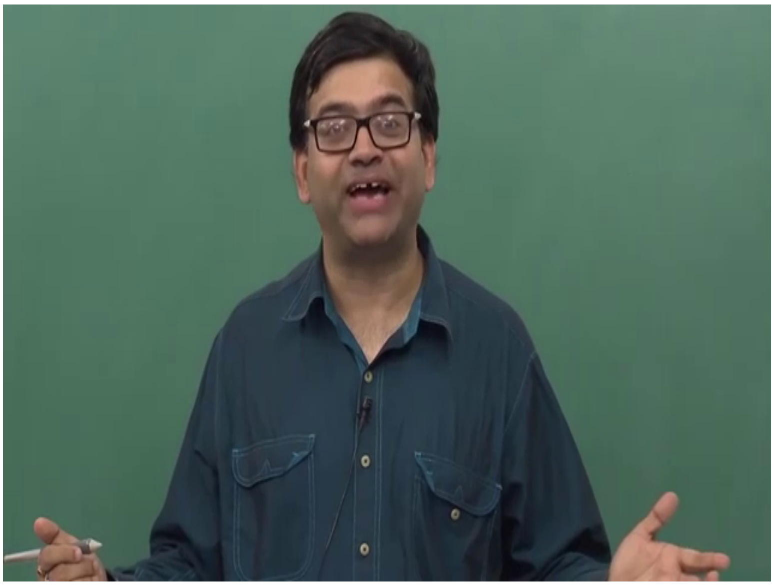

rank 4.So maximum rank possible is 3. So let's see

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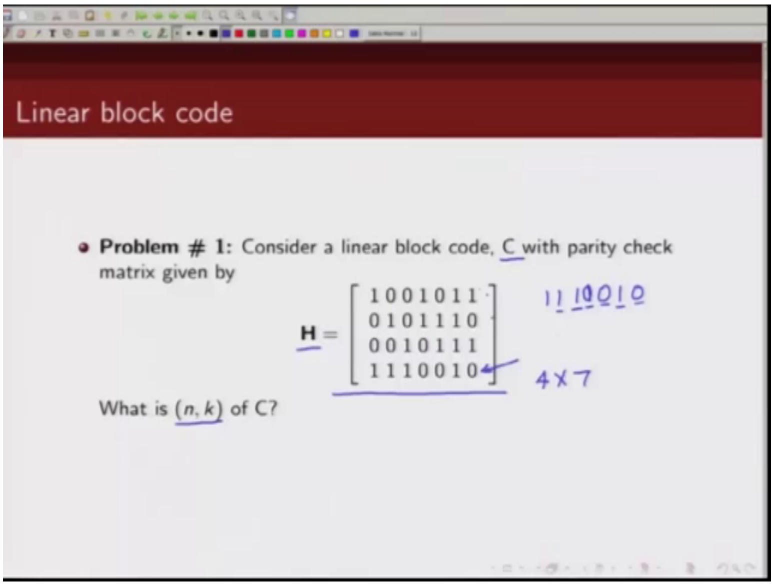

rank 4.So maximum rank possible is 3. So let's see

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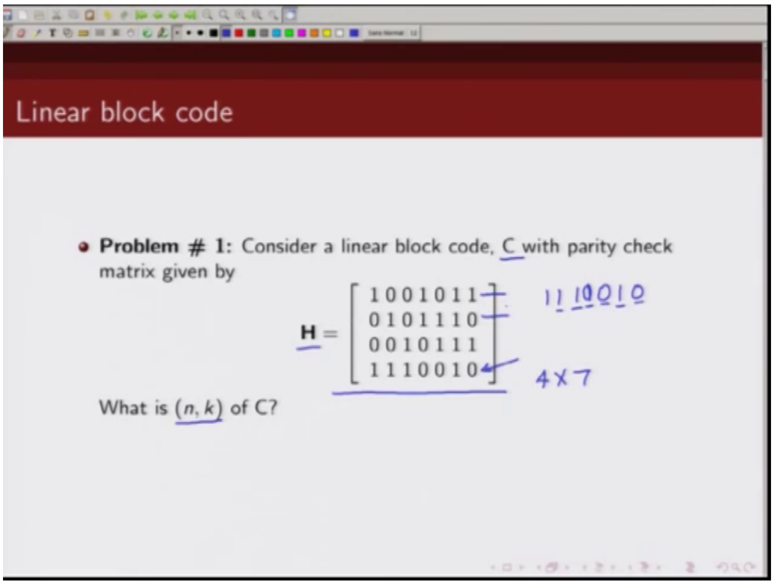

if any 3 rows combination add up to 0. So let's see, let's see if we can consider sum of these two rows. This is what, 1 1 0 0 1 0 1.

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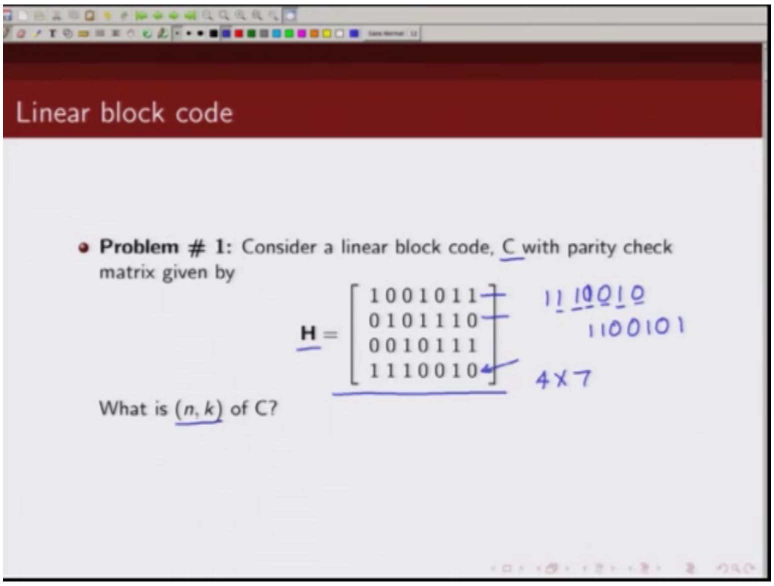

Now none of the rows are equal to this, you can see. If we consider this row and this row, we add these two rows.

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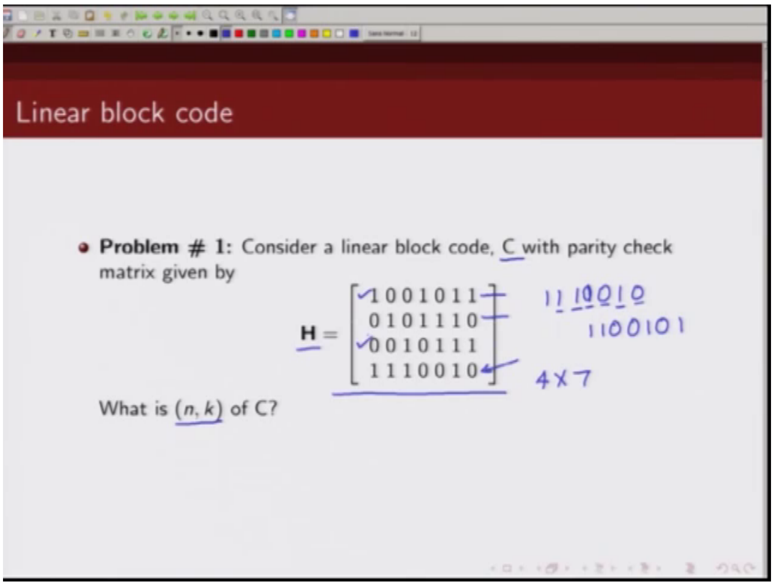

Let's see. What do we get is 1 0 1 1 1 0 0. Now note

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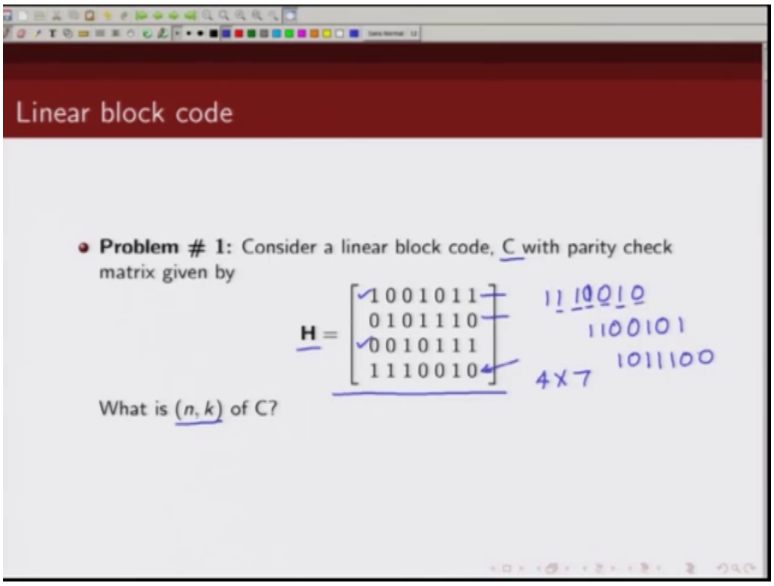

none of these rows, r 2 and r 4 is equal to this. So these set of 3 rows, basically they are independent. Let's try adding up this and

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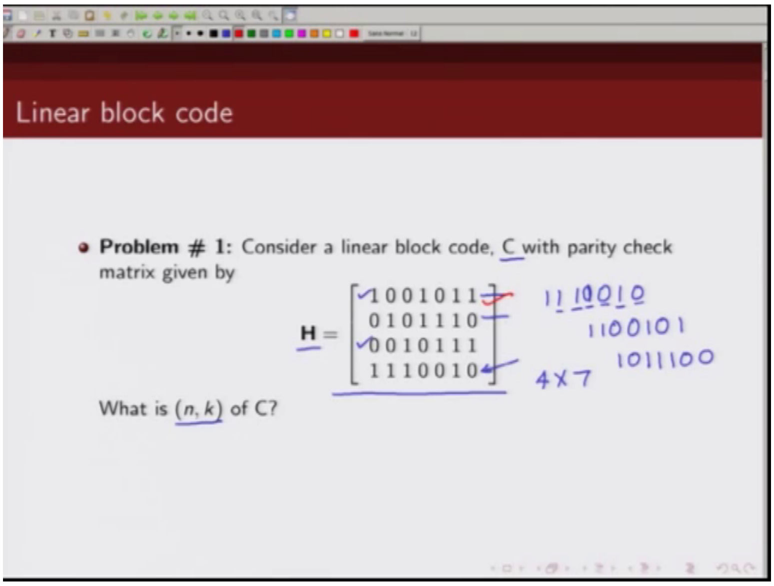

and this. If we add first row and fourth row, what do we get? 0 1 1 1 0 0 and

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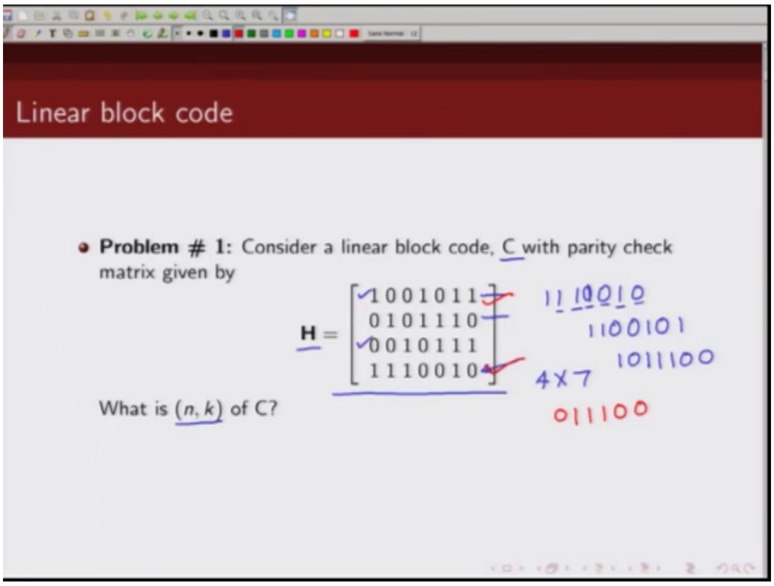

1, and note row number 3 and 2 are not same as this. Like that we can check, we can check for example row 2 and 4, we add up row 2 and 4,

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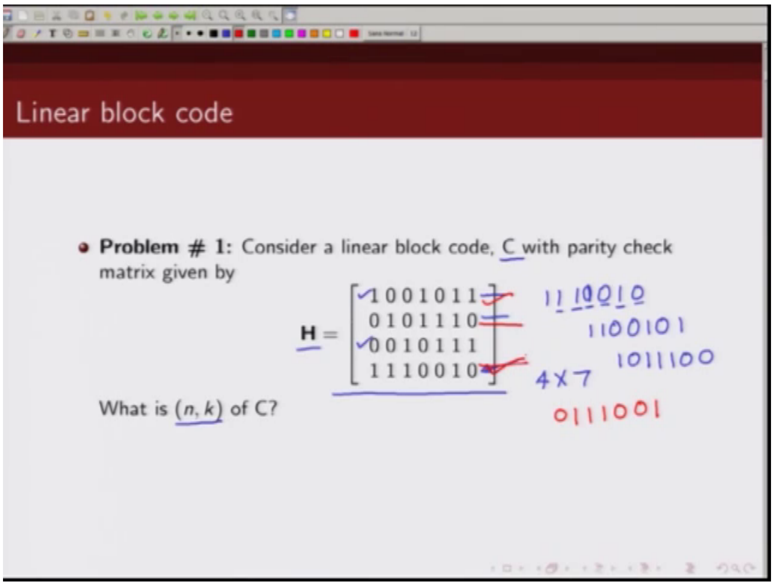

what do we get? 1 0 1 1 1 0 0, now note row number 3

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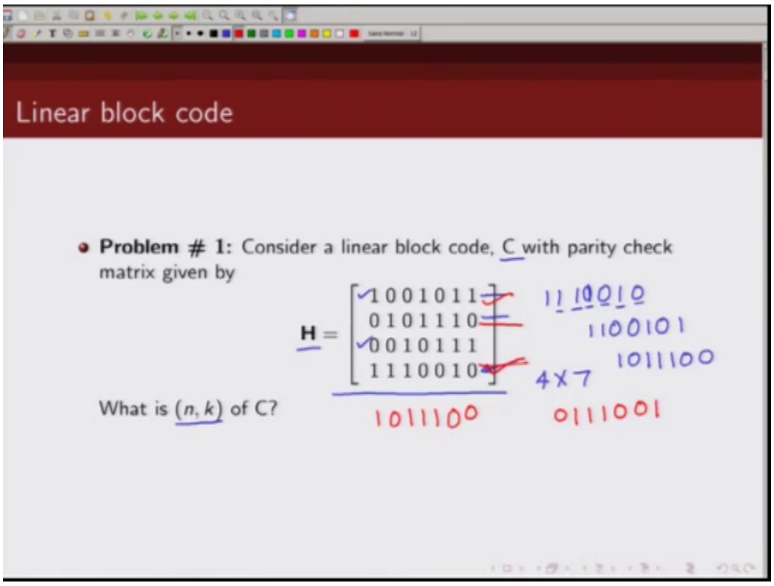

and row number 1 are not same as this. So we can see that any 3 rows do not add up to 0. So the rank of this matrix H is 3. So

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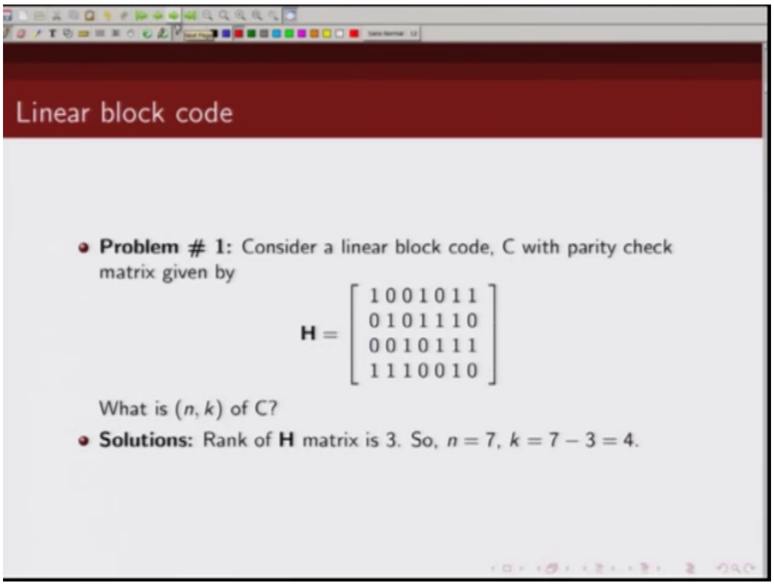

rank of this matrix is 3. Now we know parity check matrix is n minus k cross n. So

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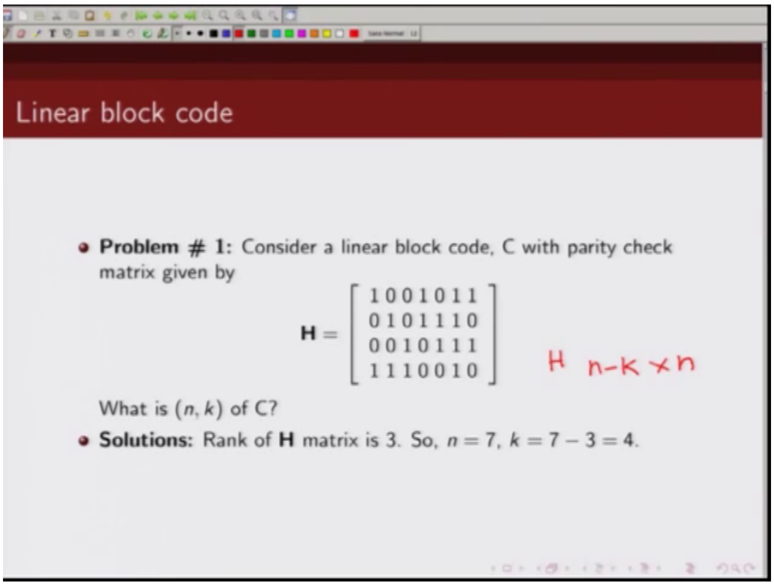

n minus k is in our case, equal to 3 and what is

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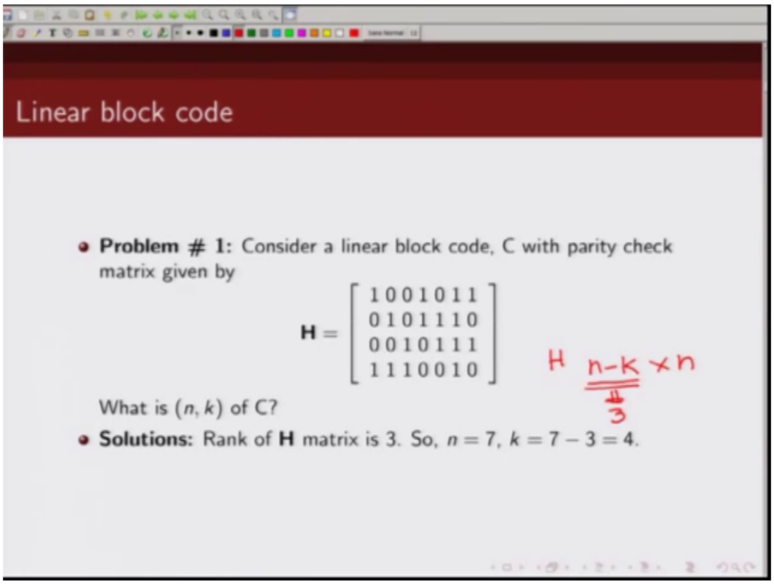

n, number of columns of this. So that's 1, 2, 3, 4, 5, 6, 7. So

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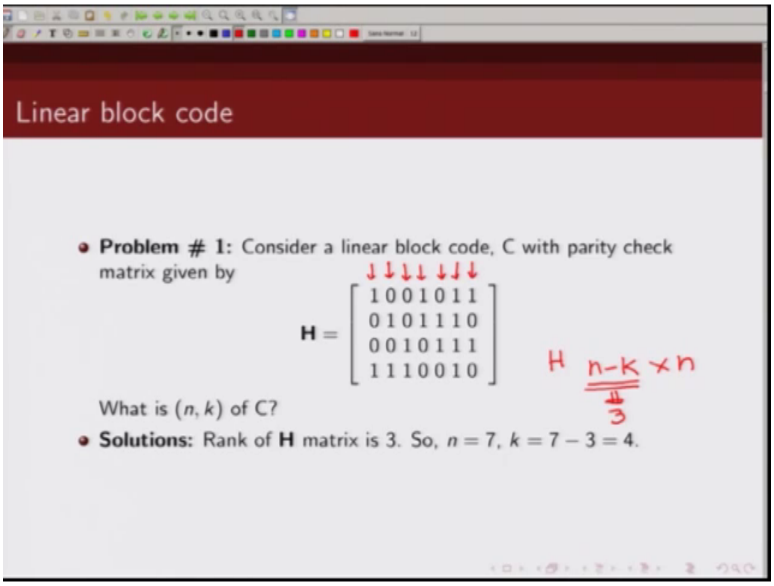

n is 7. So that would then give us k equal to 4. So this

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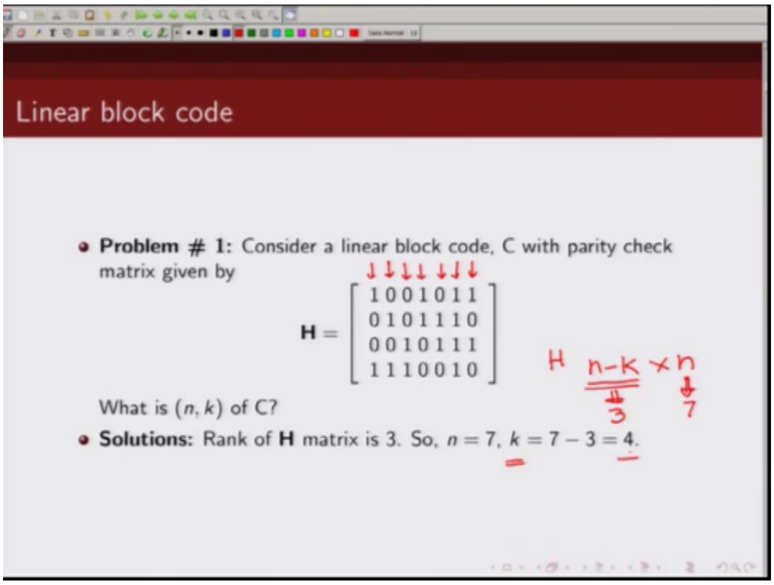

this is an example parity check matrix for a 7 4 linear block code

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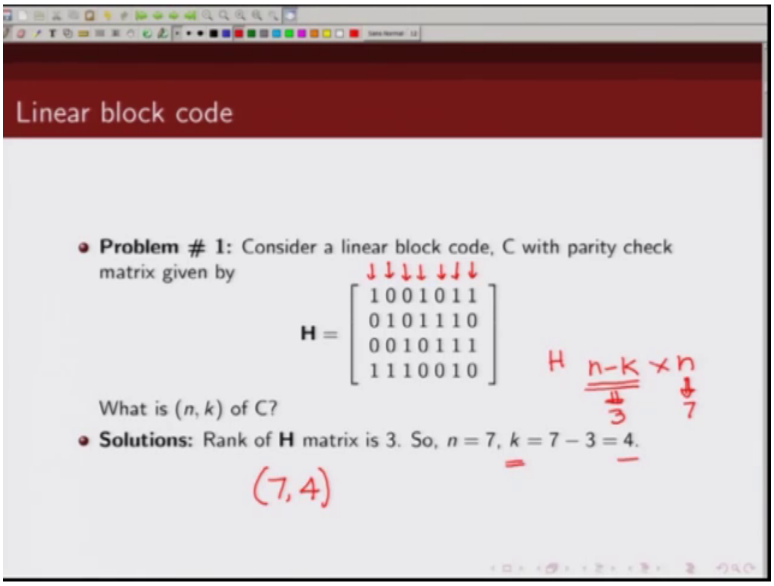

Ok.

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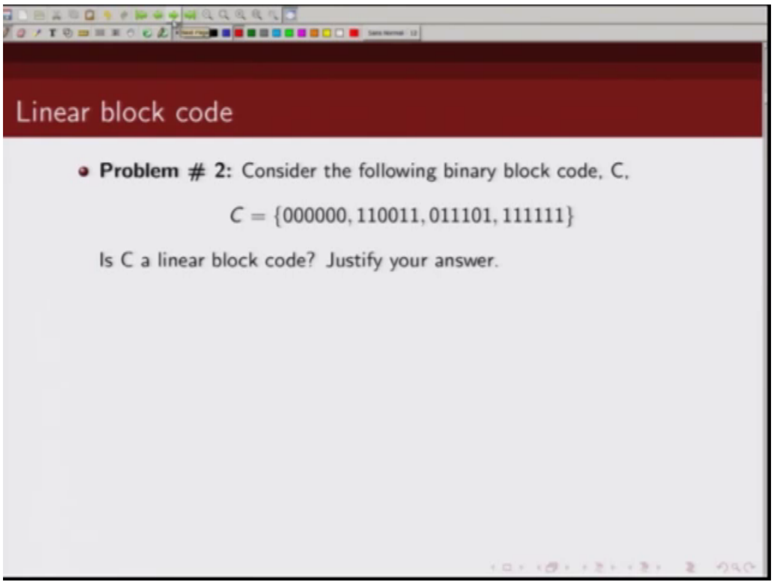

Now let's look at another problem. You are given a set of codewords. And what are these codewords? These are binary codewords. So this is all zero, 1 1 0 0 1 1, 0 1 1 1 0 1 and 1 1 all 1

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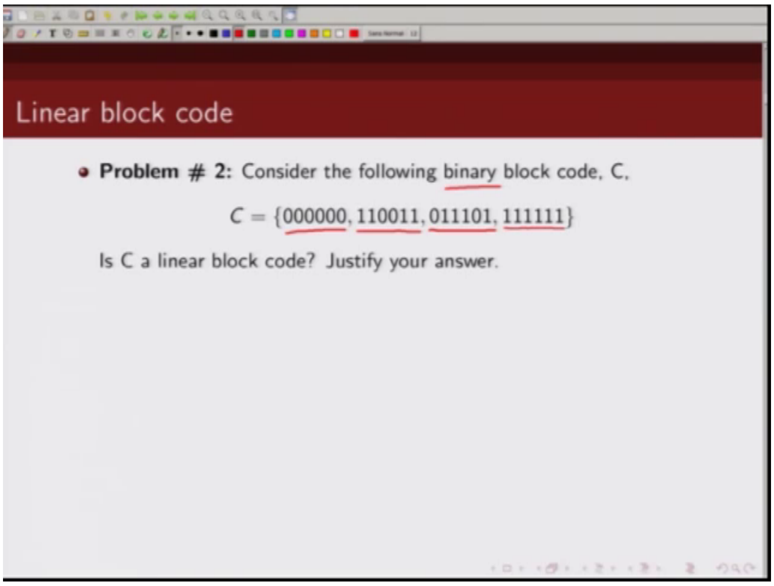

And the question that has been asked is, is this a linear code? Is this a linear code?

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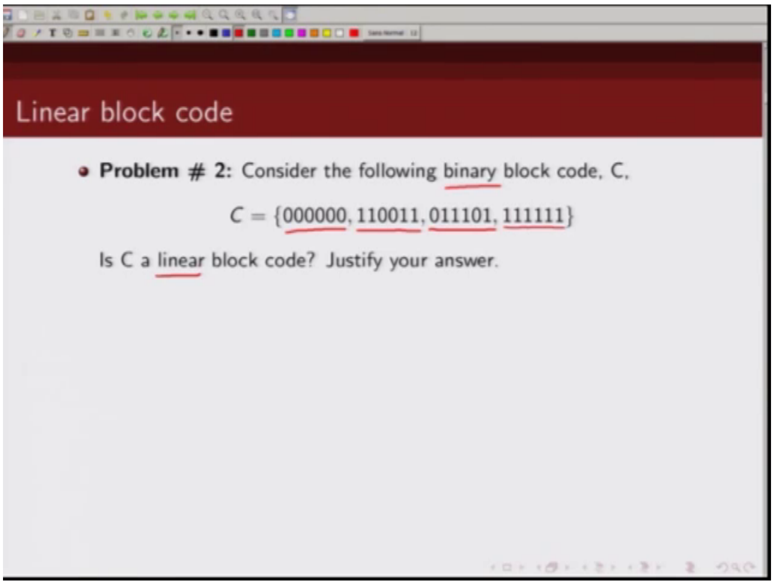

Now what do we know about linear code? A linear code should have all 0 codewords which this codeword has. And sum

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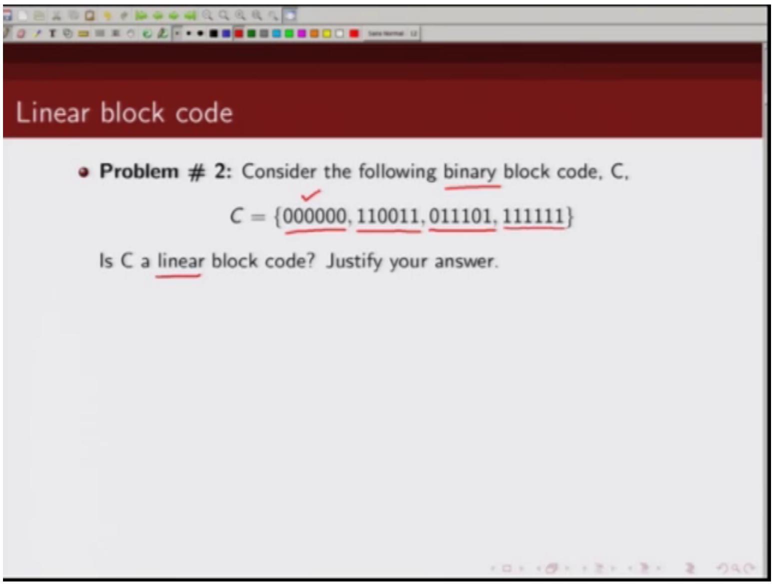

of any two codewords

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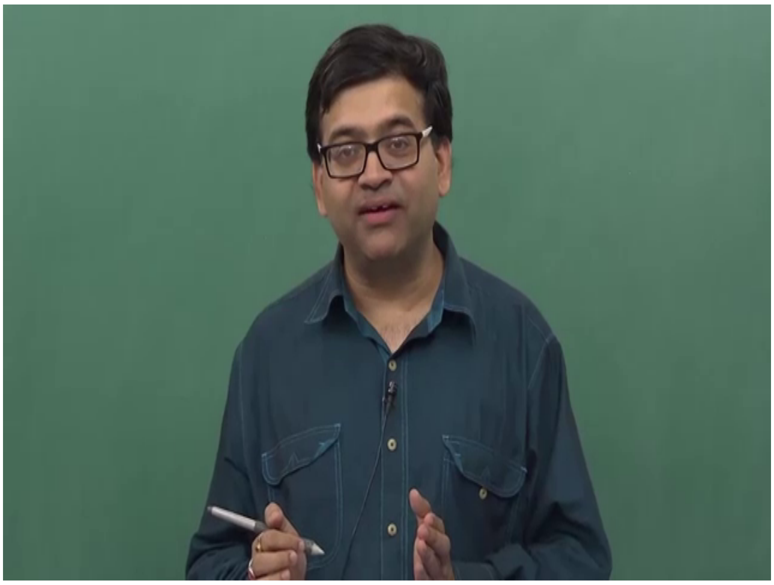

is also a valid codeword. So let's see.

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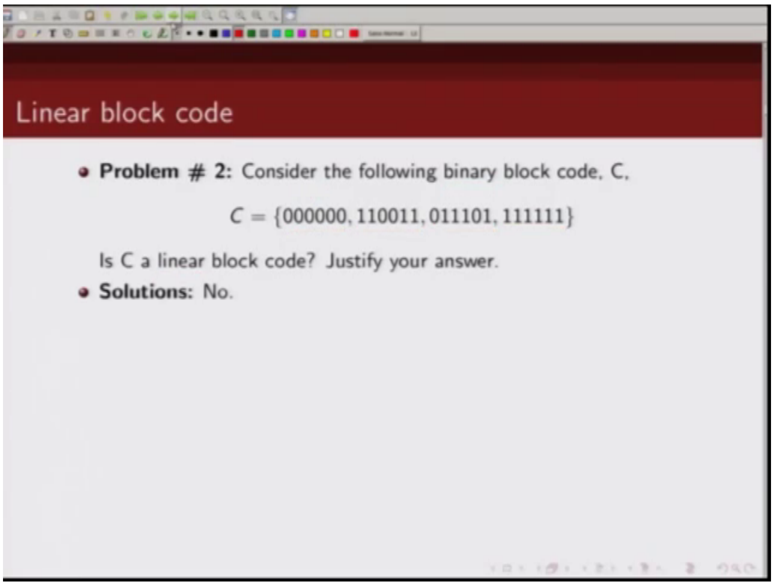

So let's see if sum of all codewords is already a valid codeword.

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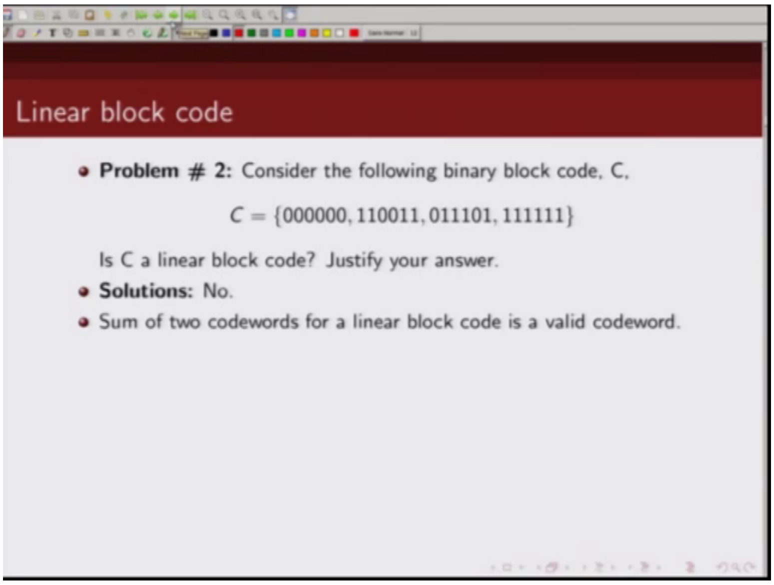

So let's see if sum of all codewords is already a valid codeword.

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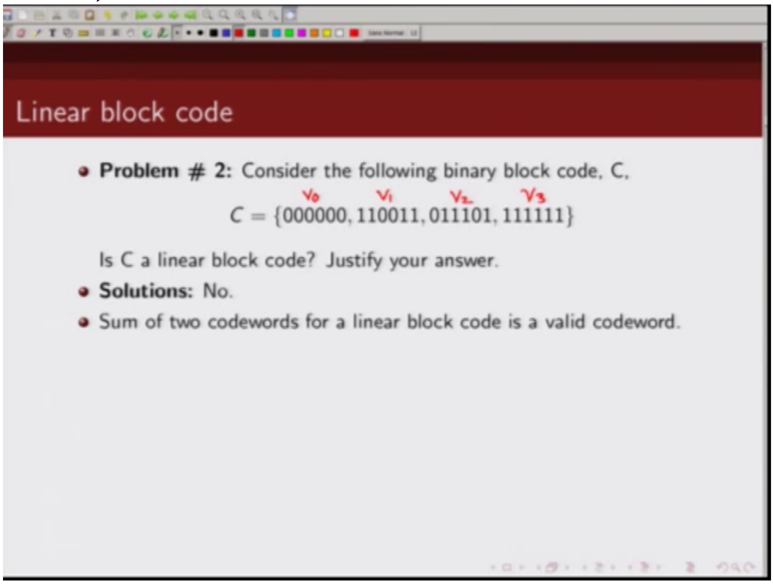

So let's call this v 0, v 1, v 2 and v 3. So what we want is all possible combinations of v 0, v 1, v 2, v 3 should also be a valid codeword. They should be in C. So let's see.

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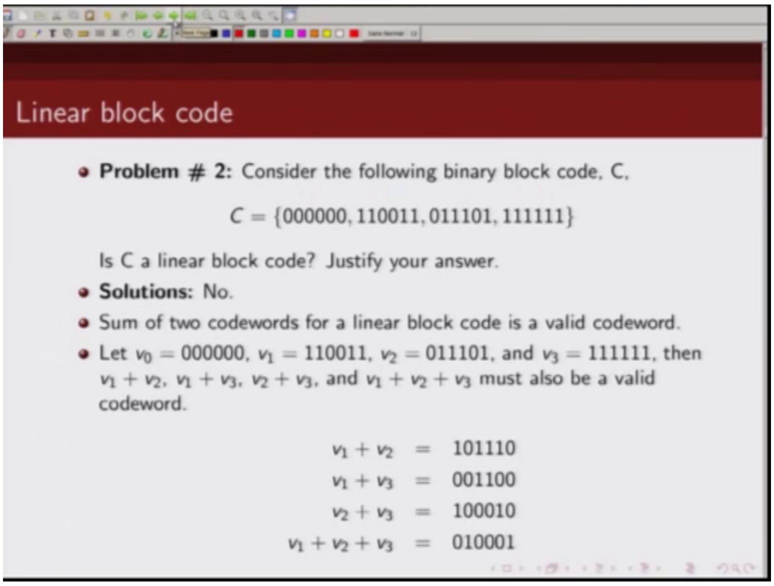

So, as I said we take v 0 to means all zero codewords,

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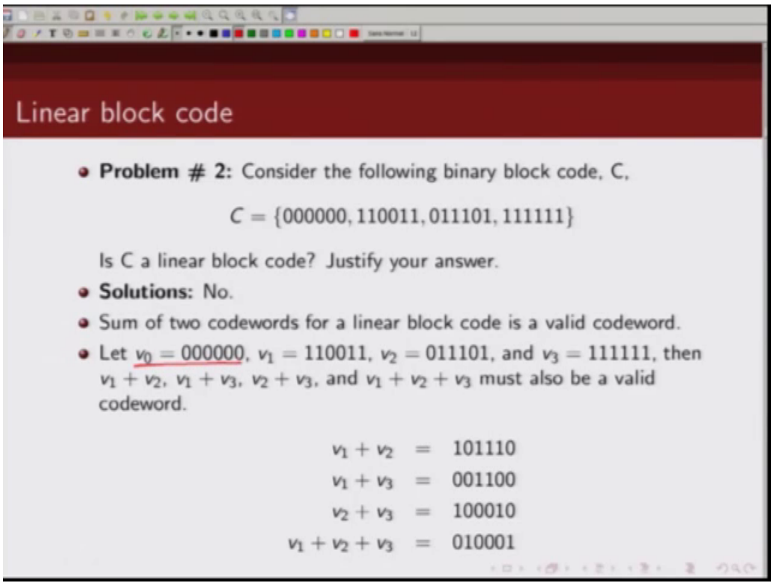

is given by this, this

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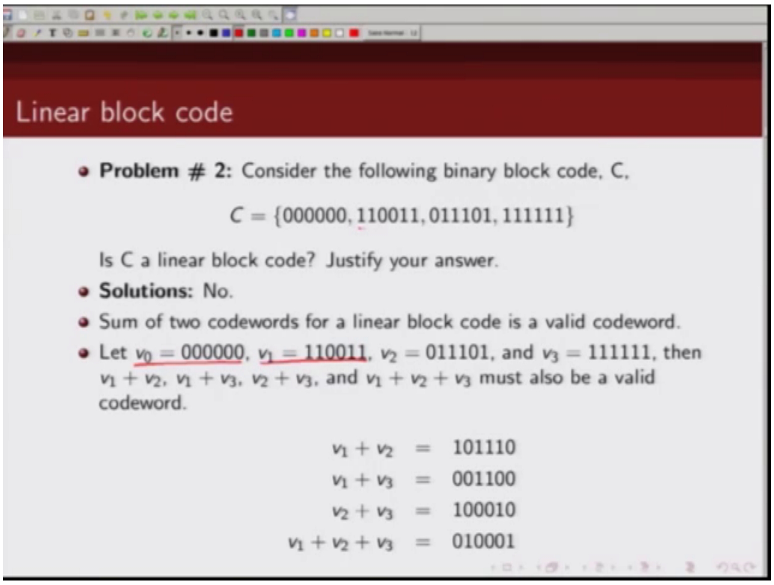

is v 1, this is v 2 and this is v 3. Now let's see all possible

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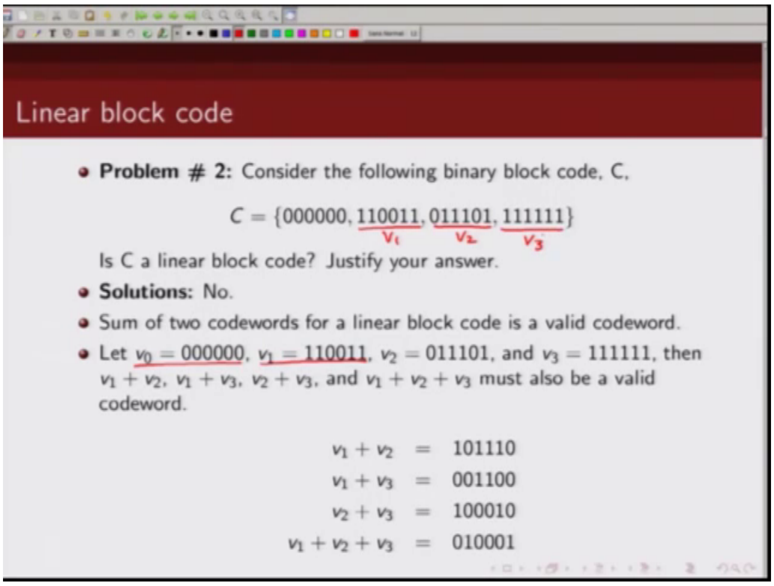

combinations of v 1, v 2, v 3, the non-zero codewords. So we consider v 1 plus v 2. What is v 1 plus v 2? v 1 plus v 2 is, you can see this is 1 0 1 1 1 0, it's given by this.

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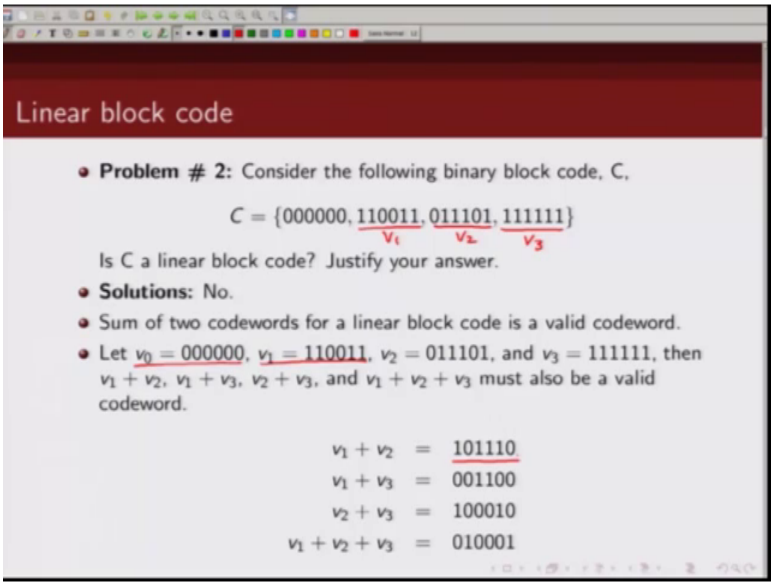

Now is this codeword in C? We don't see any codeword which is 1 0 1 1 1 0 listed here. That means this C

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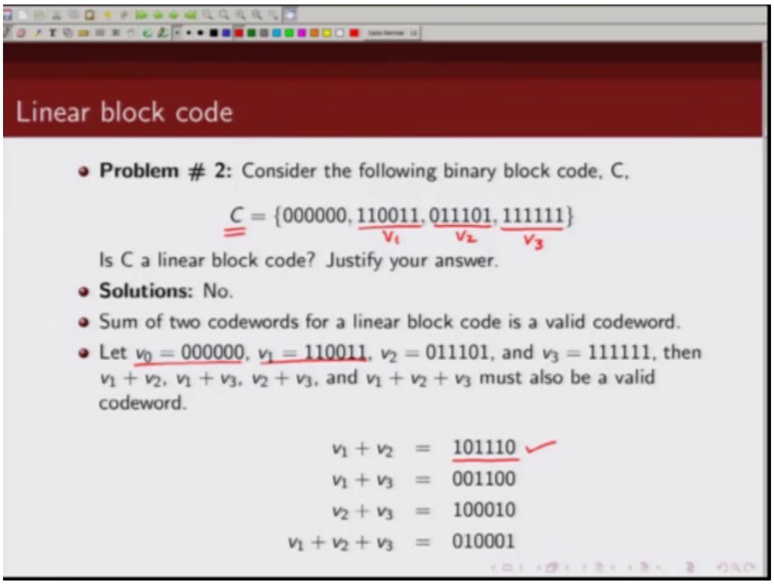

is not a linear code. Why it's not a linear code, because sum of any two codewords

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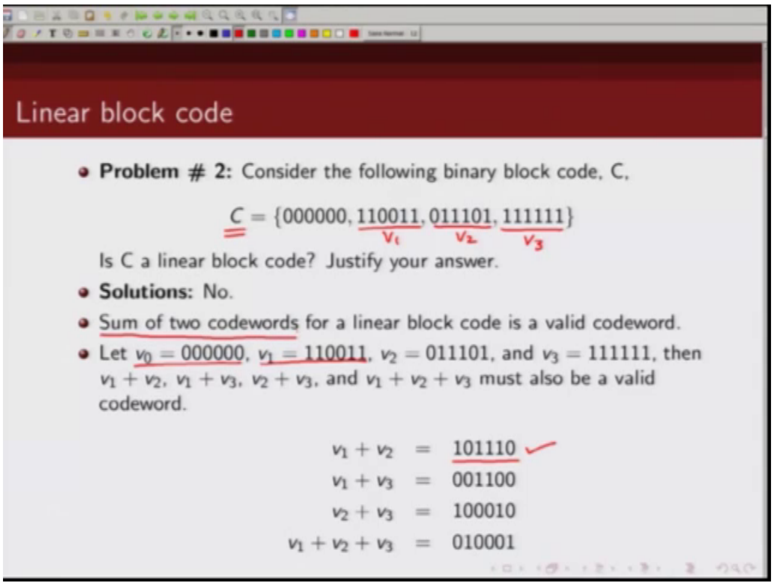

is also a valid codeword. Now v 1 and v 2 are valid codewords in C. So sum of v 1 plus v 2 should also be in C. But we notice that 1 0 1 1 1 0, which is sum of v 1 plus v 2 is not there in C. And that's why we say that C is not a

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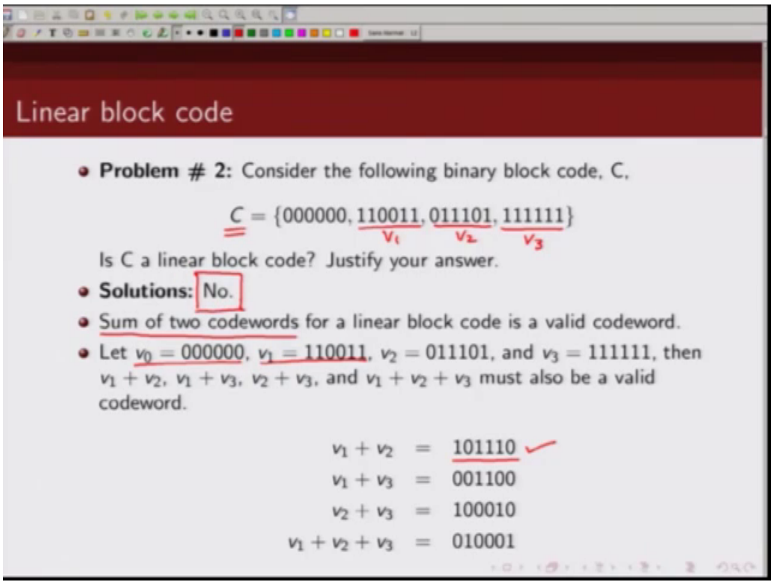

linear block code. Now my next question is can we add additional codewords here

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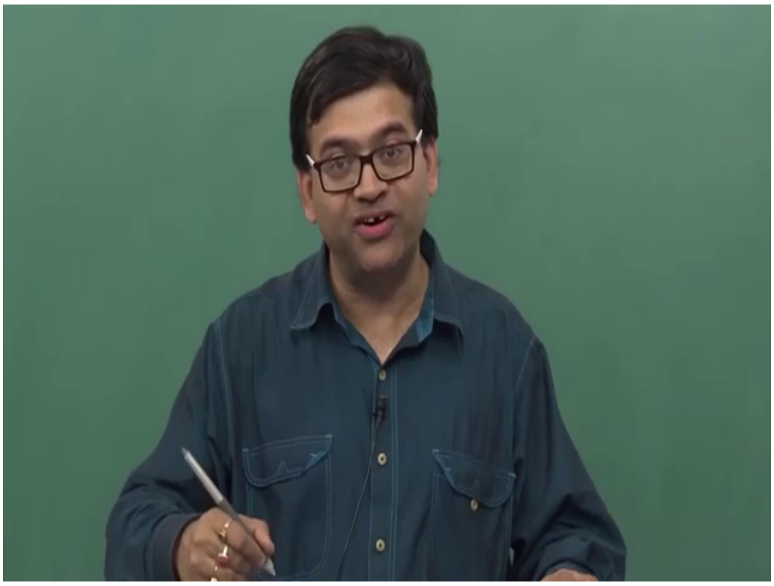

such that C becomes a linear block code? Now how do we do that? To do that, we will have to ensure all possible combinations of these

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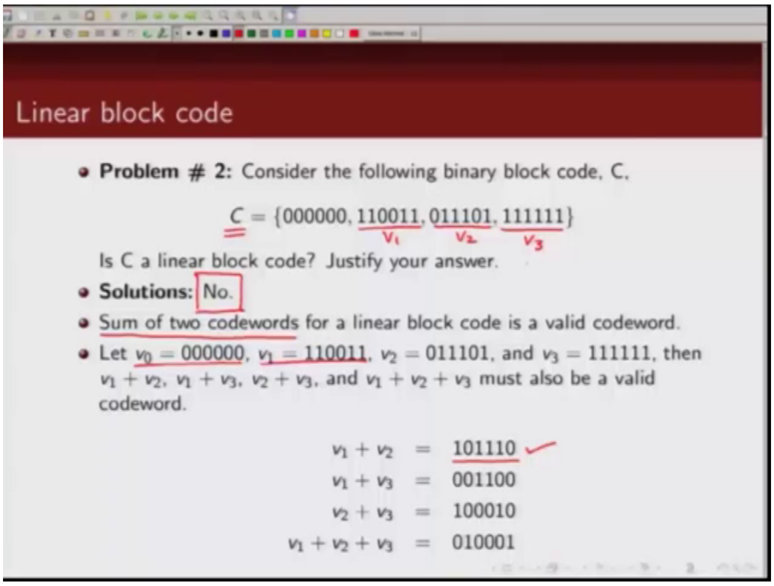

codewords is also there in C. So let's compute v 1 plus v 3 which is basically given by 0 0 1 1 0 0. Let's look at

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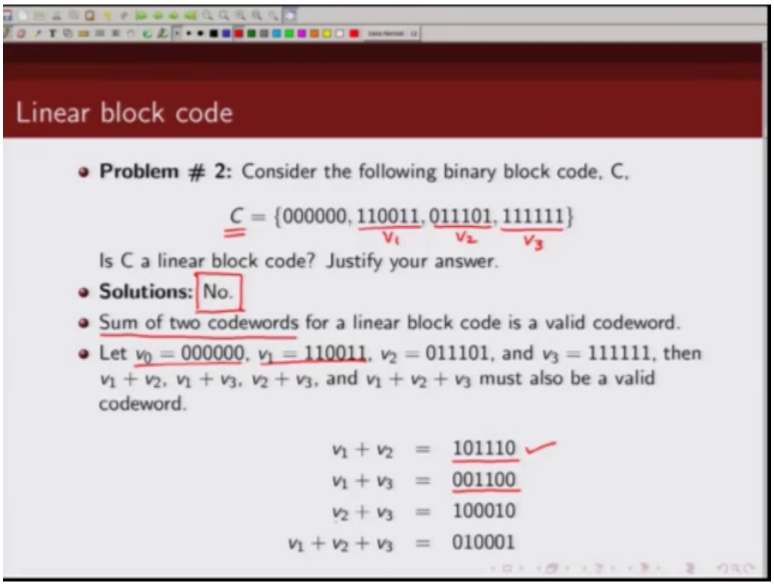

v 2 plus v 3 which is given by

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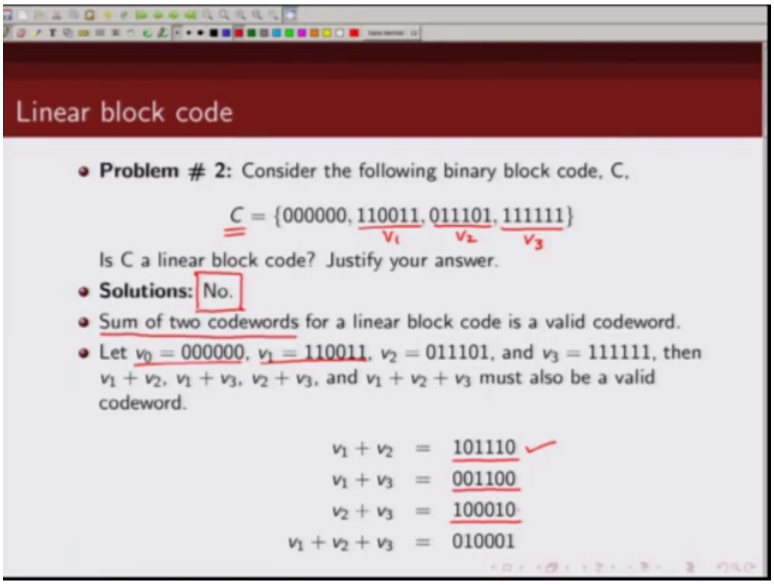

1 0 0 1 0. And let's look at v 1 plus v 2 plus v 3, is basically given by

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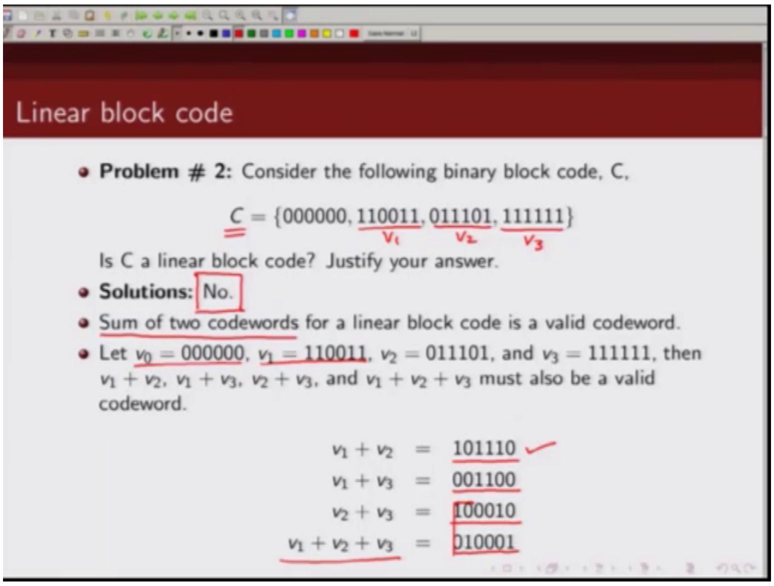

0 1 0 0 0 1. So note that I have listed all possible combinations of these codewords here. Now none of these sums are there

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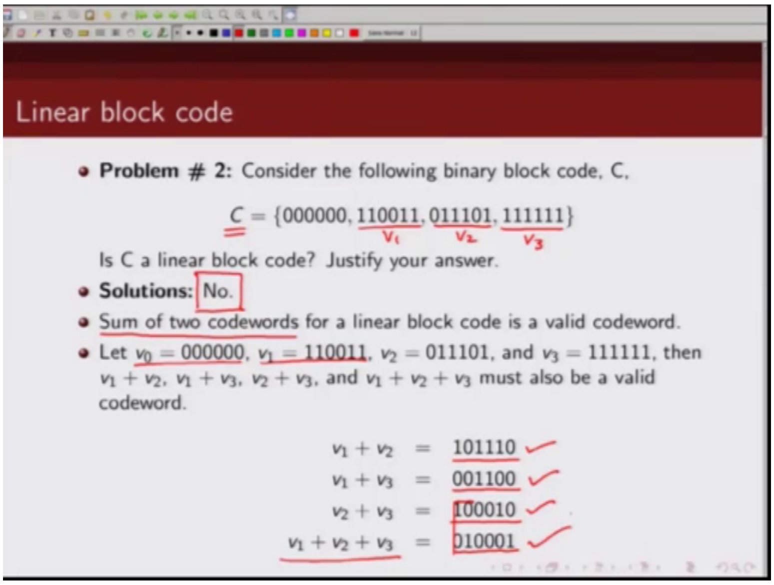

in this linear block code. So if we add them in this set of C, set of codewords, then we, our block code C will become a linear block code. So if we want to make it a linear

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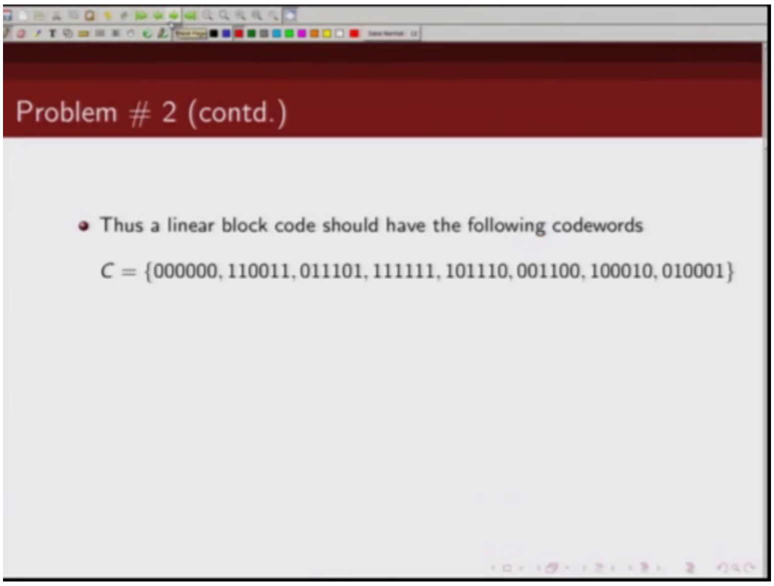

block code, what do we need to do? In this set of 4 codewords v 0,

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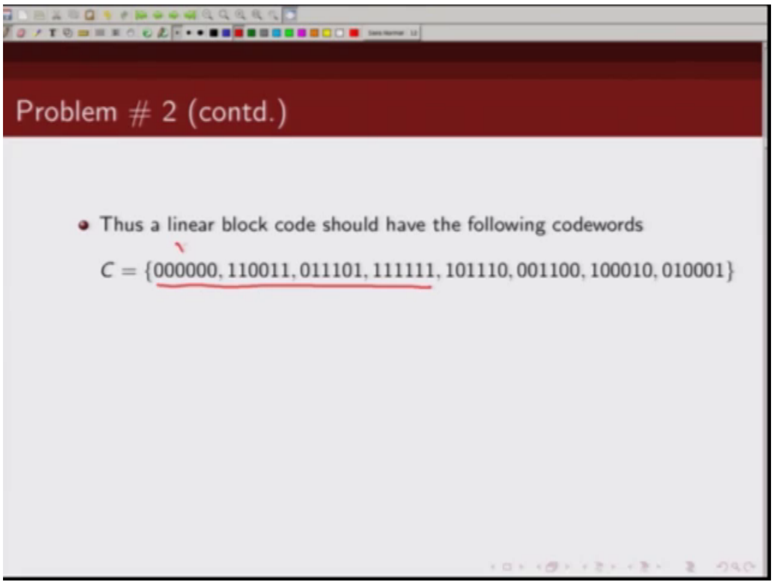

v 1, v 2 and v 3 we need to add these set of

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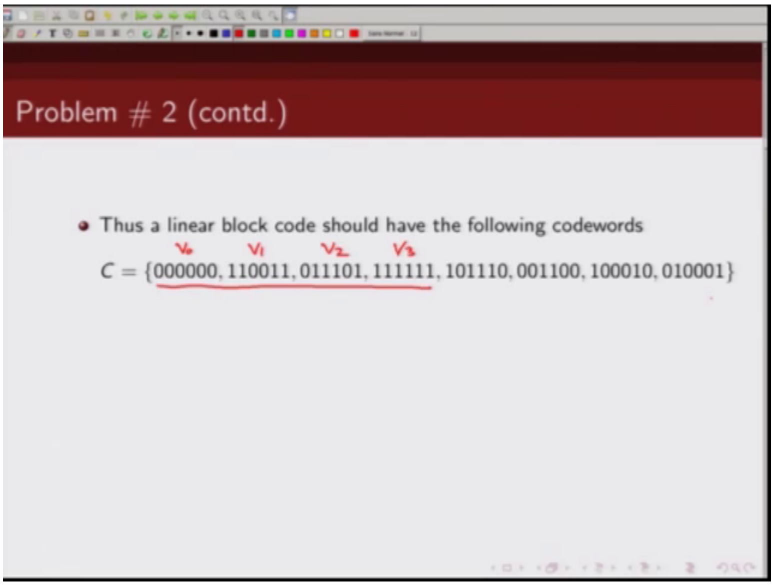

codewords which was basically

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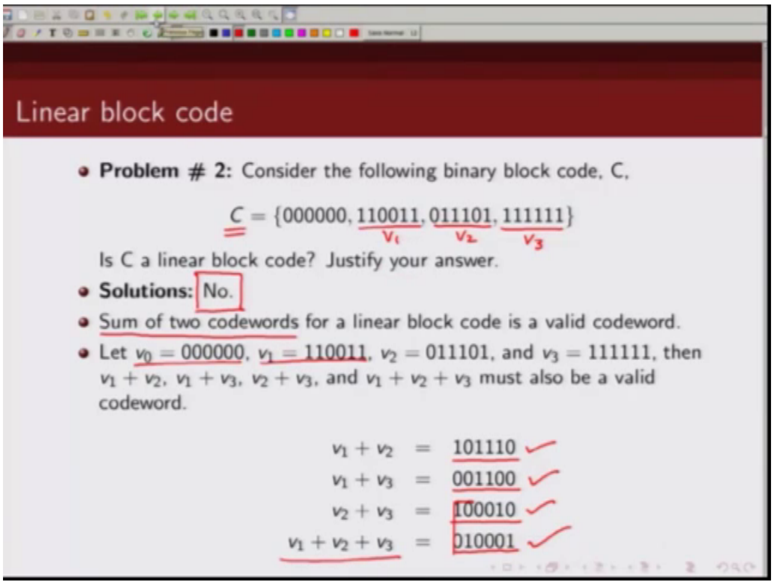

v 1 plus v 2. This is v 1 plus v 2.

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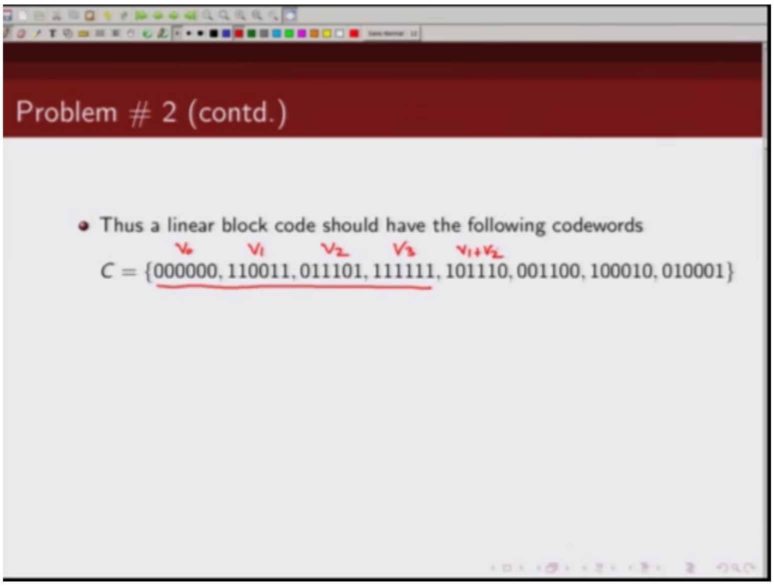

This is

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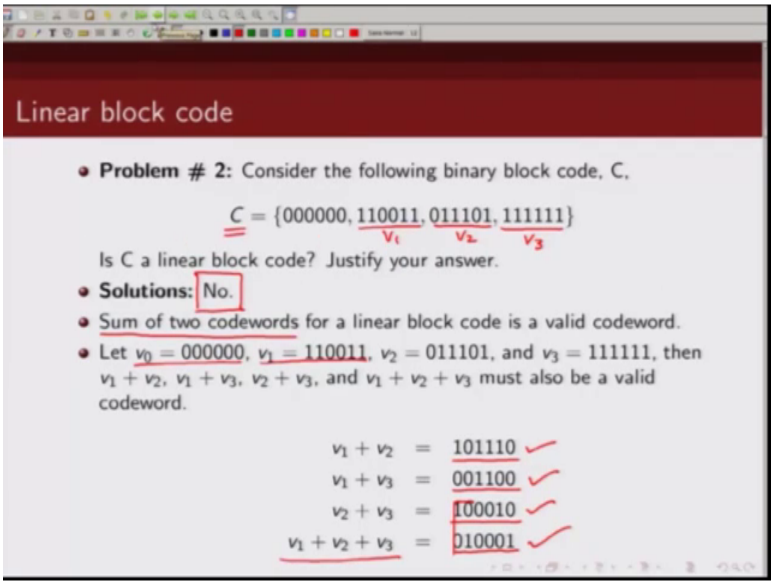

v 1 plus v 3, v 1 plus v 3.

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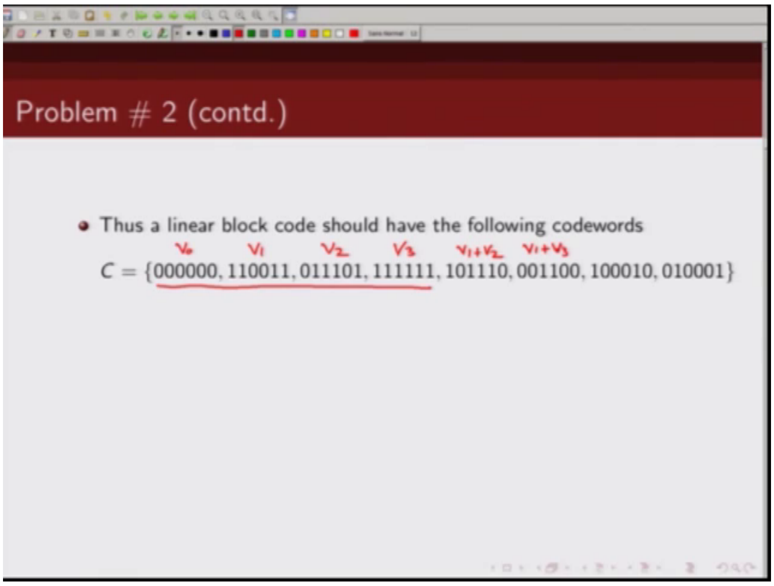

Then this one is

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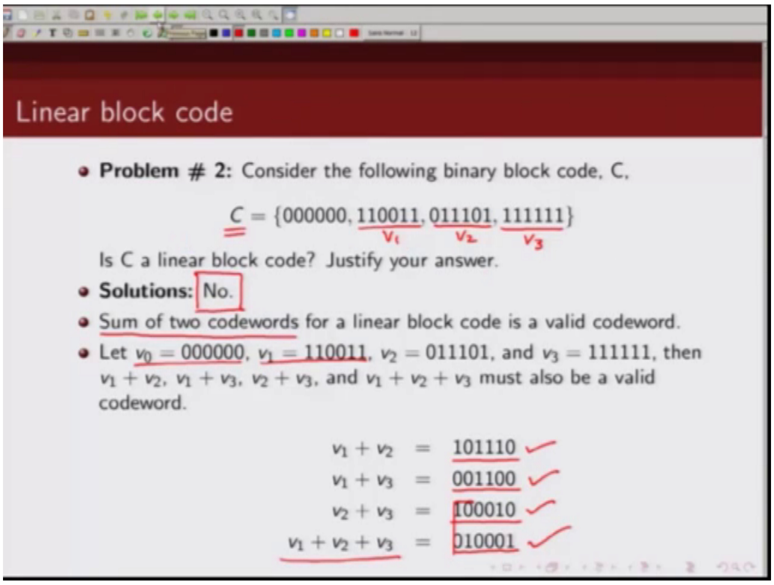

v 2 plus v 3, v 2 plus v 3 and this one was

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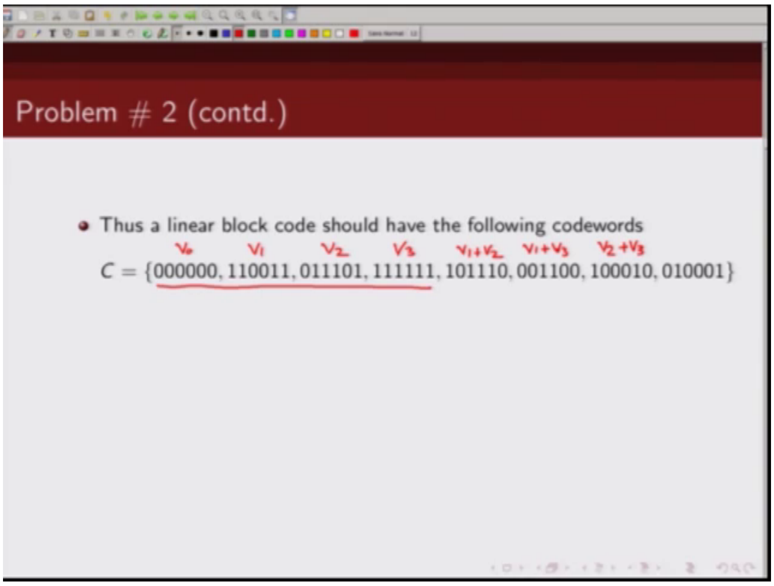

v 1 plus v 2 plus v 3.

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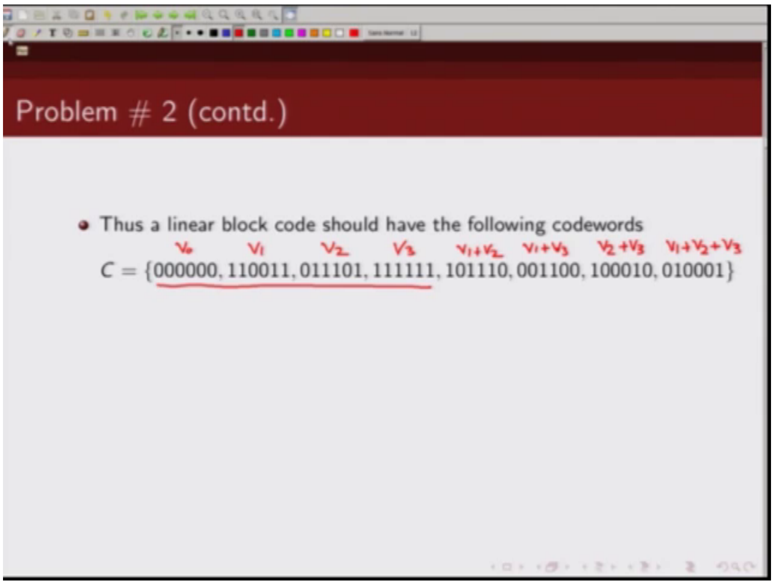

So let's look at these 2 codewords. This is v 1 plus v 2 and this is v 1 plus v 2 plus v 3. So if we add these two, what we will get is v 3. We can double check. So if we consider add these two, the first bit will be 1, this 0 plus 1 will be 1, then 1 plus 0 will be 1, then 1 plus 0 will be 1, then 1 plus 0 will be 1 and 0 plus 1 will be 1. And this is already there in this set of codewords. This is v 3, Ok. Similarly take these two.

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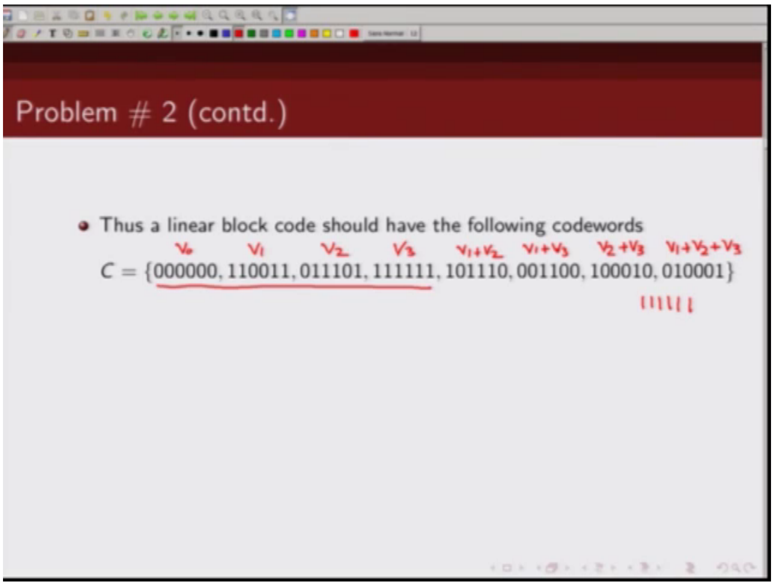

This one is v 1 plus v 2 and this is v 2 plus v 3. If we add them, what we get is v 1 plus v 3. We will get this. If we consider these two we will get v 2. We will consider this, we will get v 3. If we consider these two, sum of these two, we will get v 3. If we consider sum of these three, what we will get, we will get v 3. So you can see basically, linear combinations of all these codewords is already there in

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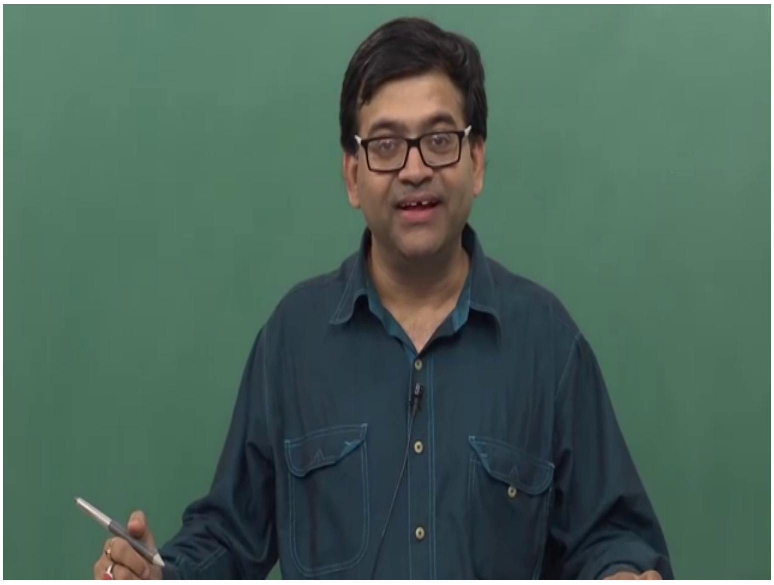
this C. So this C which contains the set of 8 codewords

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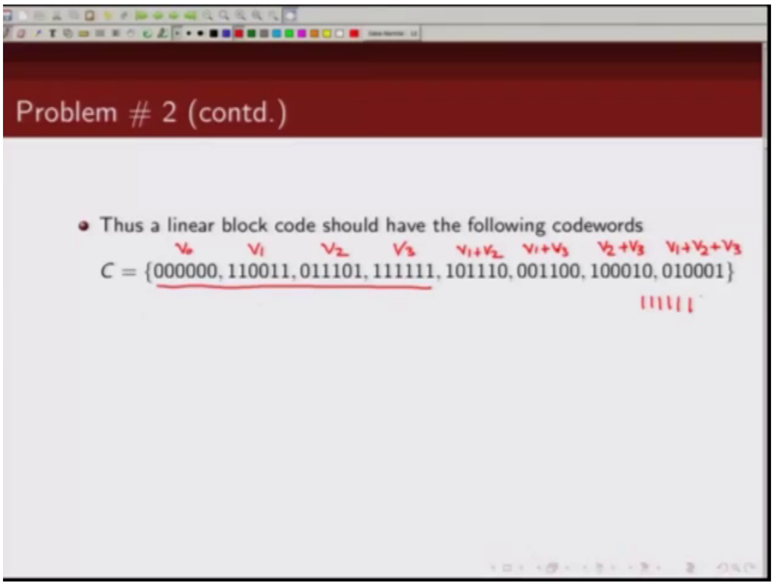

is a linear code.

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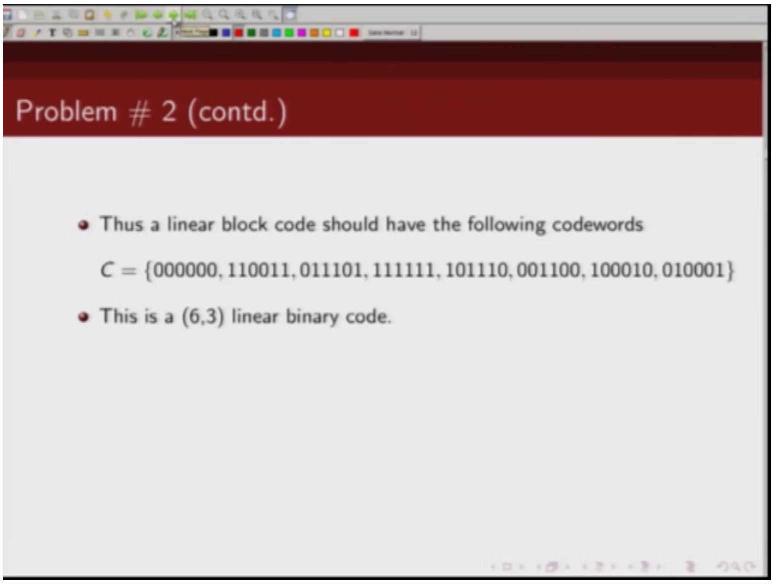

And what are the parameters n and k? Now the length of the codewords is 6.

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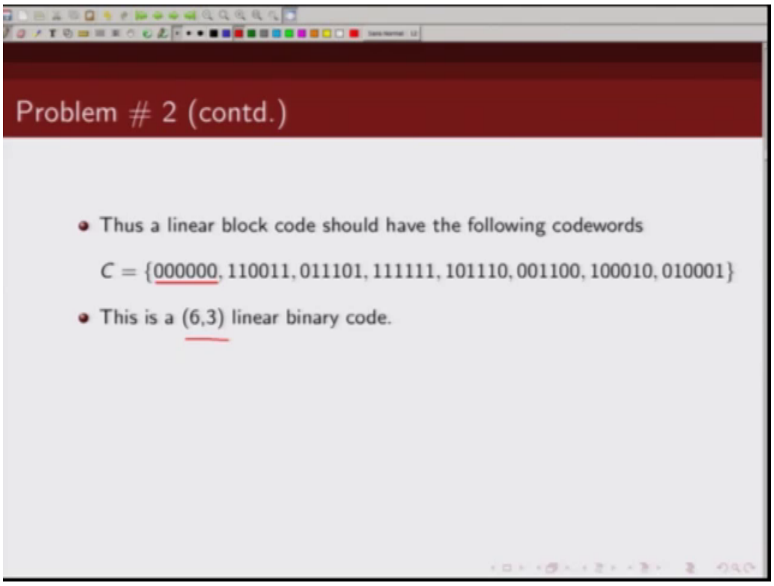

Each of these codewords are 6 bits. So that's why n is 6. And there are total

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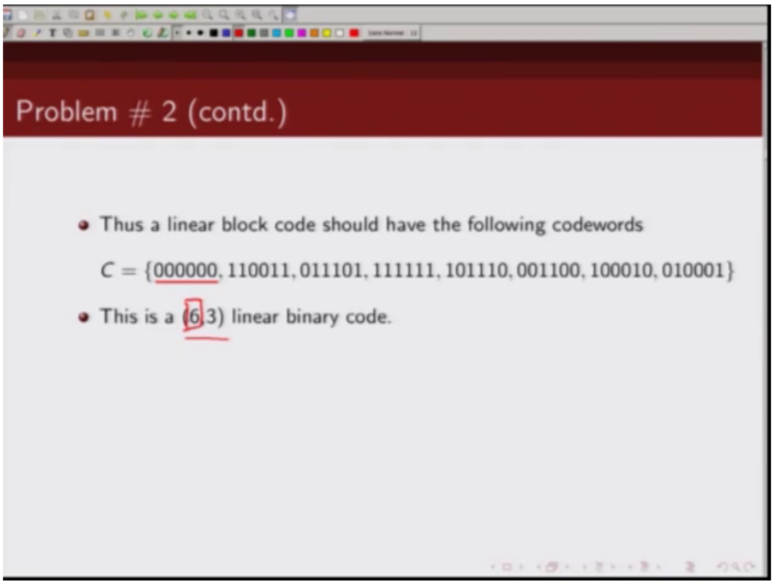

2 k codewords. And in our case 2 k is basically 8. So k is 3. So this

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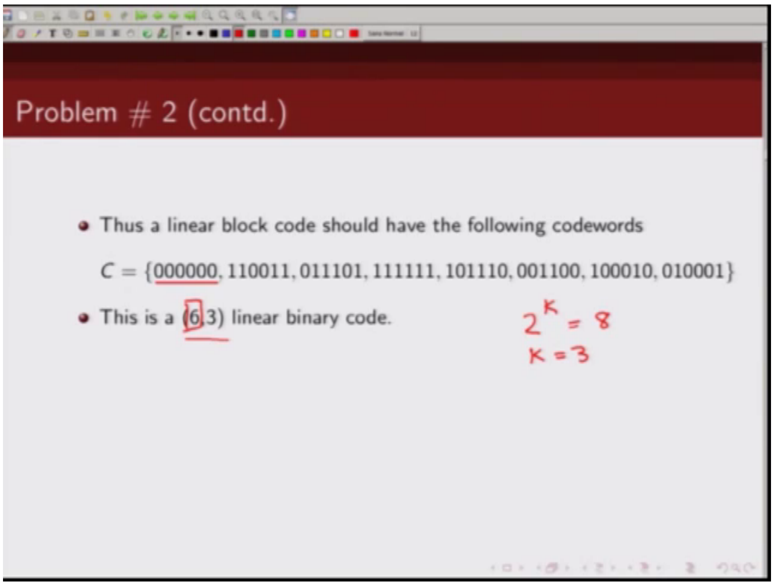

is basically a 6 3 linear

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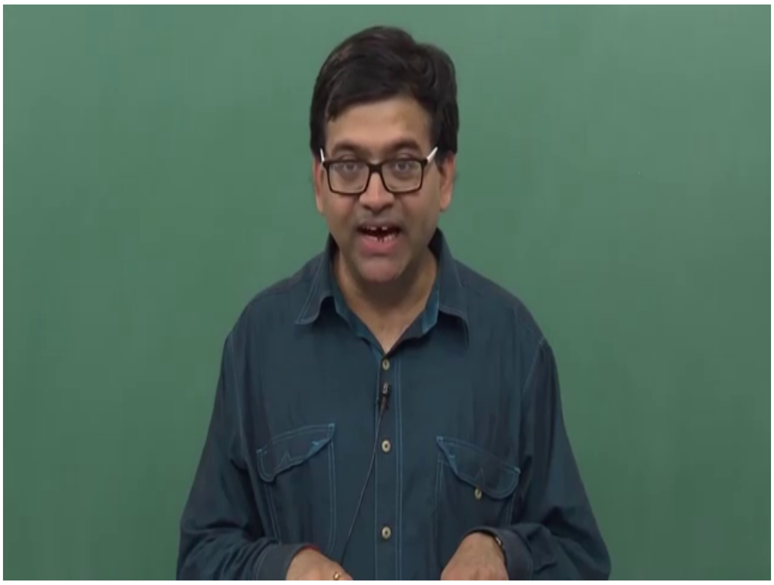

binary code.

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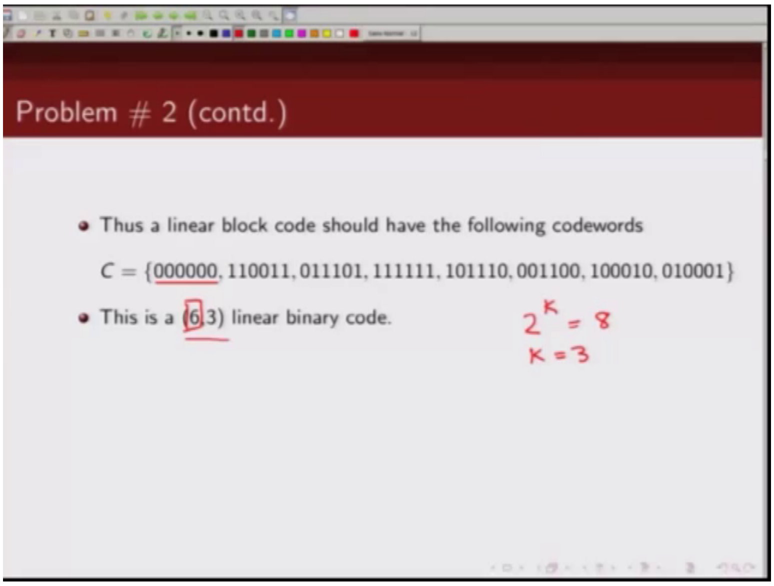

Now if I ask you, tell me what is a generator matrix that will generate this set of codewords?

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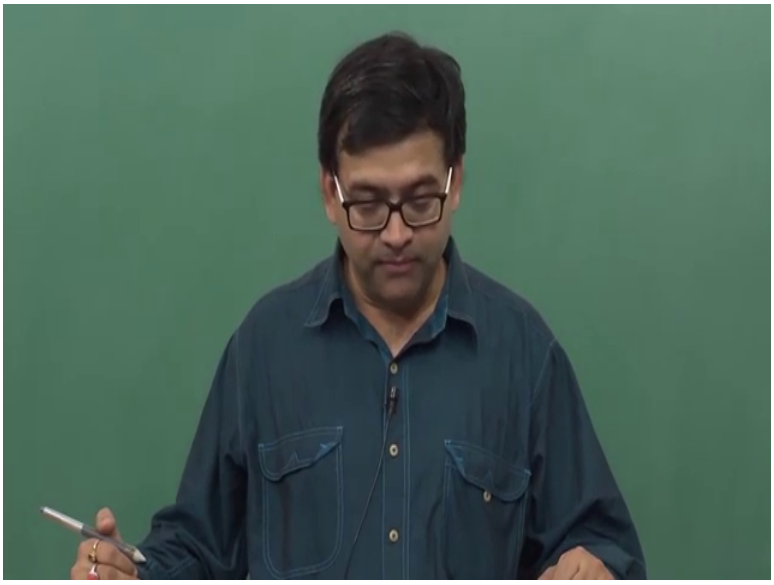

Now how can you do that?

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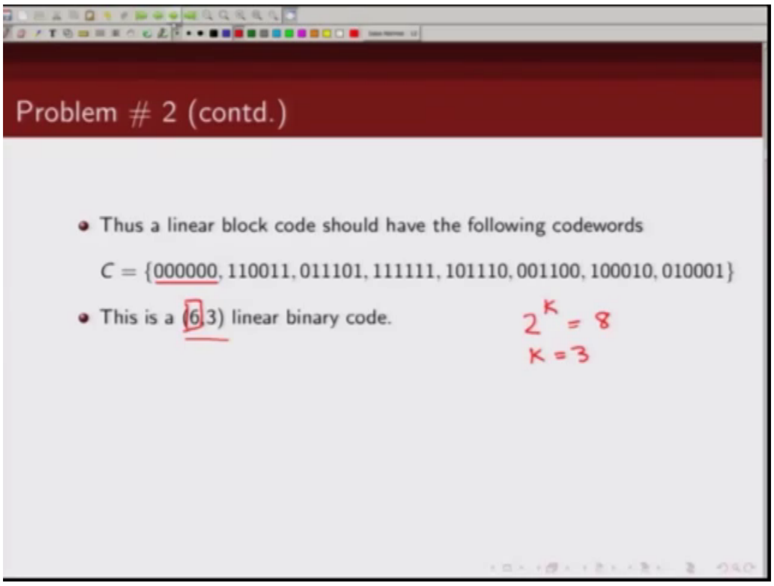

So we know the generator matrix. It's basically

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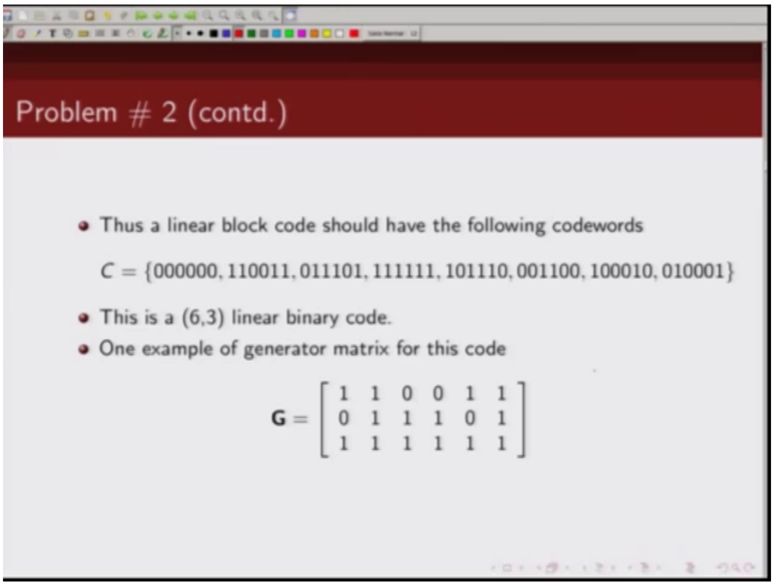

a k cross n matrix,

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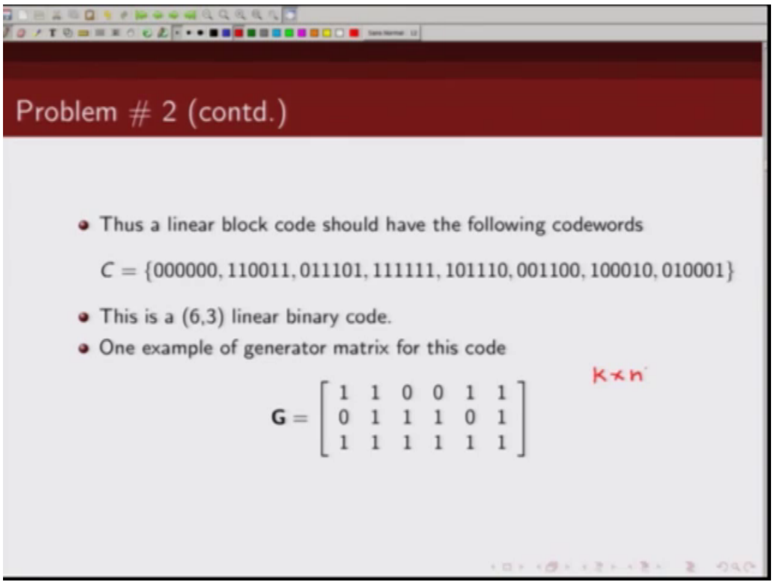

right. So if you take basically 3, k in this case is 3, if you take 3 codewords which are linearly independent basically, if you take them and form them as rows of your generator matrix, then you get your generator matrix. So I just took this v 1, v 2 and v 3

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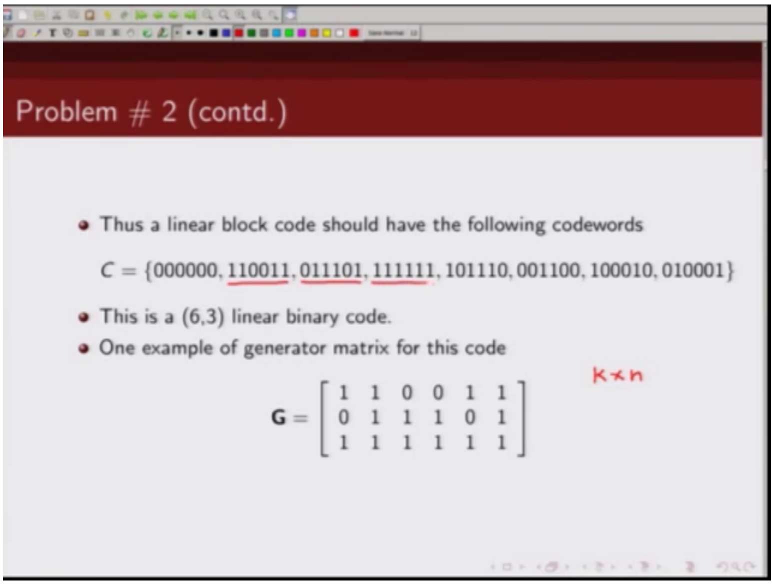

and you can verify that rank of this matrix G is 3. It's full rank,

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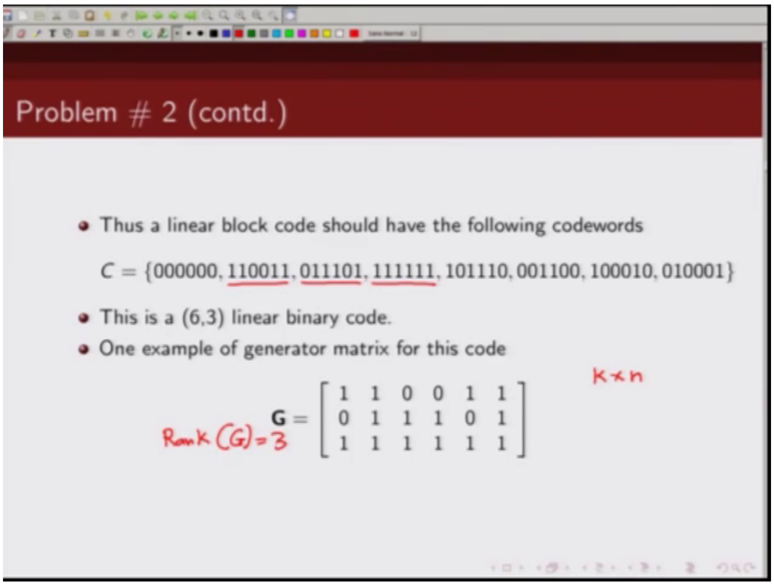

Ok. So then this G will be able to, this generator matrix will be able to generate this set of codewords.

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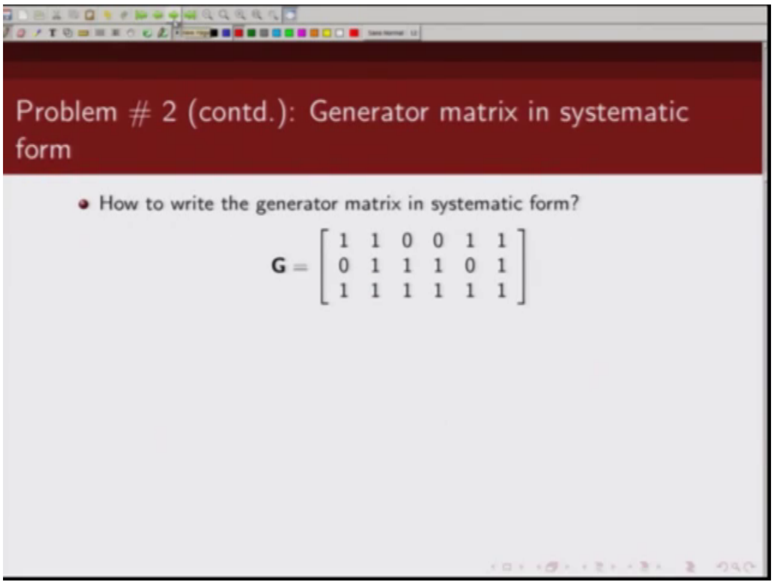

Now can we put this, is this generator matrix in systematic form? The answer is no. Because to get it in systematic form, what we need is our generator matrix should be of the form like this, or

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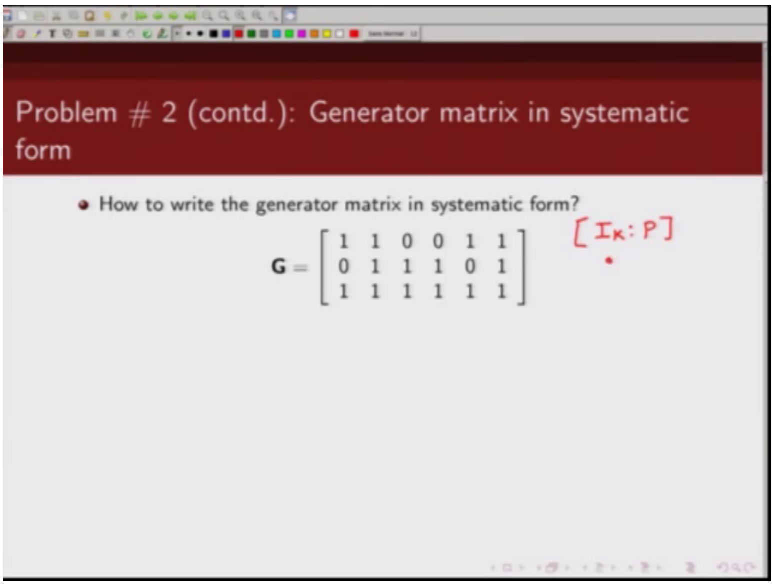

something like this, Ok.

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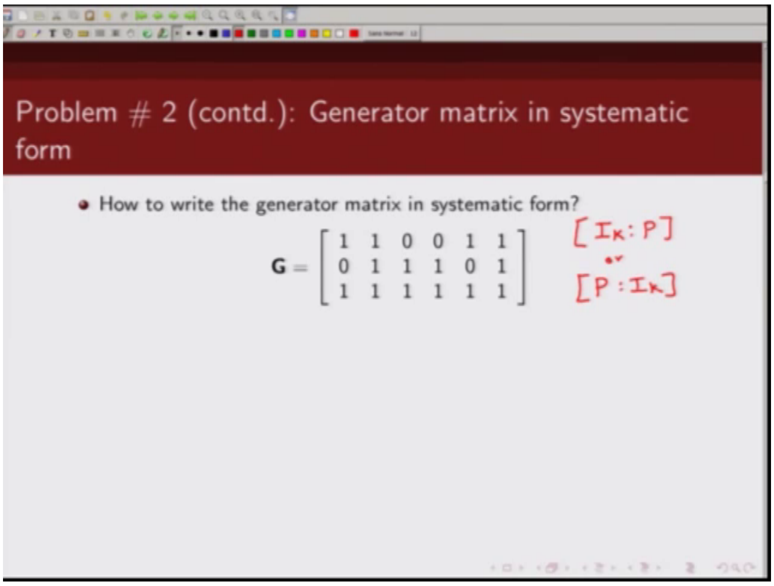

But this is not in this particular form. So we will have to get some identity matrix and some matrix P. Now by doing elementary row operation, we can put this in systematic form. So let's do that.

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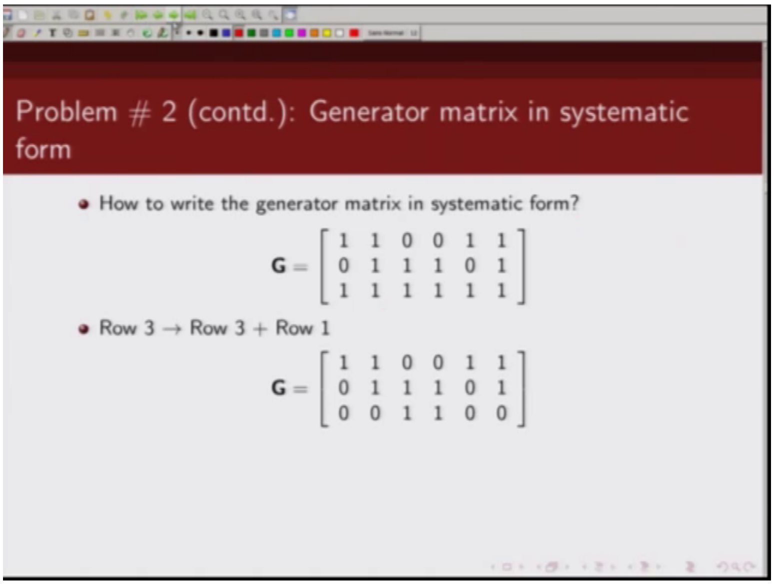

So note, if we want to get, let's say this in the form of identity

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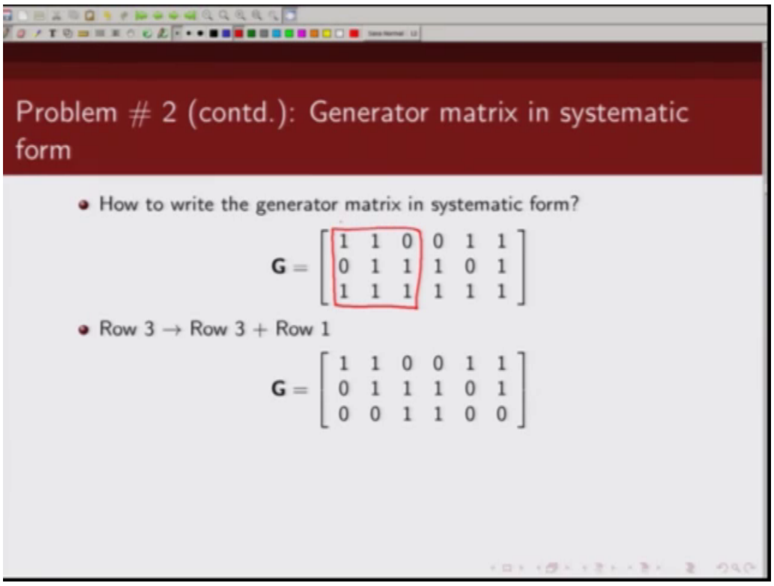

what do we need? We would need basically here, we would need a 0, here we would need a 0, here we would need a 0, here we would need a 0, right?

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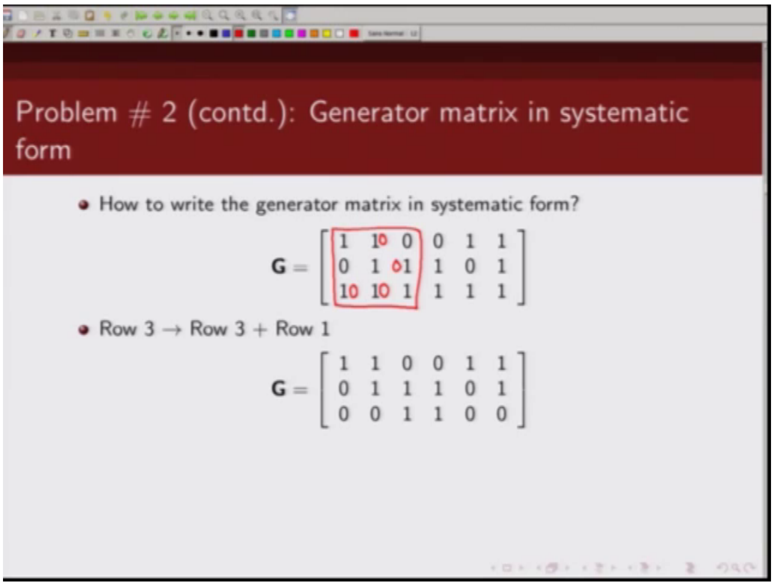

So first let's try to get this 1 to 0. Now how can we make this 0? So if we do this transformation that row 3 is row 3 plus row 1.

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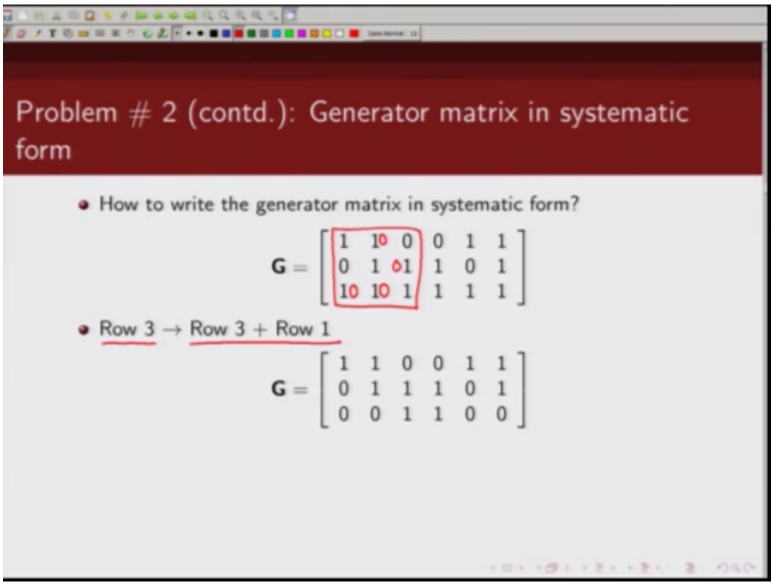

So row 3 is row 3 plus row 1, if we do that then 1 plus 1, this will be 0. 1 plus 1, this is 0.

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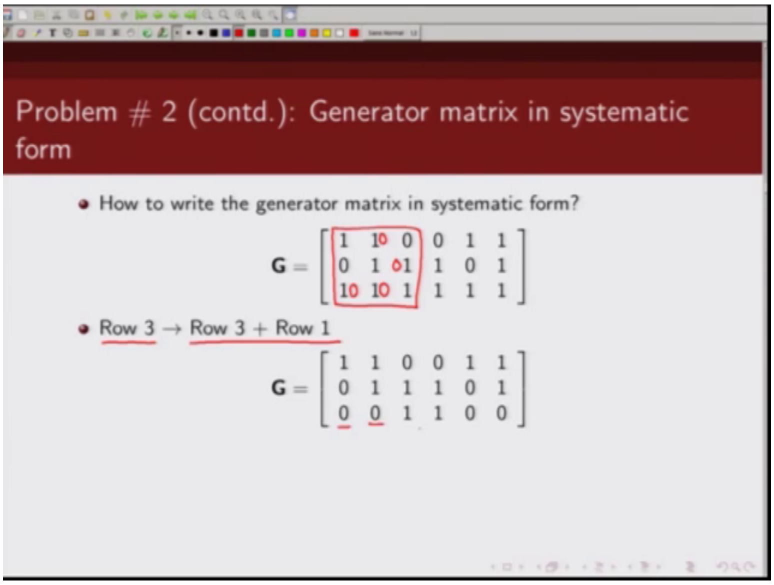

. 0 plus 1, this is 1, 0 plus 1, this is 1, 1 plus 1, this is 0 and 1 plus 1, this is 0, Ok.

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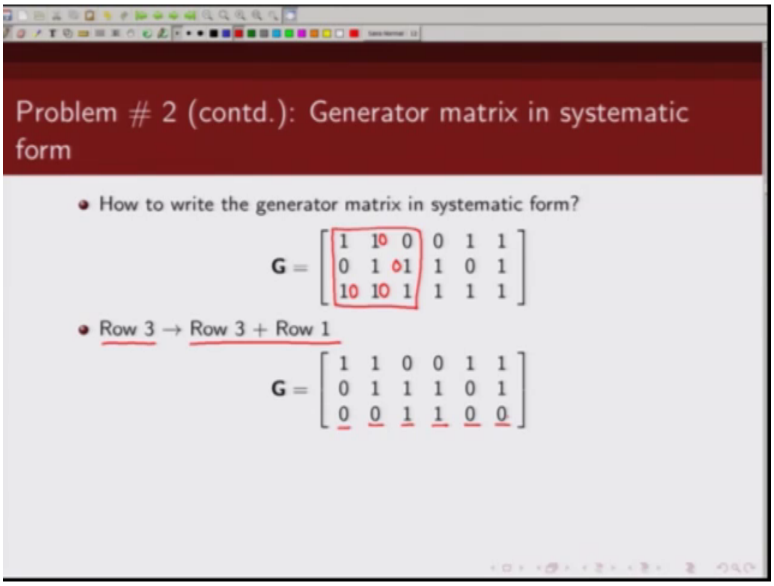

So we got a 0 here, right?

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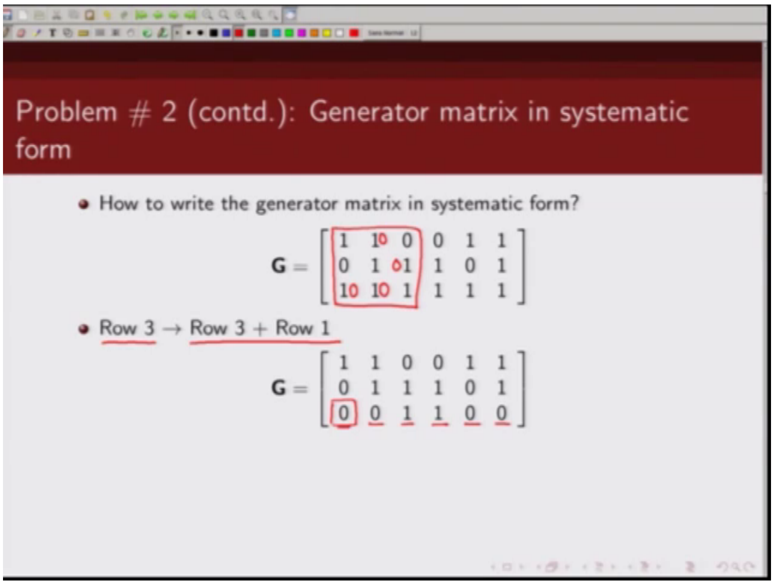

Next, we want a 0 here. We want

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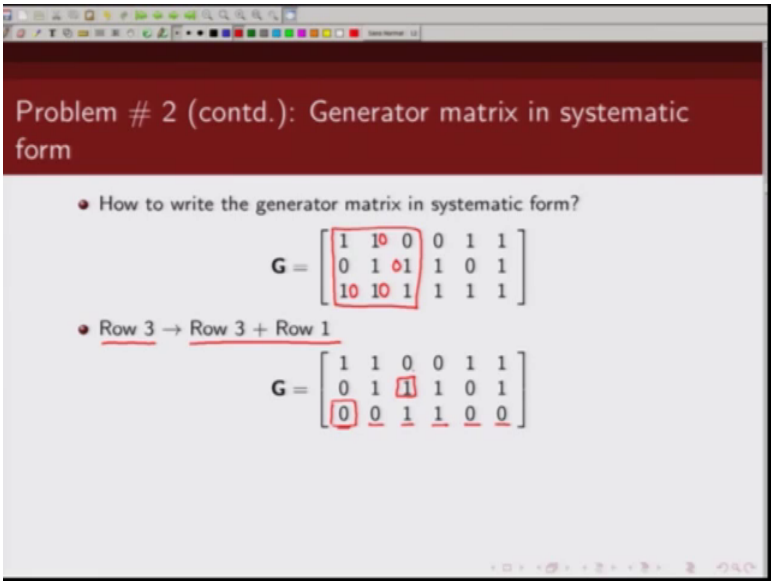

this; we want to make this 0. So how can we do that?

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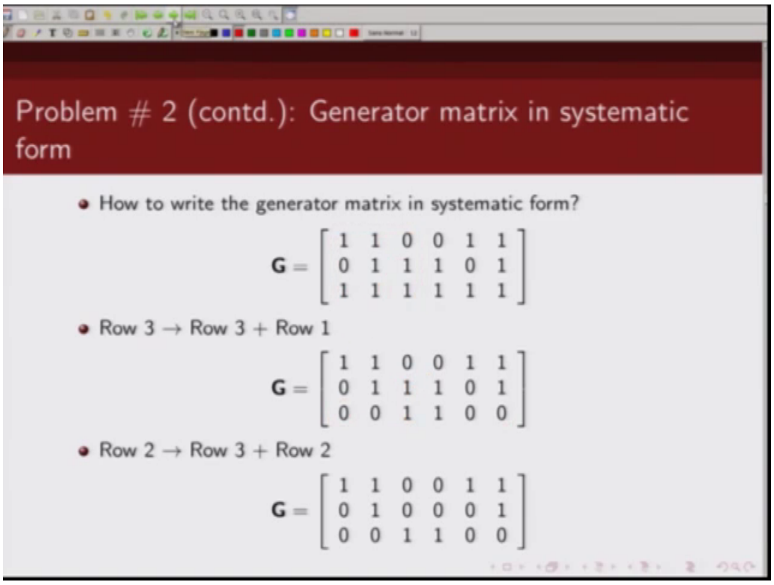

We do this transformation that row 2 is

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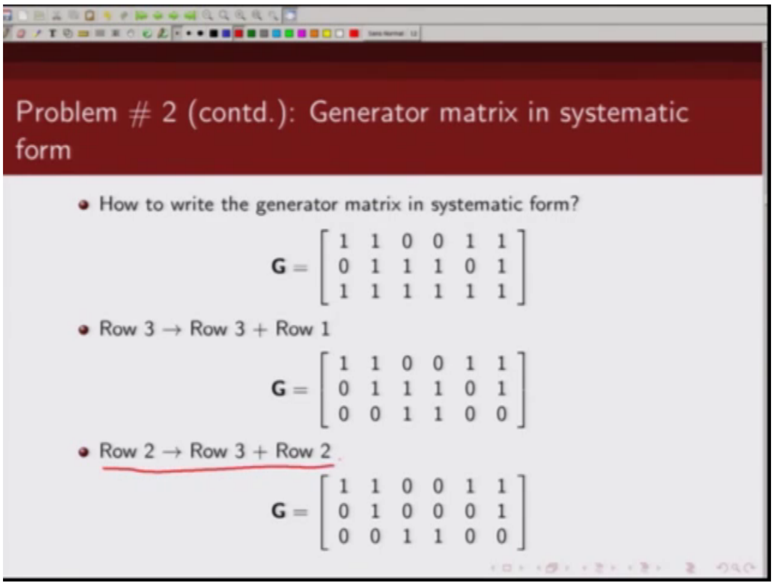

row 3 plus row 1, row 2. So if row 2 is row 2 plus row 3, then what's going to happen? This will remain 0, this will remain 1 but this 1 will become 0. So let's do that. So this is 0 plus 0 is 0. 1 plus 0 is 1, 1 plus 1 is 0, 1 plus 1 is 0, 0 plus 0 is 0, and 1 plus 0 is 1, Ok.

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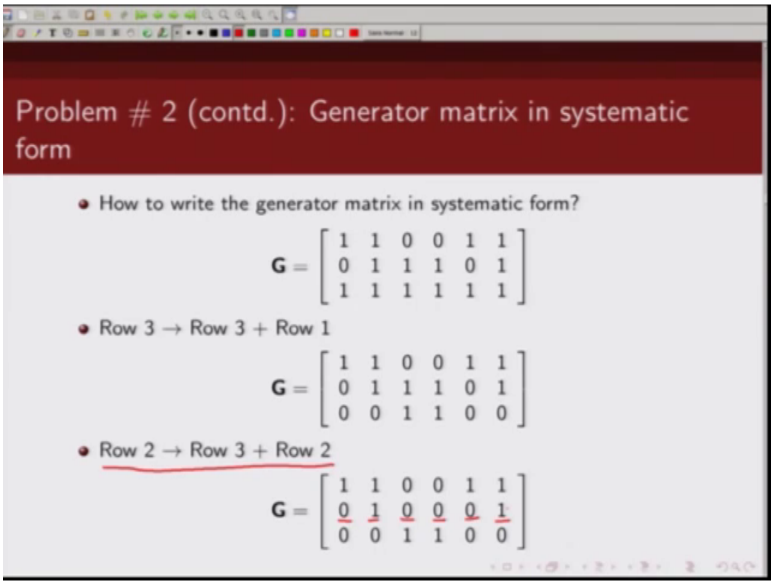

So we got these 0's, we got this 0, Ok now what do we have to do? We will have to get

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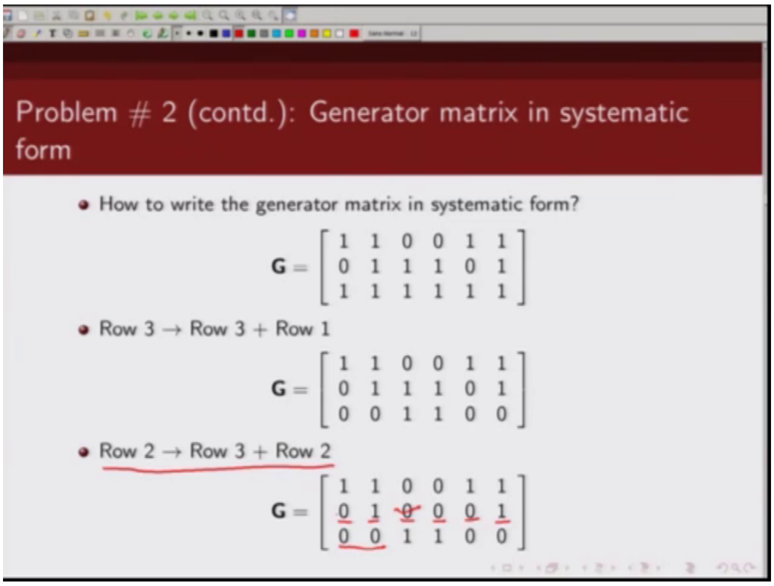

this; here we will have to get a 0.

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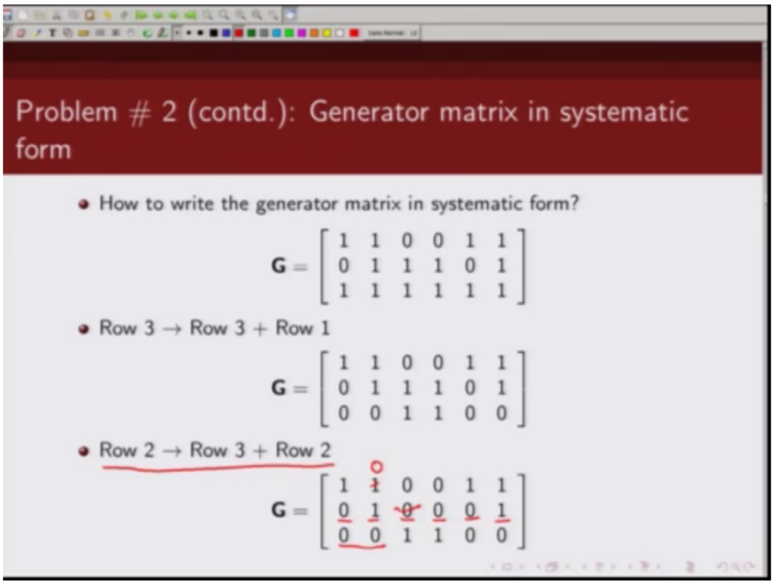

So how can we get a 0 here? We will do

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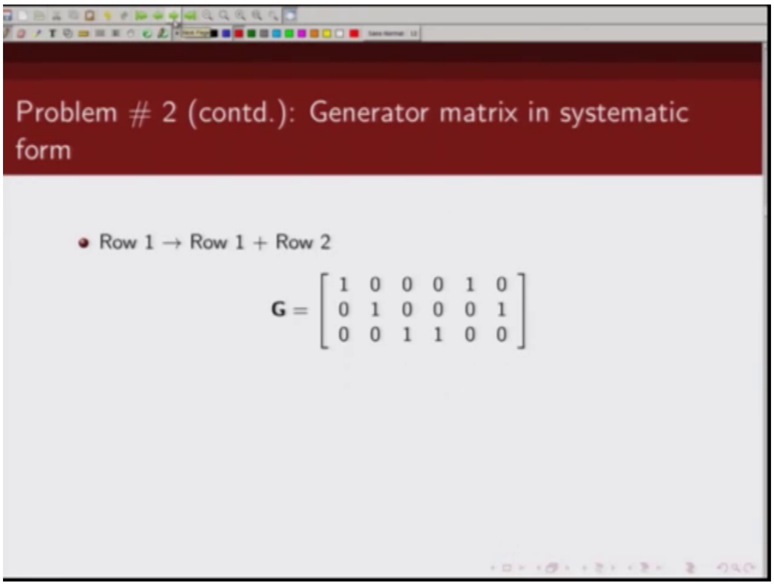

this transformation. We will add row 1 and row 2 and replace row 1 by this. So we are going to add these 2 rows.

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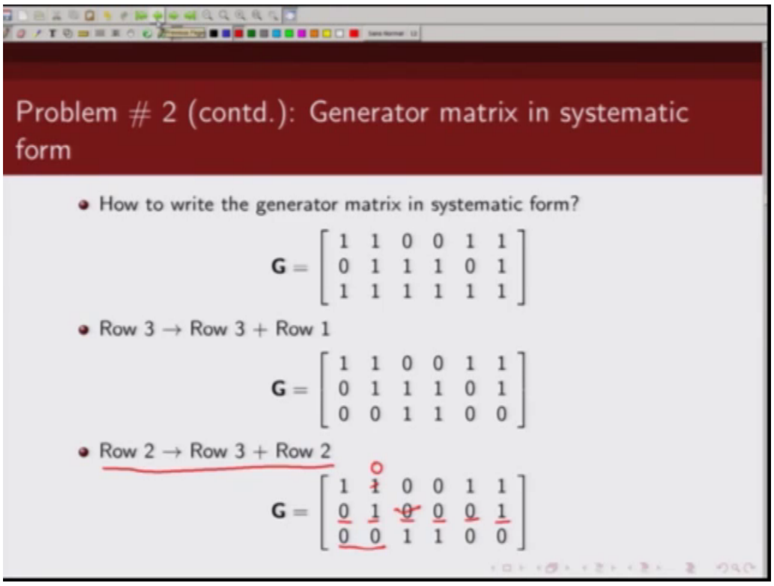

If we add these 2 rows, what's going to happen? This 1 will remain 1. 1 plus 1, this will become 0 and this will remain 0. This will be 0, this will be 1, and this will be 0. So if we do this transformation, what

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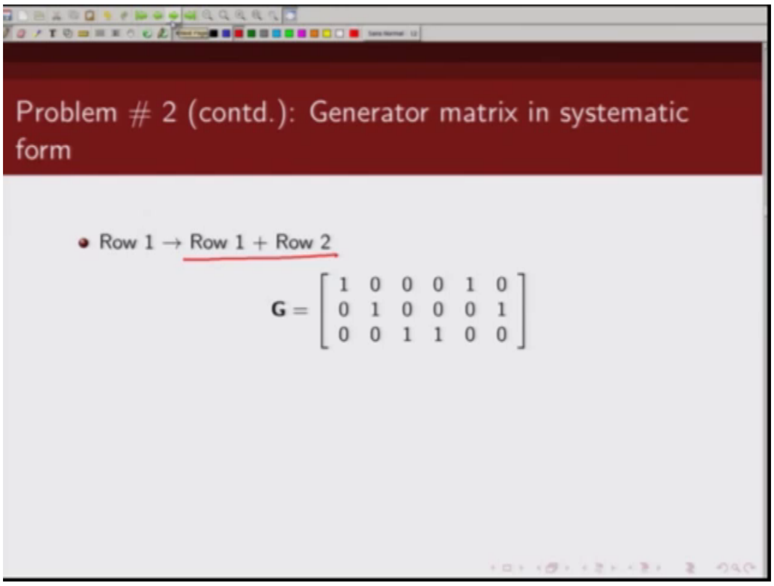

we get is this. Now

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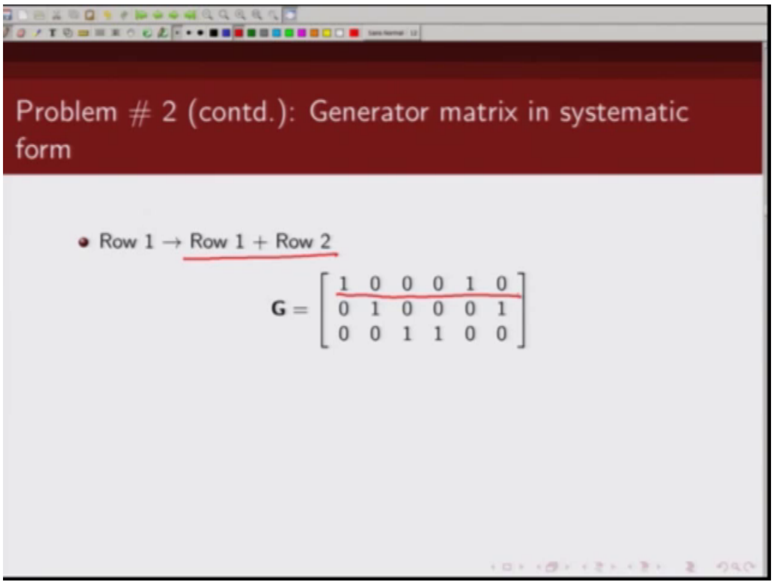

note that this is our identity matrix.

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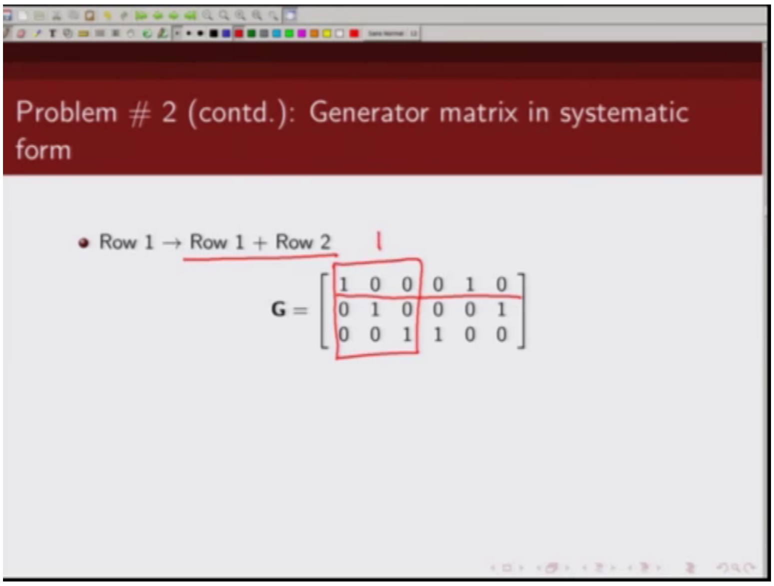

This is 3 cross 3 identity matrix and then this is your another matrix P, Ok. So

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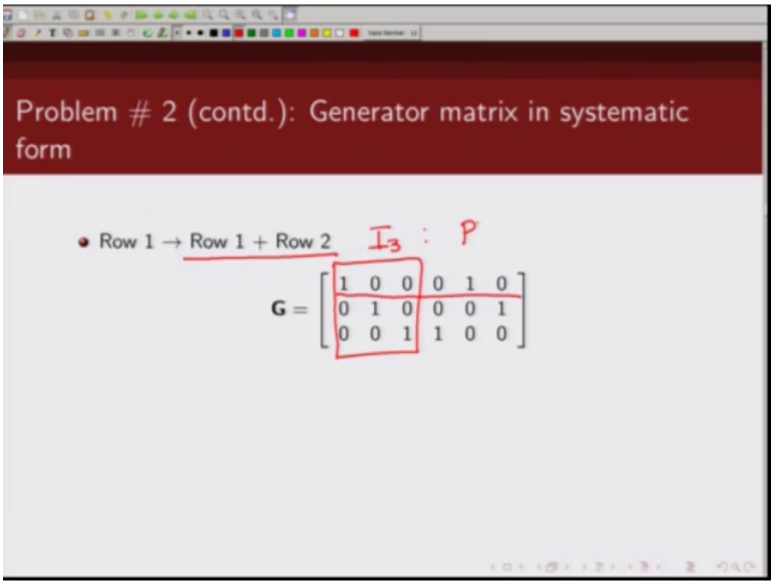

by doing elementary row

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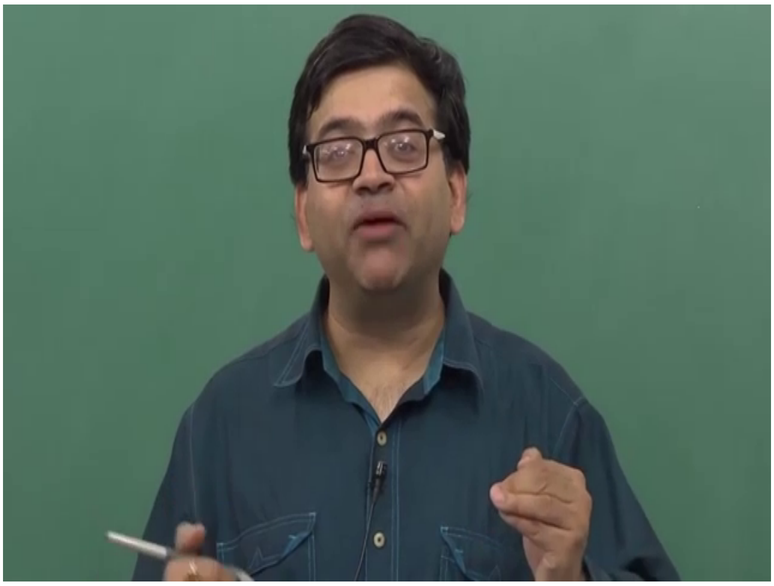

operation, we are able to get our generator matrix in a systematic form. And

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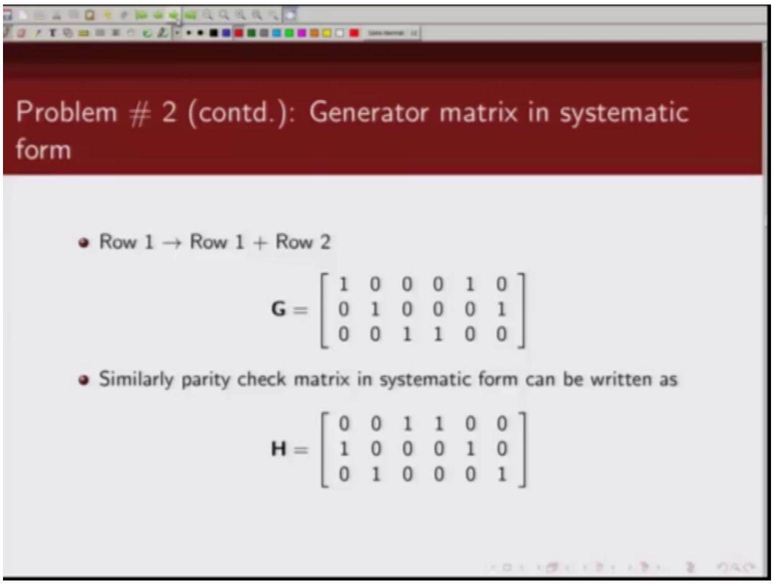

if we have a generator matrix in a systematic form we can very easily find the parity check matrix in systematic form. So this is like I k P then this H matrix

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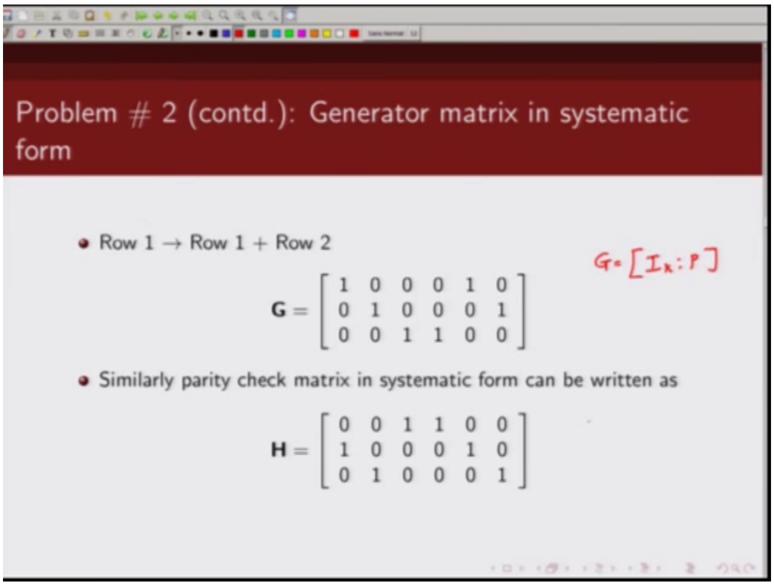

will be P transpose I n minus k.

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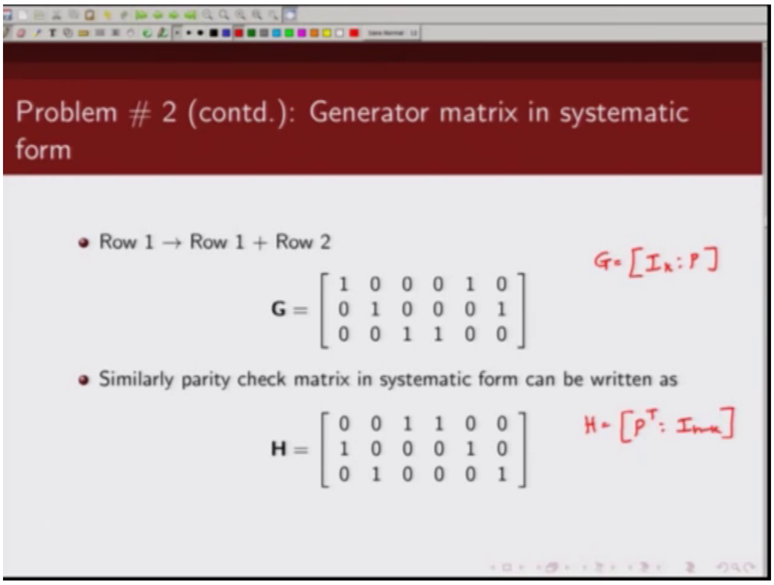

So this, this is basically your P transpose. So this

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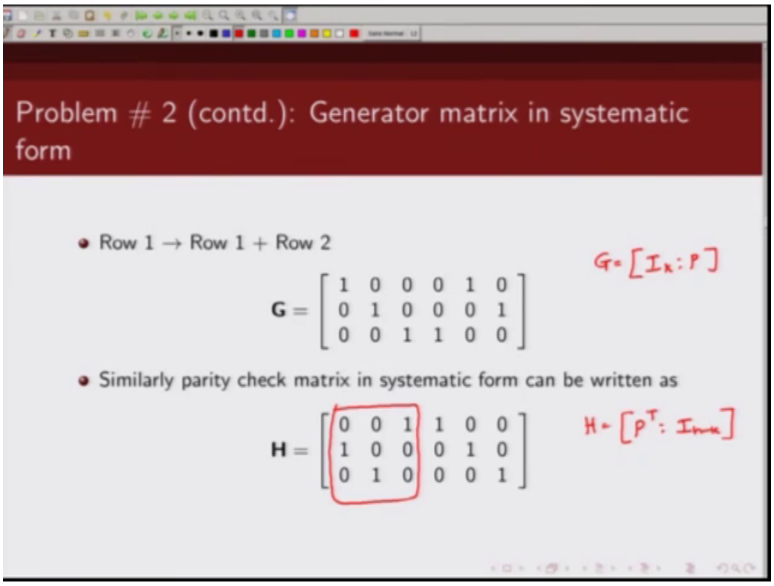

is 0 1 0, this will come here, 0 1 0, 0 0 1, this is 0 0 1. And 1 0 0 is this, 1 0 0.

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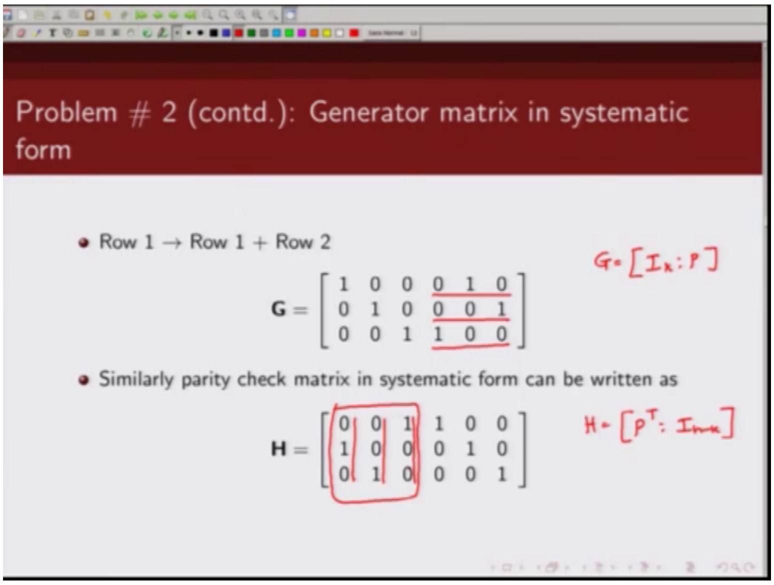

And then you have this identity matrix which is here,

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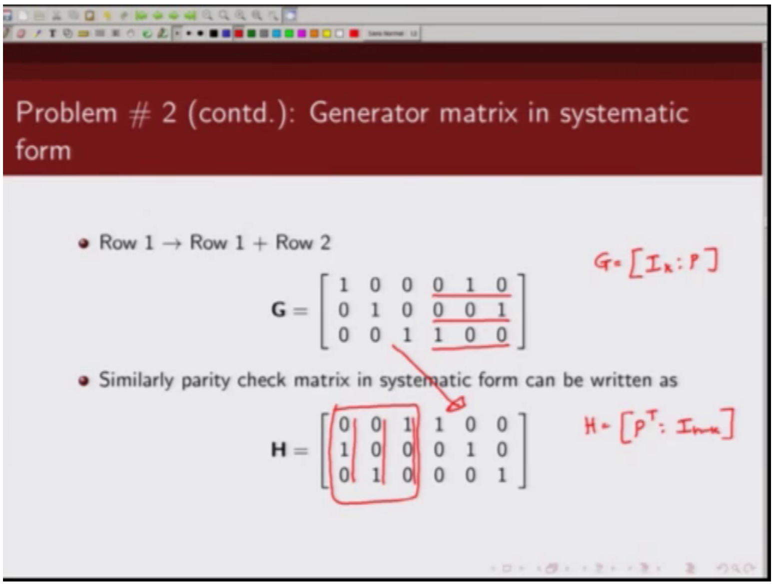

Ok. Next

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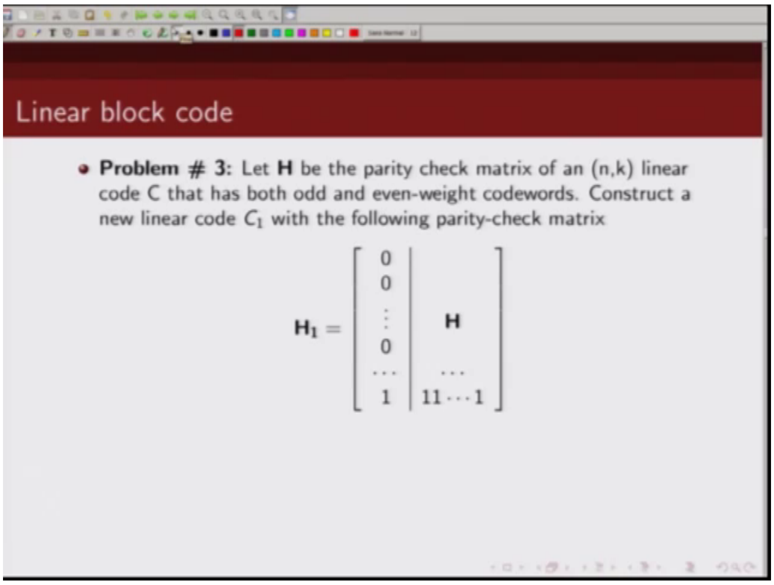

we are given a parity check matrix H of a linear

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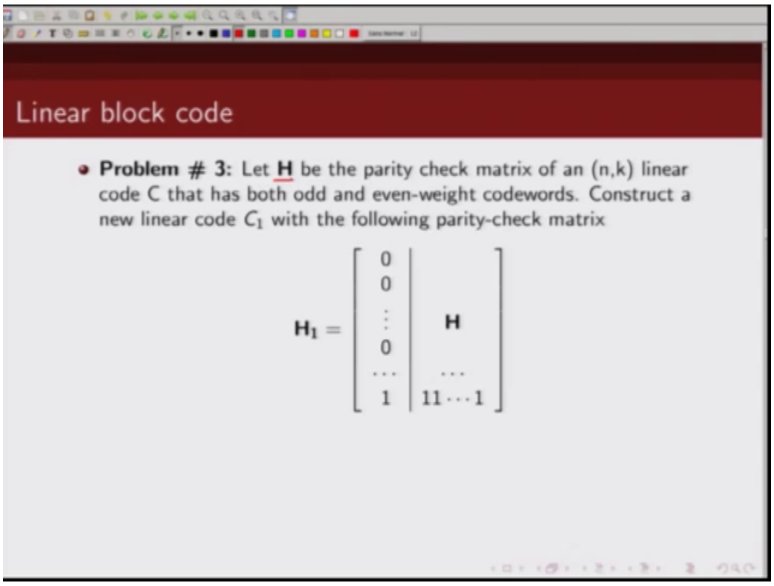

block code

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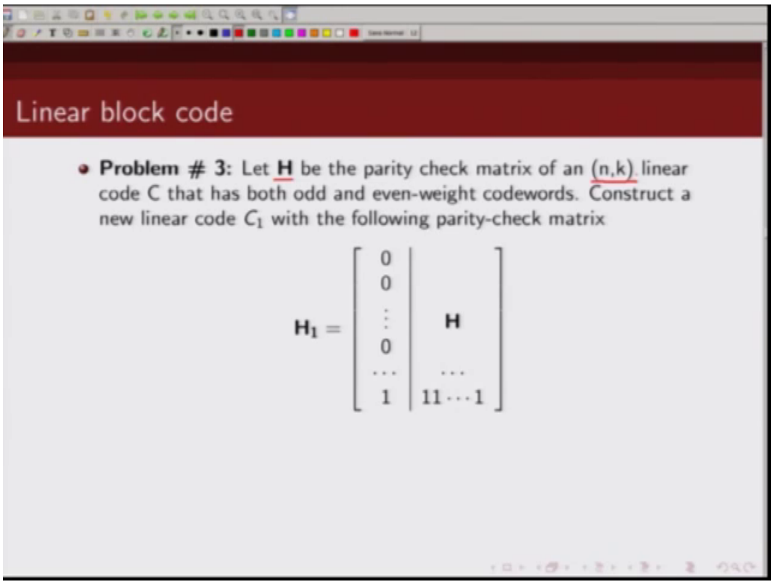

with parameter n and k. And it is given that this code C has both odd weight codewords and even weight codewords. In

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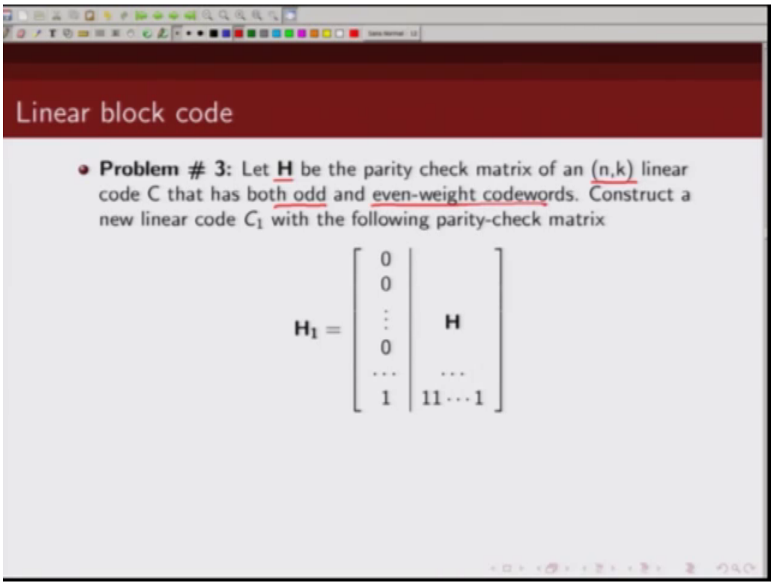

other words, the number of 1's in the codewords, it contains both odd number of 1's as well as even number of 1's. And we are constructing a new code that we are calling as C 1

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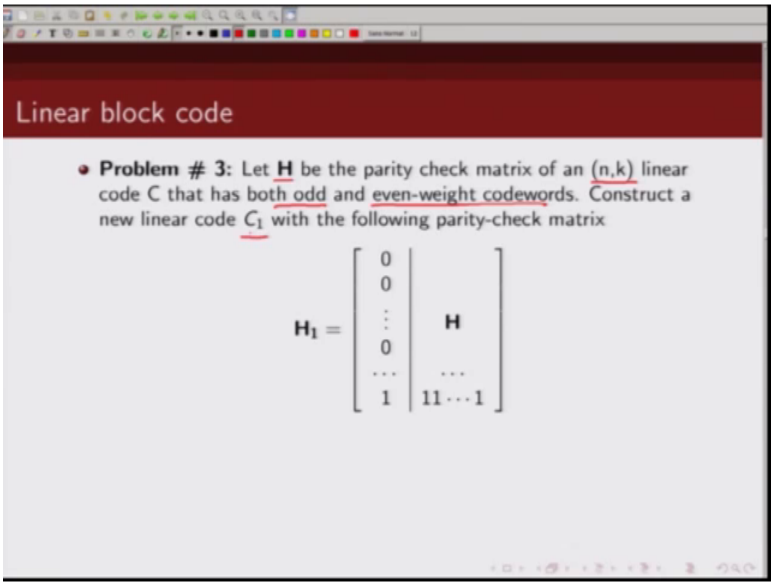

and the parity check matrix of the new code C 1 is given by this.

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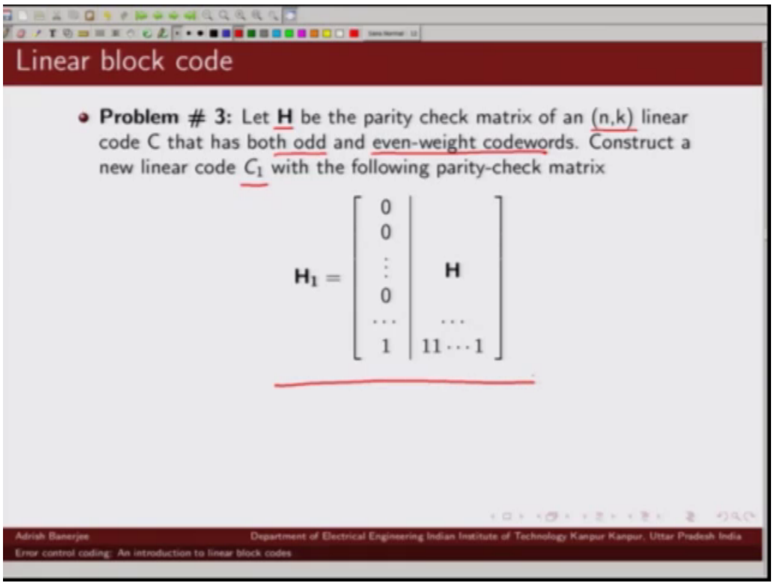

So how do we find this new matrix, parity check matrix H 1? We are adding a new column which is 0 in the initial rows except in the last row which is a 1 and here we have put our original n minus k cross n matrix. And the last row

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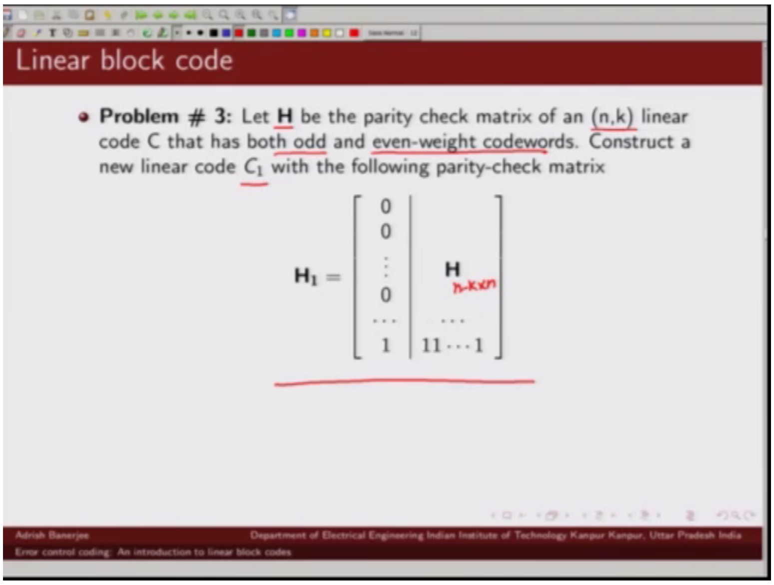

is basically all 1's,

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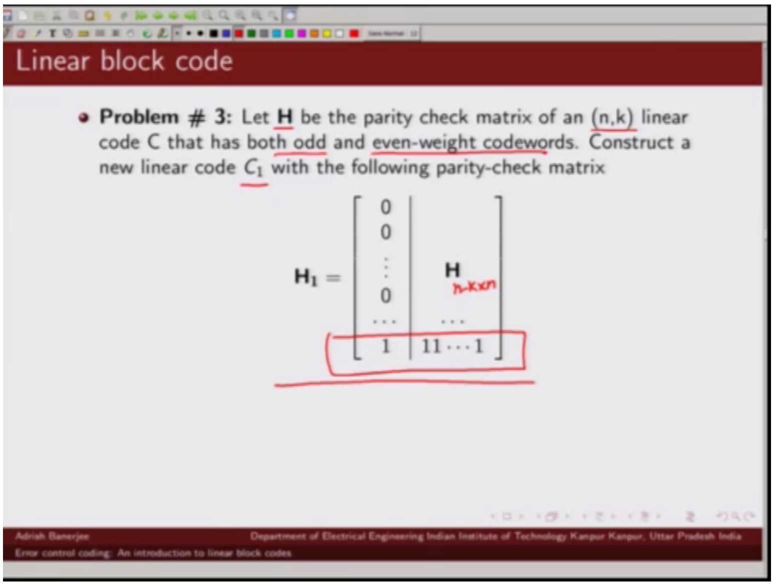

Ok. So the dimension of this matrix is, so number of rows is n minus k plus 1 and number of columns are

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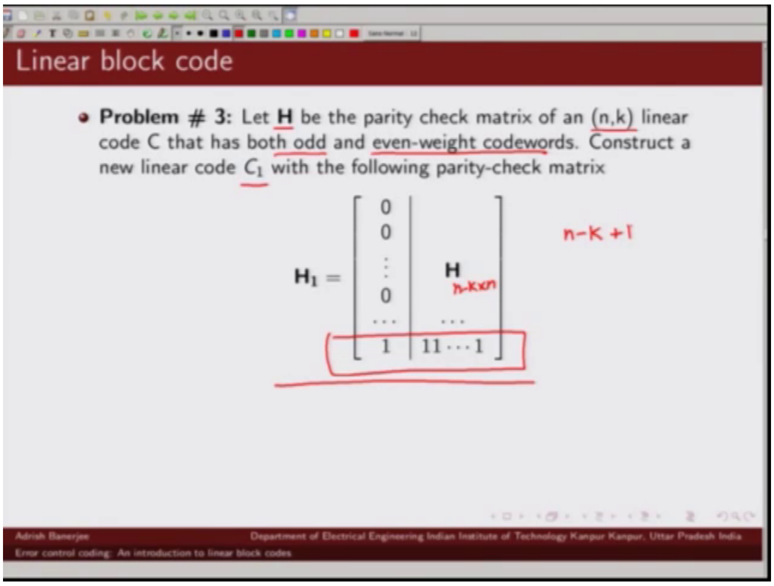

n plus 1

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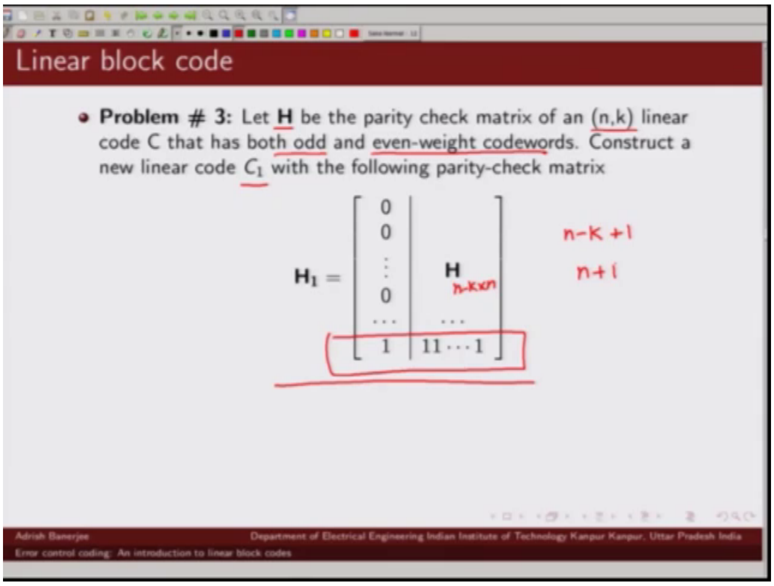

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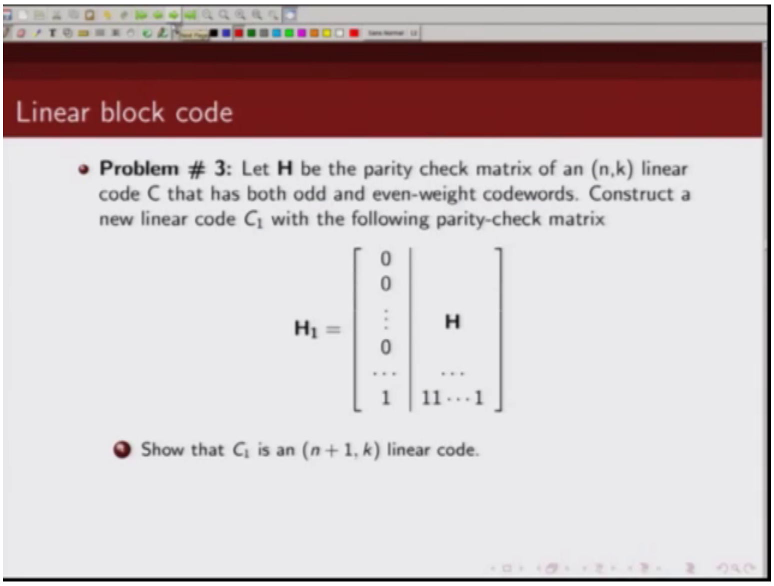

Now you are asked to show that the code generated by this

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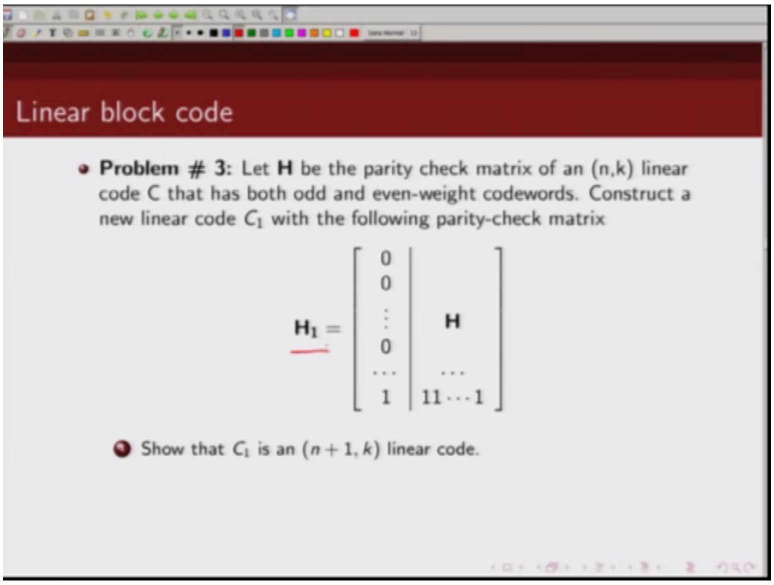

parity check matrix H 1 is a linear code with parameters

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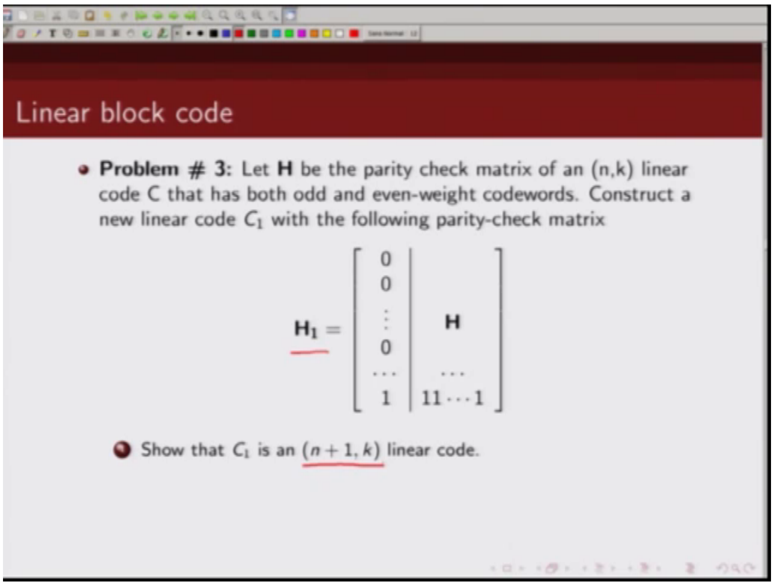

n plus 1 and k. Second thing you are

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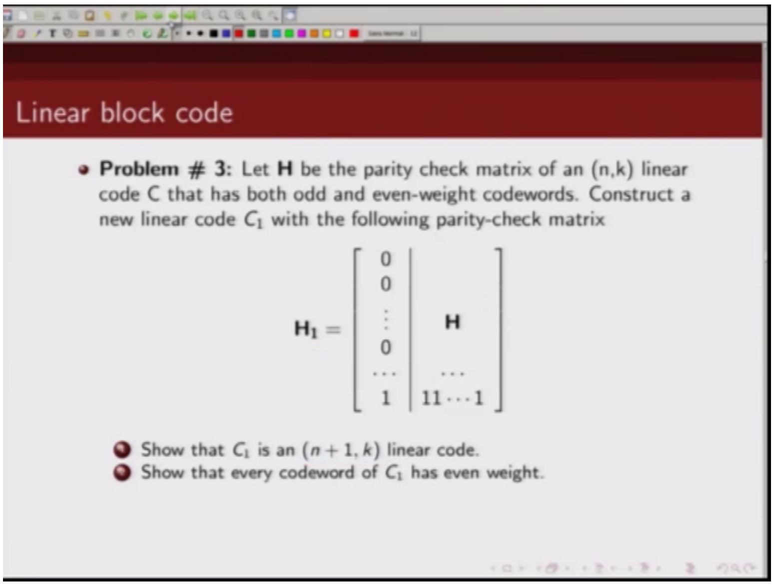

asked to prove is that all the codewords

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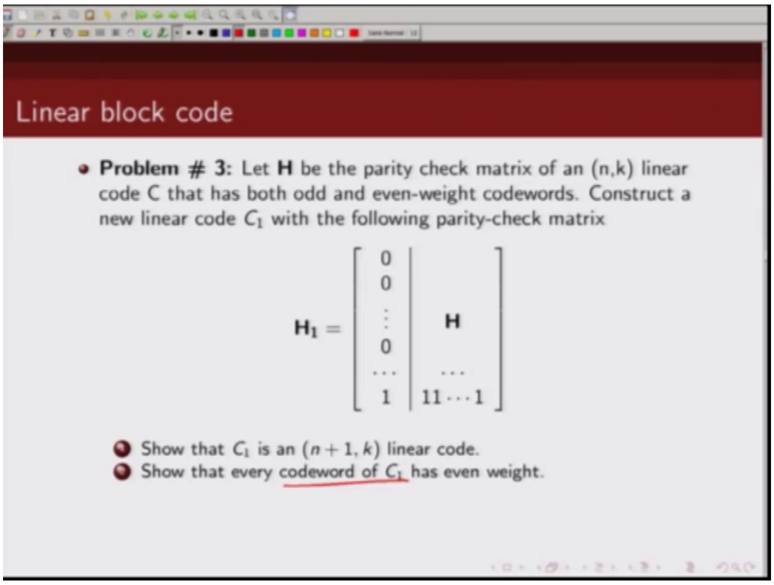

of this new code C 1 will have even weight. That means they will have even number of 1's in them.

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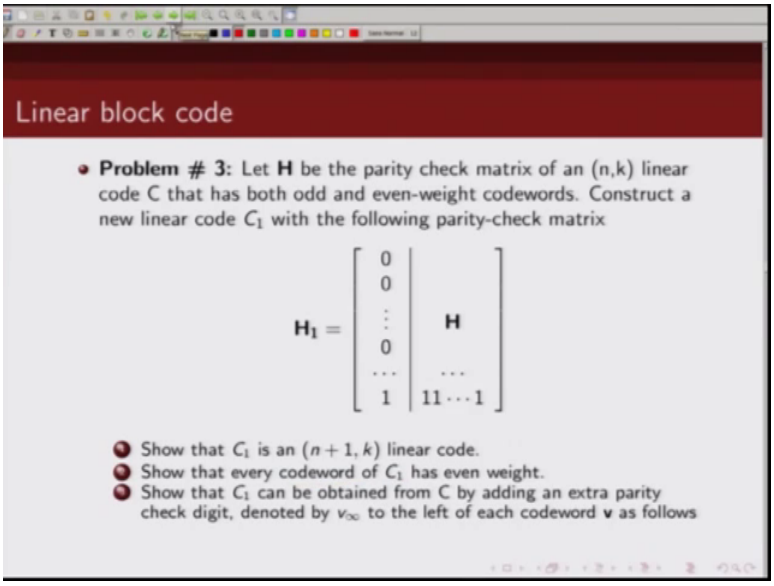

The third thing you have to prove is, this new code C 1 is obtained from old code C by adding an additional parity bit which we are denoting by v infinity

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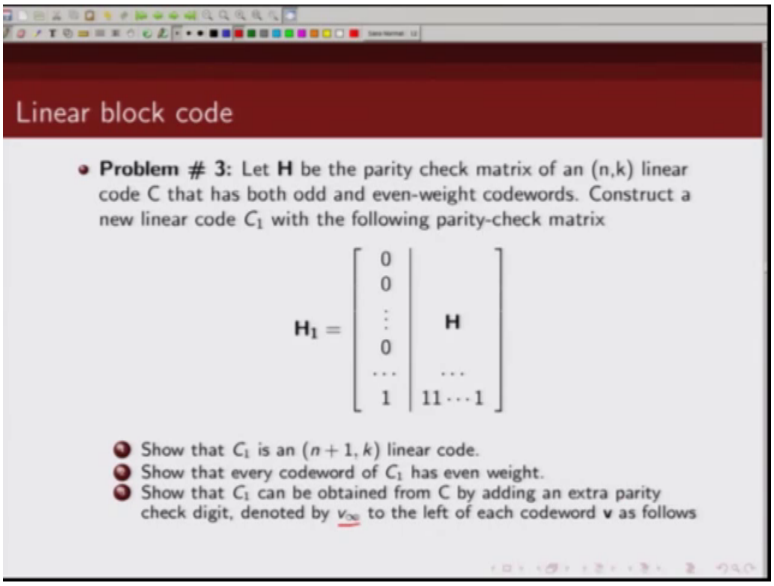

to the left of this codeword and how do you select this parity bit v infinity?

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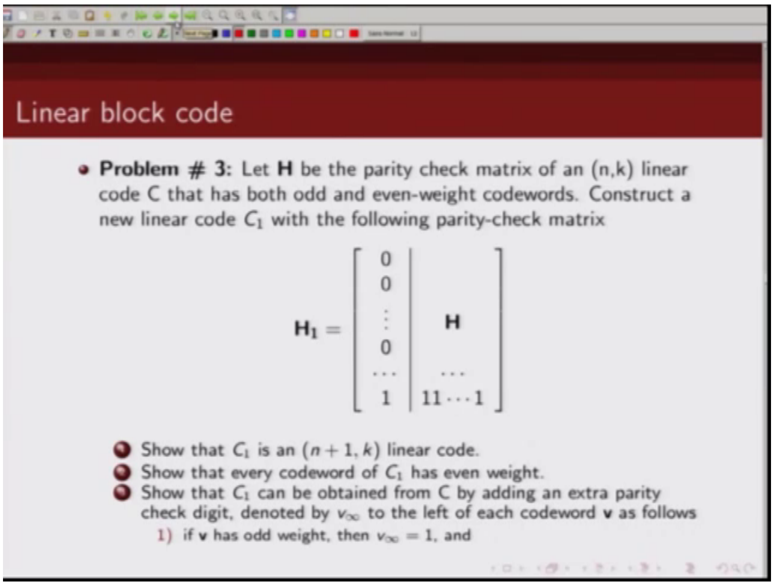

If the original codeword has odd weight, then you put v infinity as 1 otherwise

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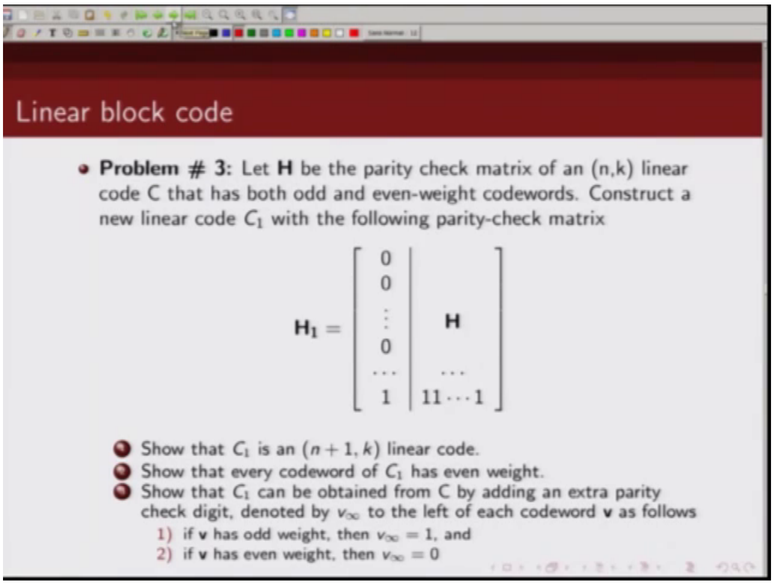

if the original codeword has even weight then you put this v infinity as 0.

So let's prove one by one. Let's first prove this that code generated by this new parity check matrix

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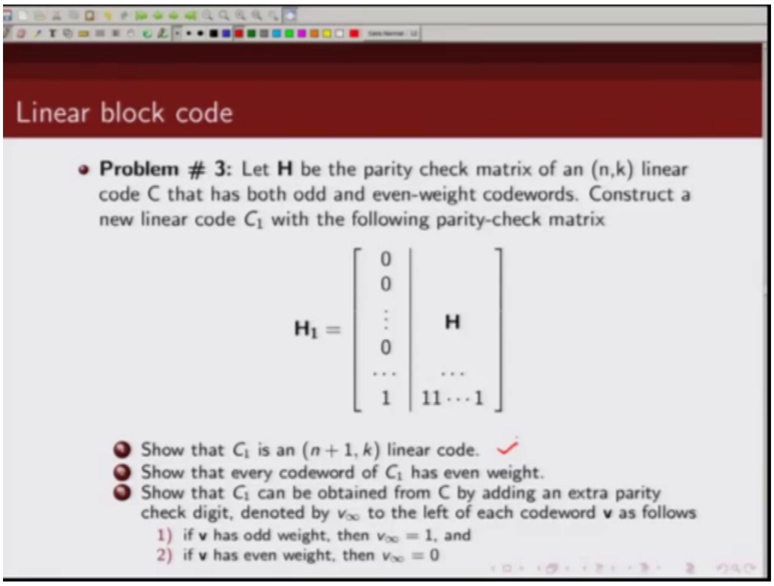

is basically a new code with n given by n plus 1 and k given by k. So

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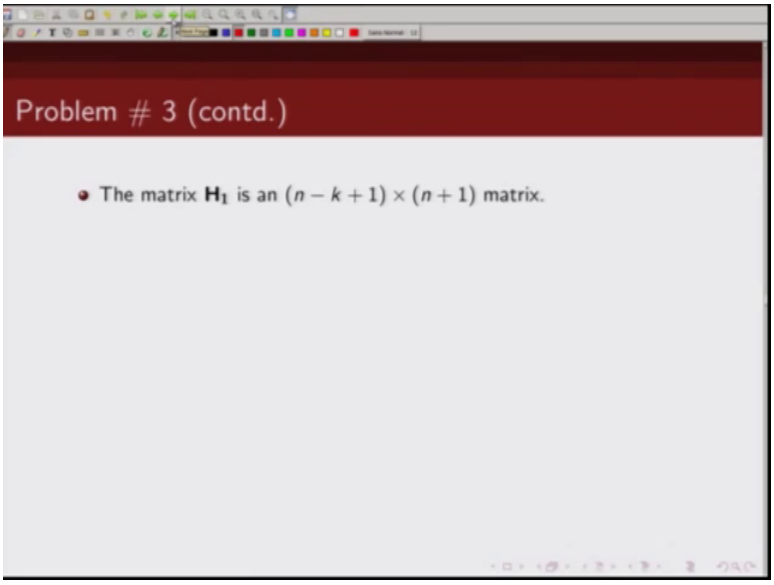

as we know that this H matrix has these dimensions because we are adding

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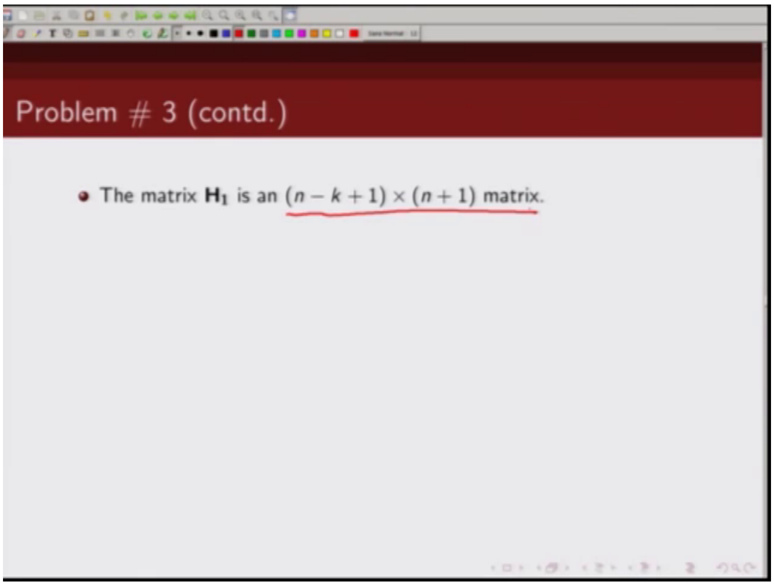

a new column and we are adding a new row. Next,

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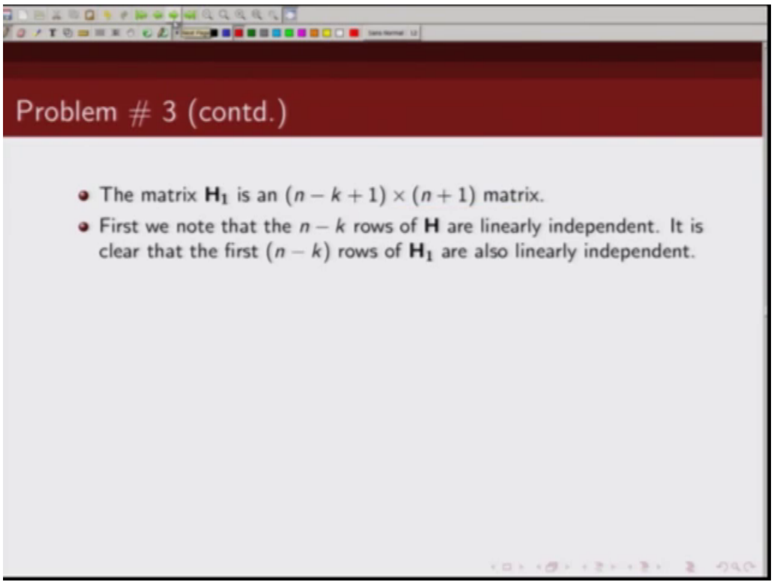

now what is the rank of the original matrix H? The rank of the original matrix H is

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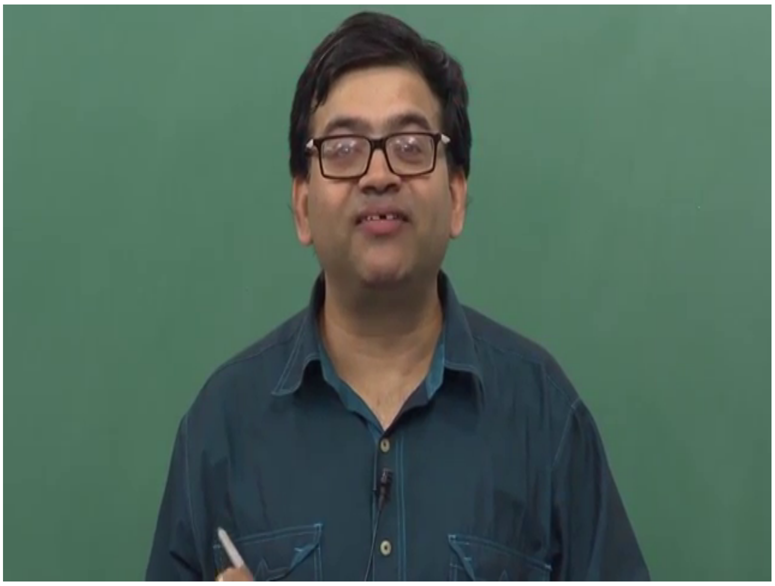

n minus k. That means

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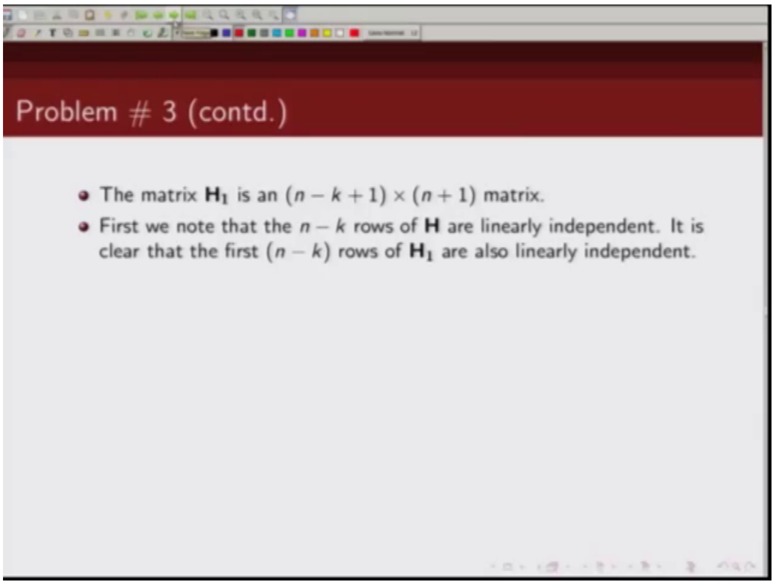

the n minus k rows of the original

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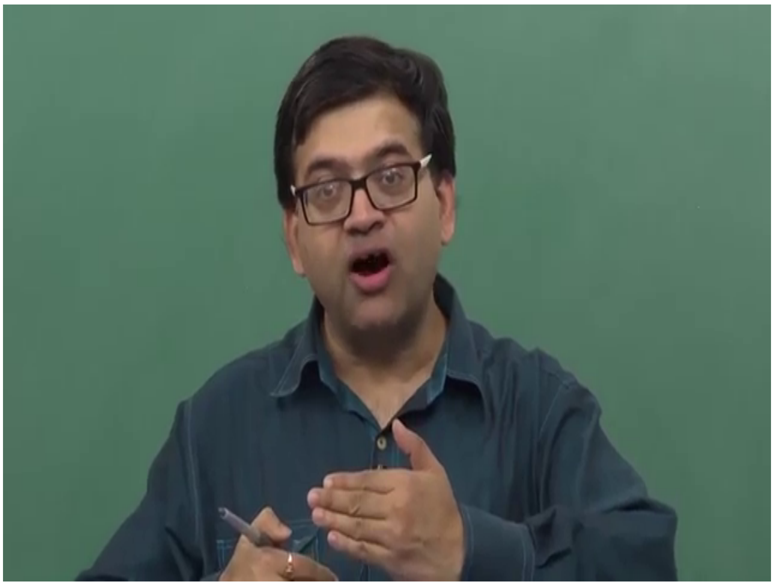

parity check matrix H are linearly independent, Ok. Now go back

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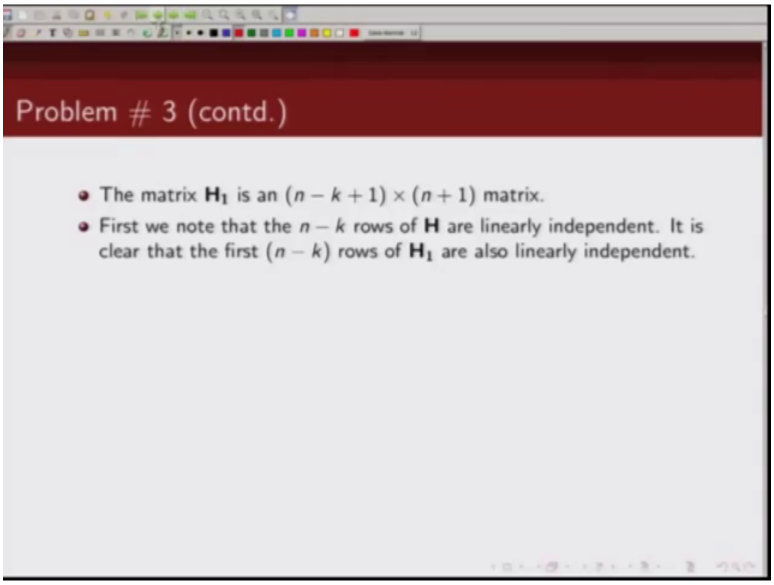

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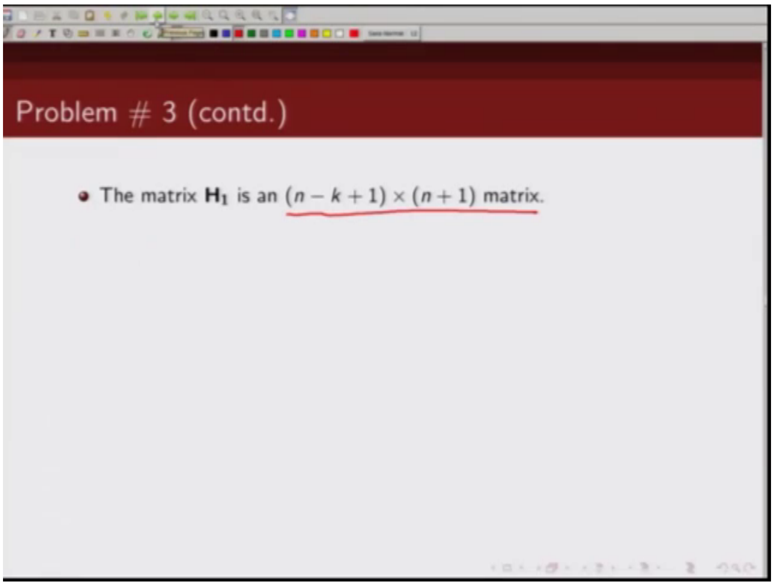

and look at the new

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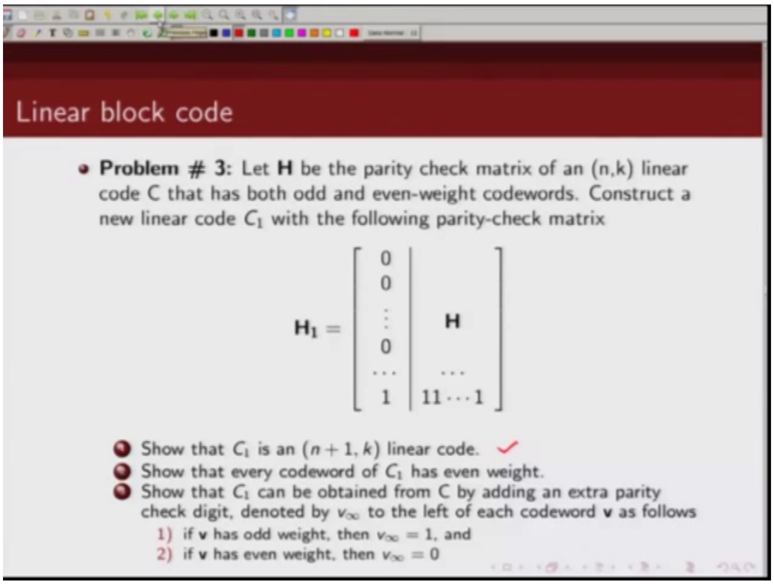

construction. So these n minus k rows are linearly independent.

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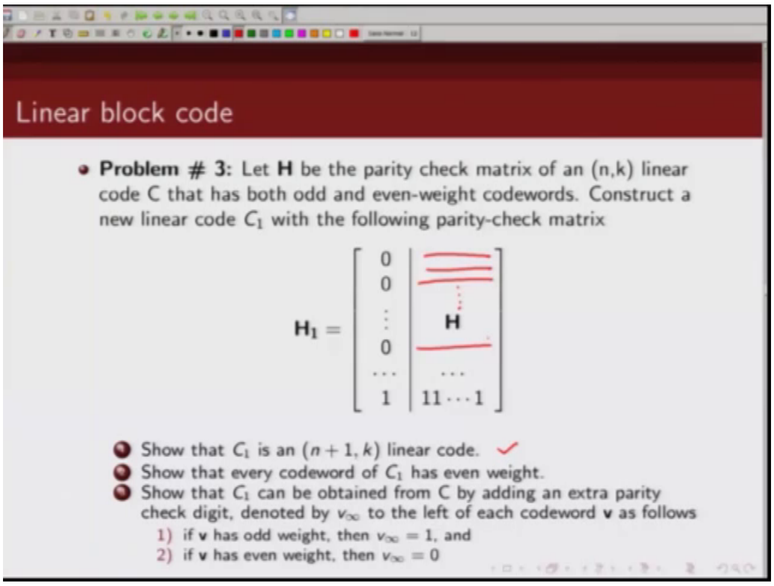

And what have we added here? We have added 0 here. So these new rows, these new n minus k rows will also be

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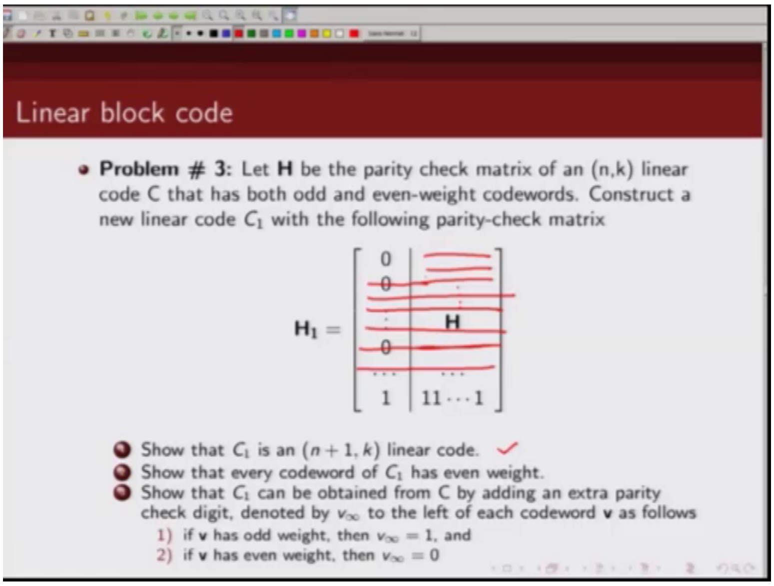

linearly independent.

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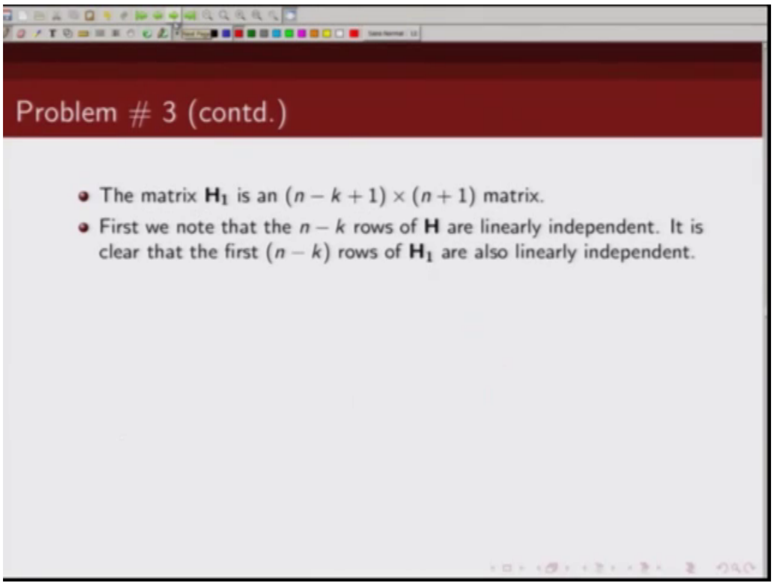

So that's what we are saying that since n minus k

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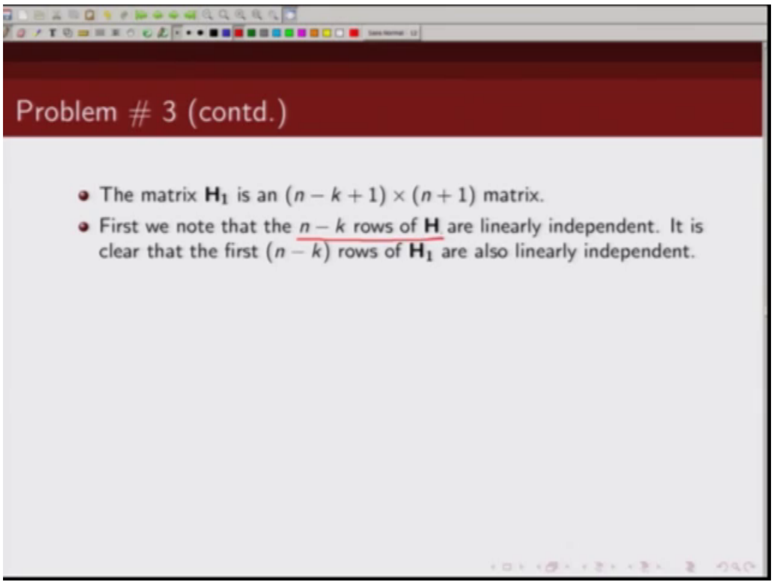

rows of the original parity check matrix H are linearly independent, so

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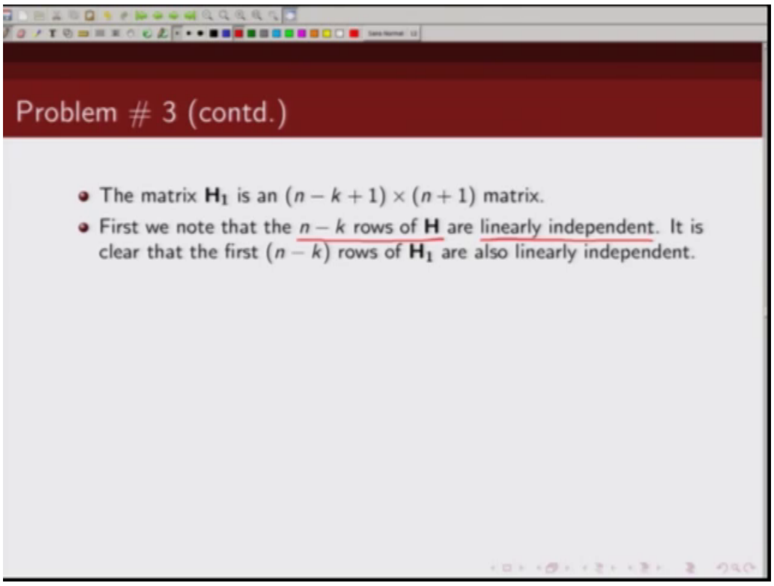

the first n minus k

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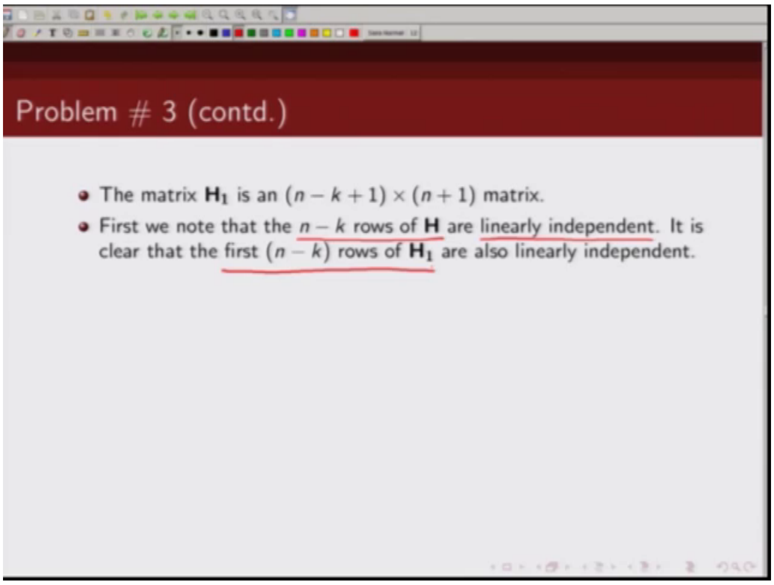

rows of the original parity check matrix H 1 will also be linearly independent.

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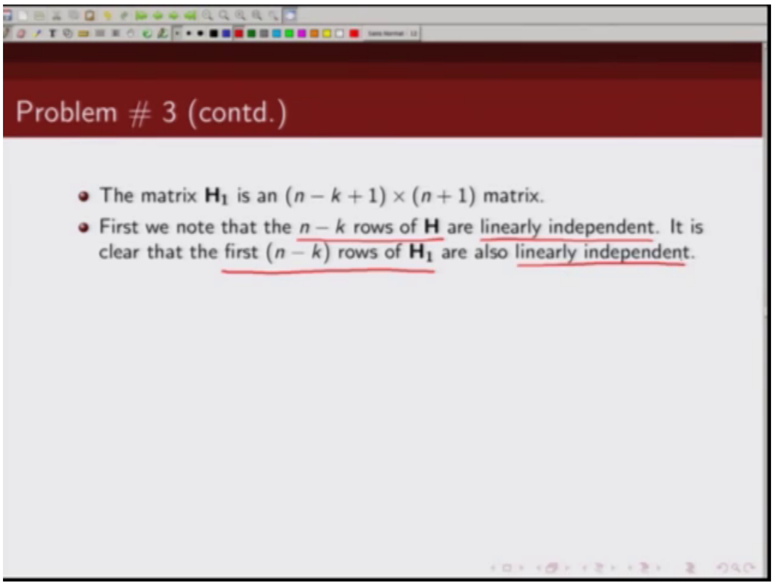

Now let's look at

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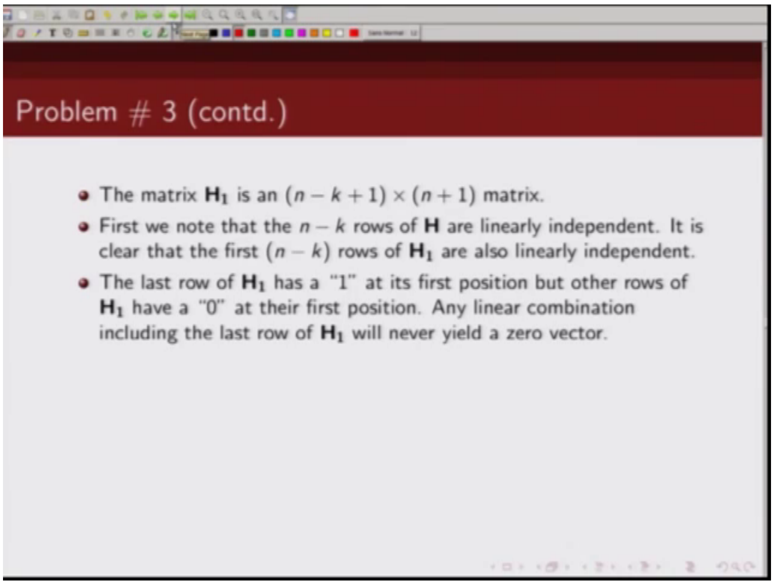

the last row of this new parity check

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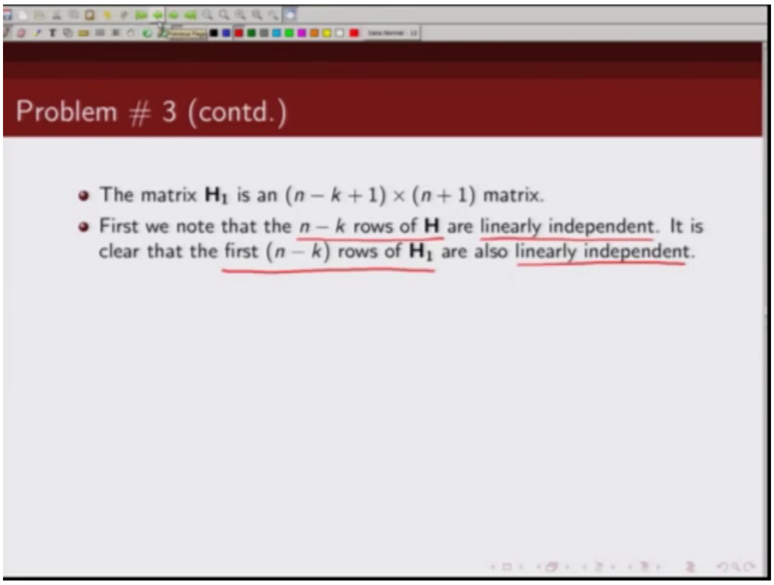

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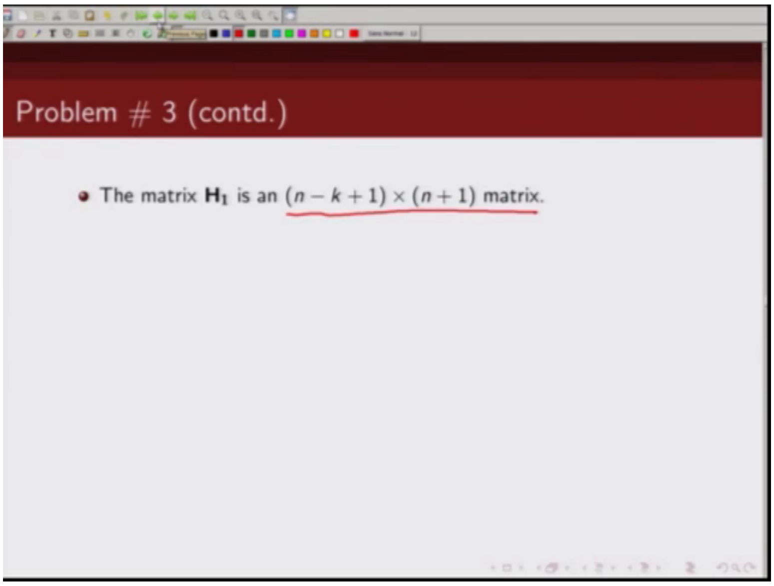

matrix H 1.

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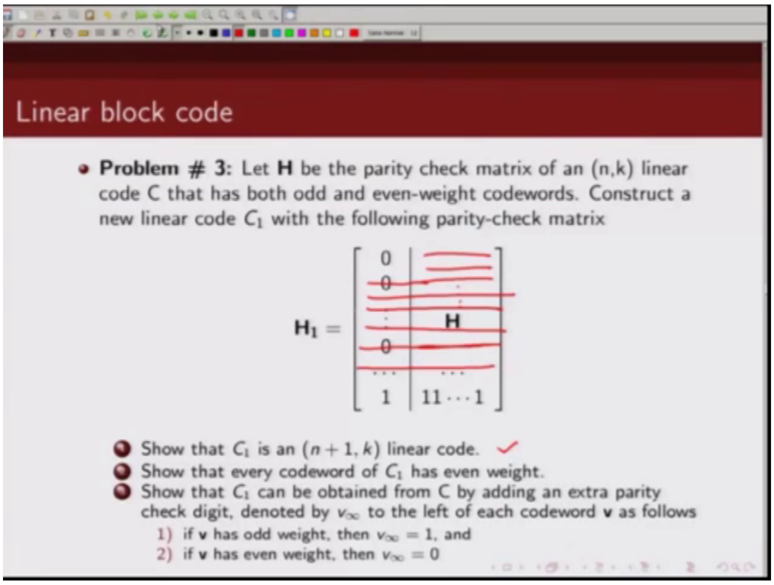

Note that we have 1 here. And these are all

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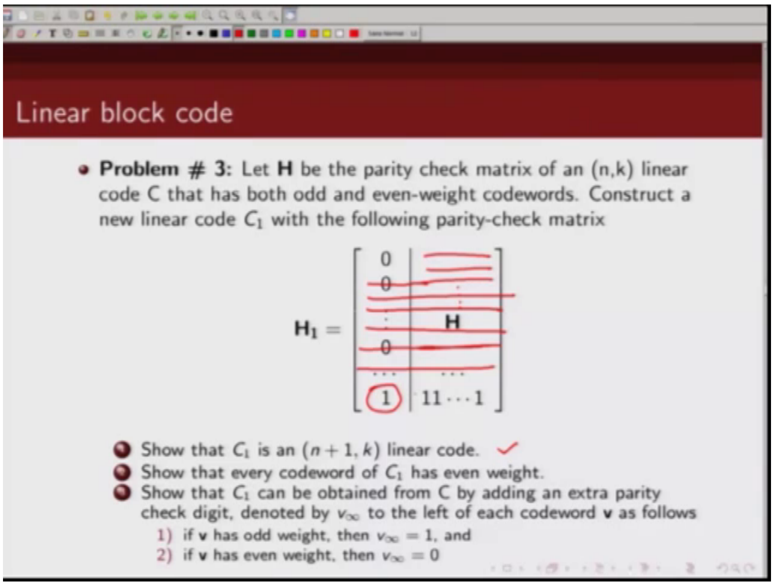

all 1's here.

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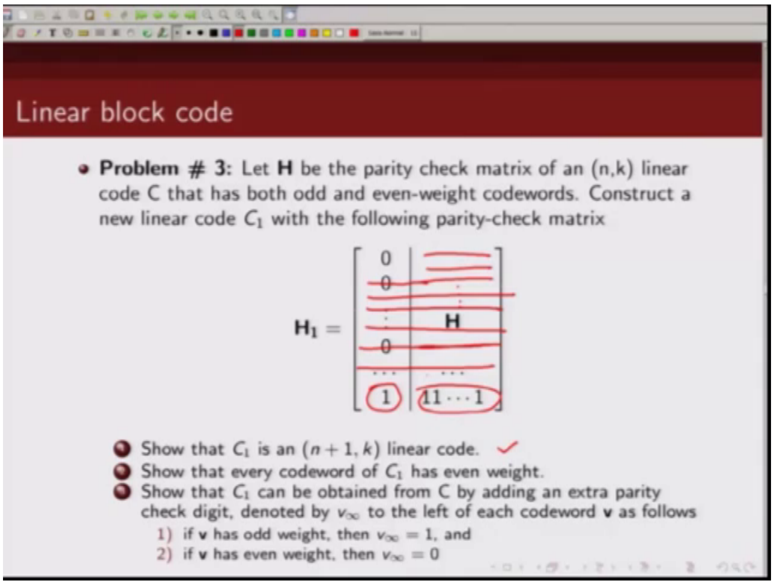

Whereas here, all of these are 0's. So this new row will also be linearly independent from any of the other rows of this parity check matrix H 1.

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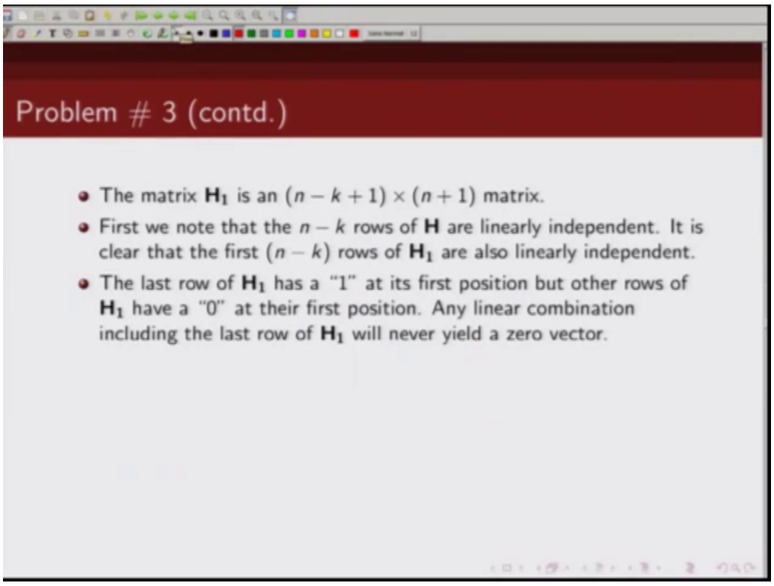

So any linear combination including the last row of H 1 will never result in a all zero vector. So what does it mean? It means that

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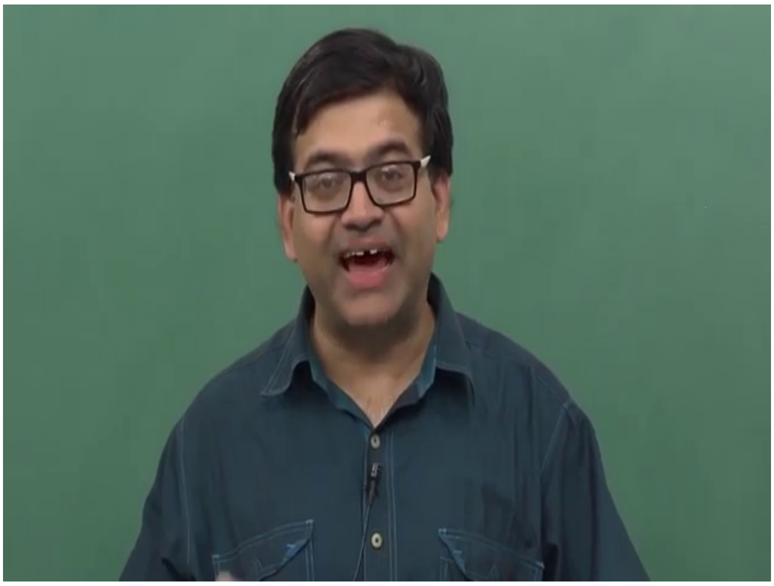

n minus k plus one rows of our new parity check matrix H one are linearly independent.

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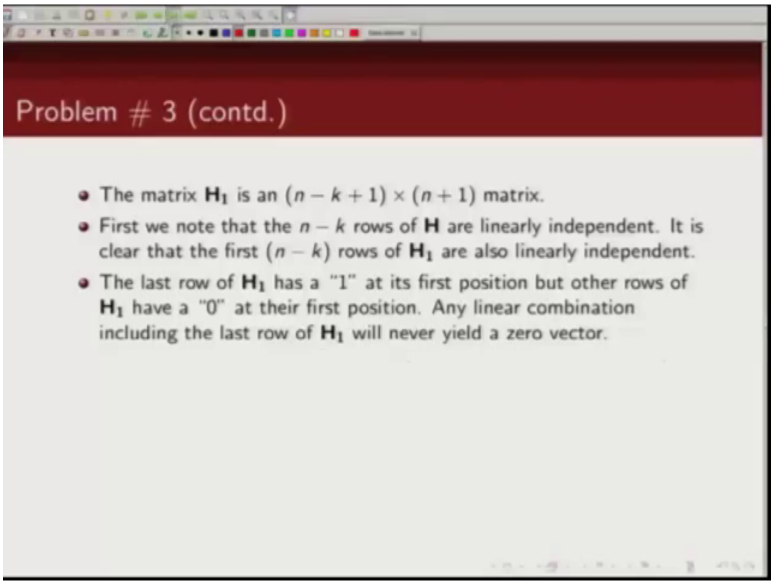

Hence

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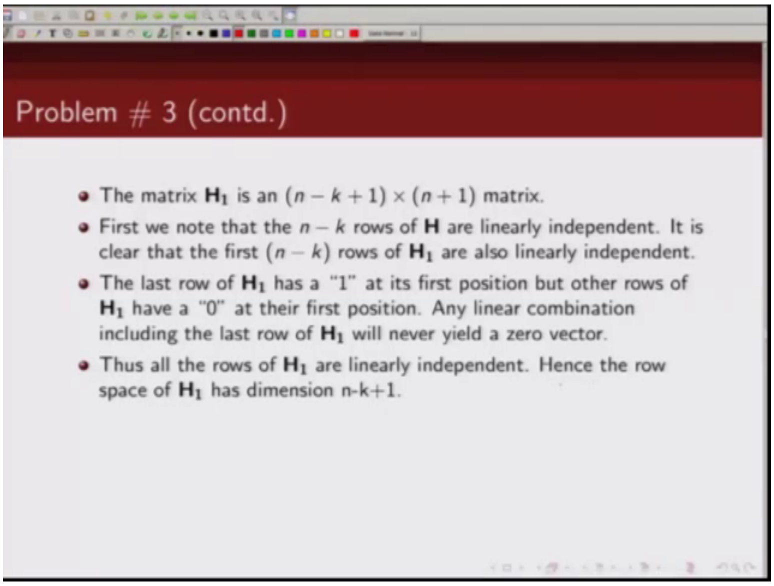

the dimension of H 1 is

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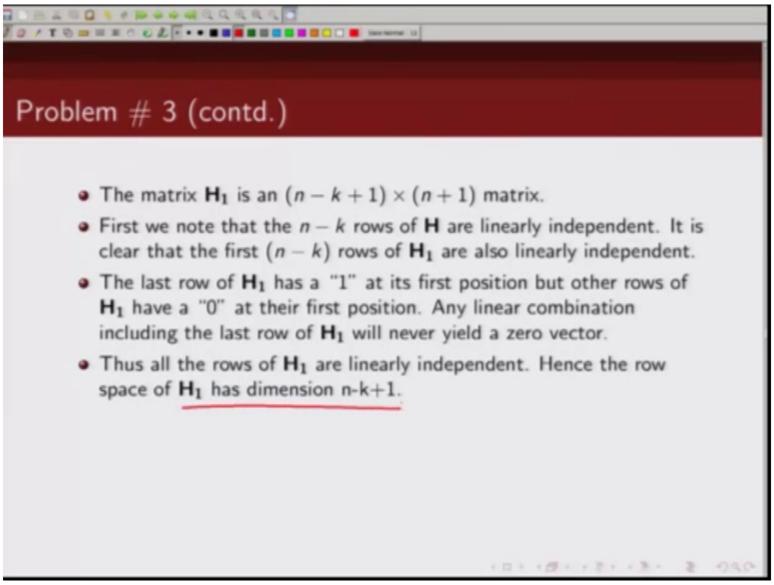

n minus k plus 1. Now how do we find

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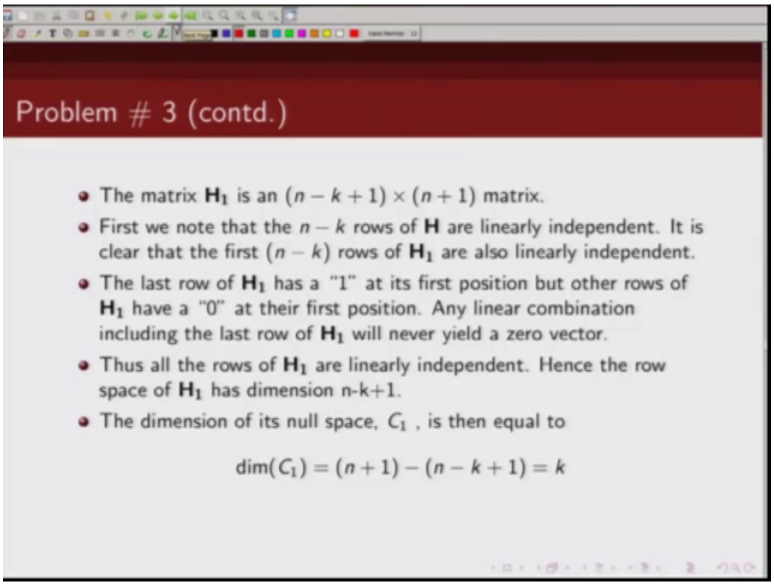

the dimension of basically, the null space of this parity check matrix H 1? It is given by, so number of columns is n plus 1. The dimension of H 1 is given by this.

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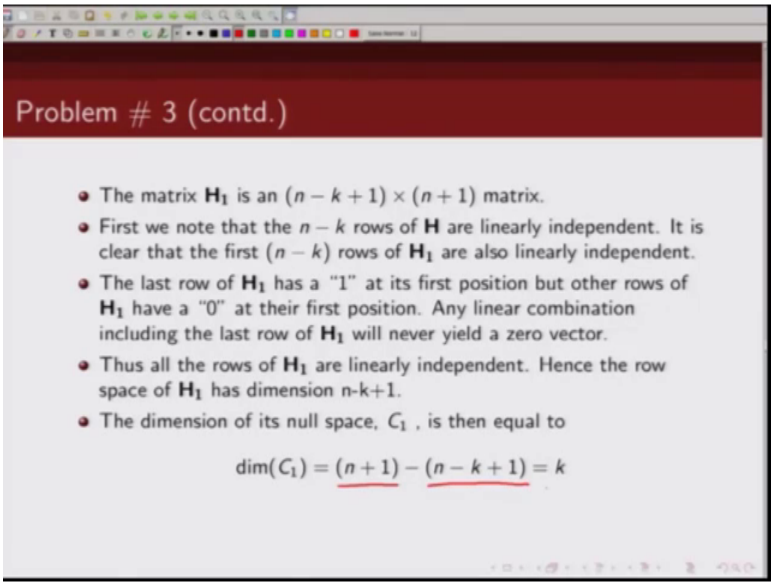

So this is the dimension of the null space of this parity check matrix. So then basically number of information bits is then k and number of coded bits is n plus 1. So this proves that C 1 is an

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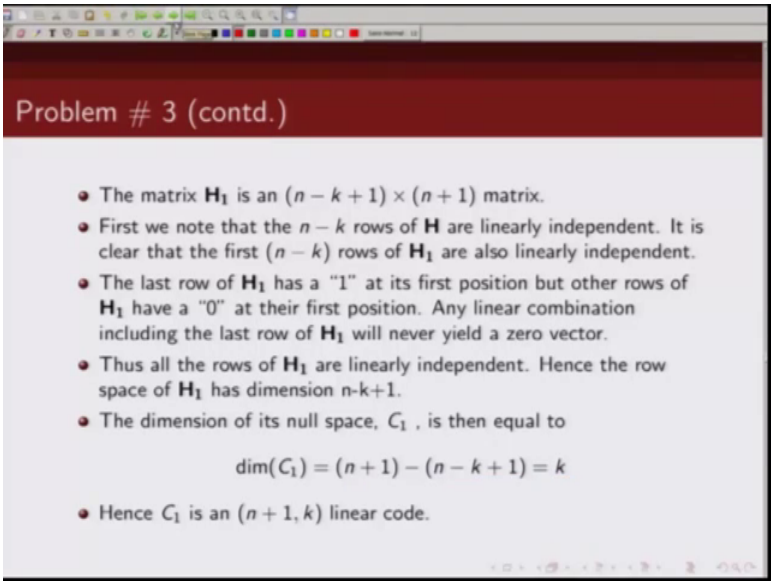

n plus 1 k linear code.

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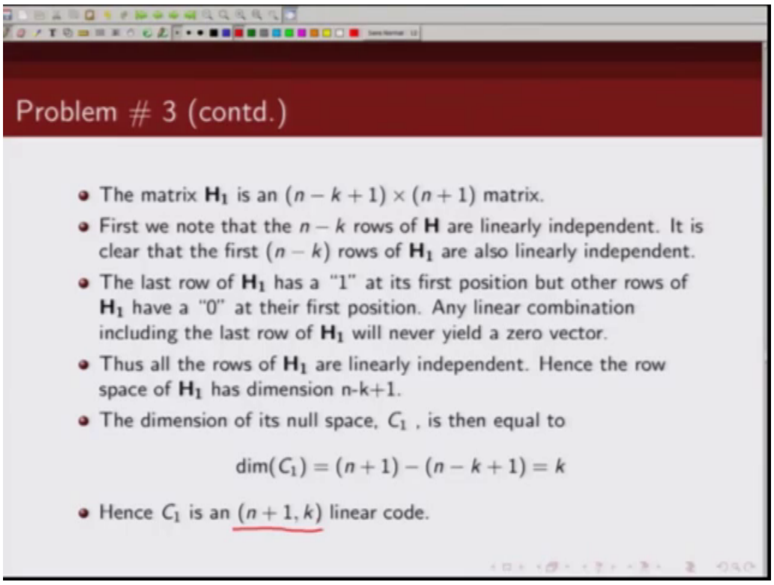

Next we are going to show is

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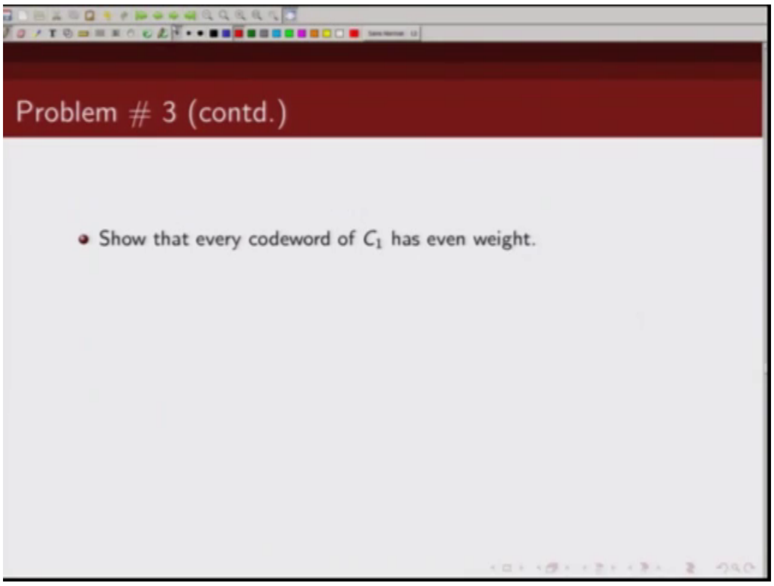

every codeword of C 1 has even weight. So how do we prove this?

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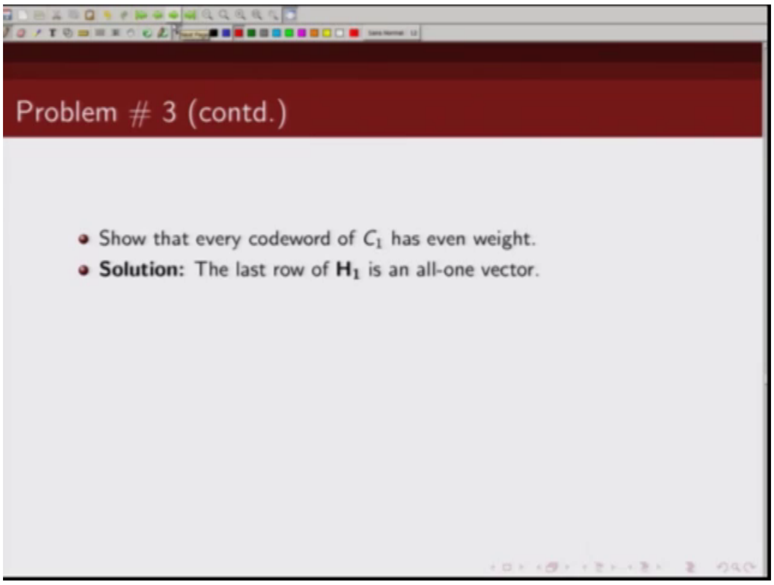

Please note that the last row of this parity check matrix H 1 contain all 1 vector. If you go back, recall the last row of this parity check matrix has all 1's.

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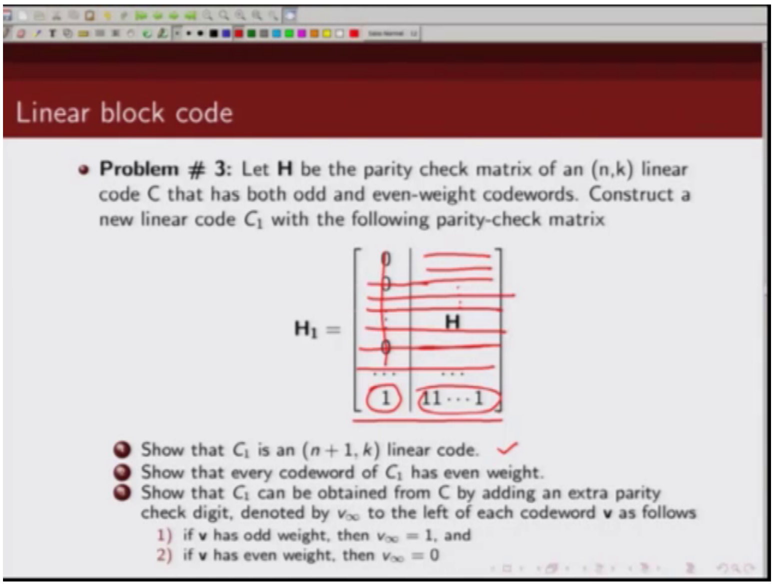

And if v is a valid codeword what property does it satisfy? If v is a

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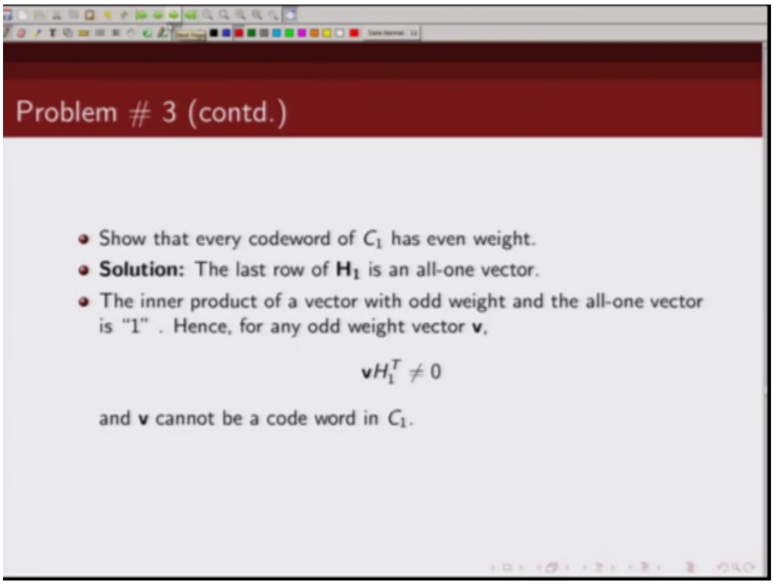

valid codeword then

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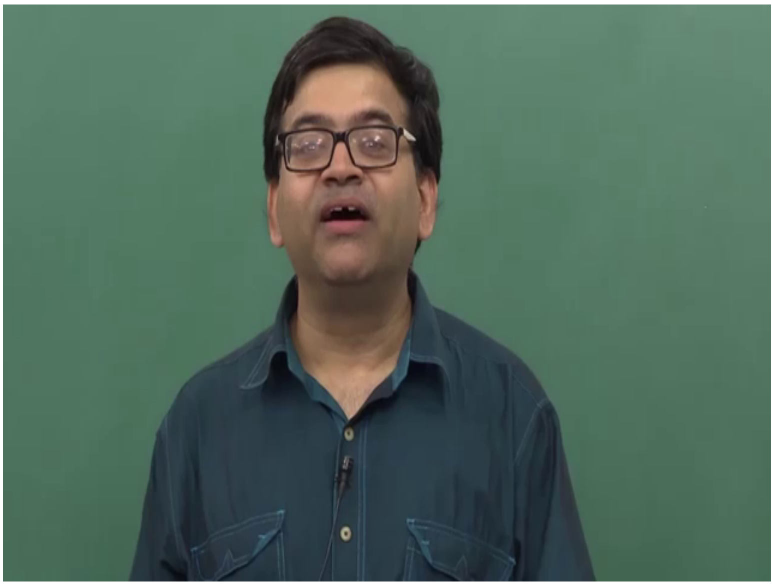

v H transpose should be 0. Now let us take

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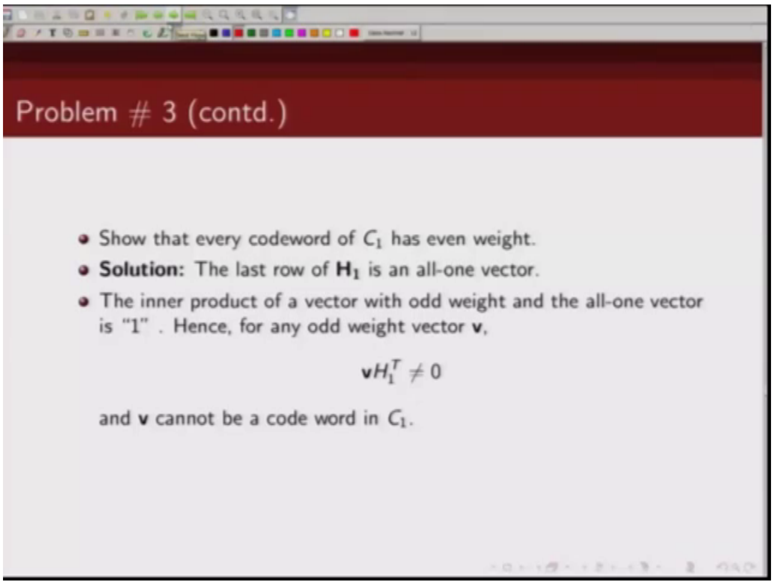

a codeword, let's say there exists a codeword with odd weight which is generated by, described by this parity check matrix H 1. Now if we do v H transpose, so when you are

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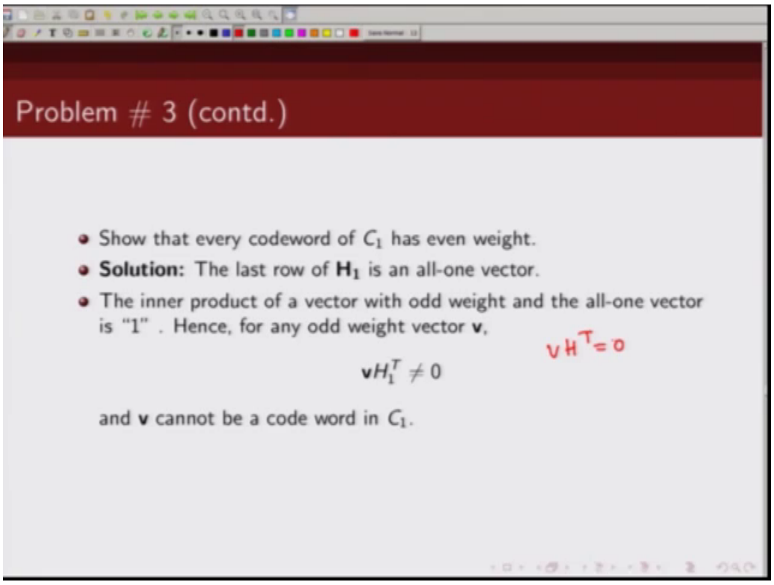

going to take the inner product of this code vector v with the last row of this parity check matrix what will you get?

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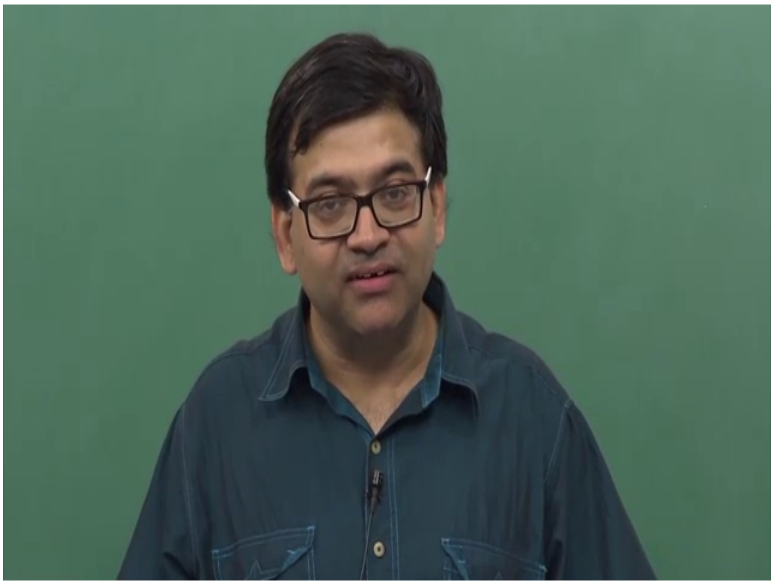

You will essentially get sum of, so basically

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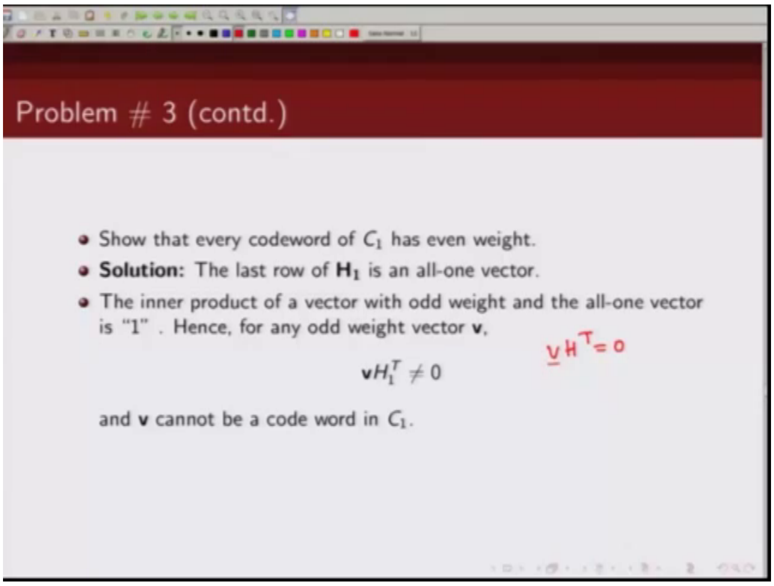

if you do v H transpose so your H is of the, so v H transpose where H is, H 1 transpose where H 1 is given like this. This is all 0's. You have your parity check matrix H and you have here all 1 matrix,

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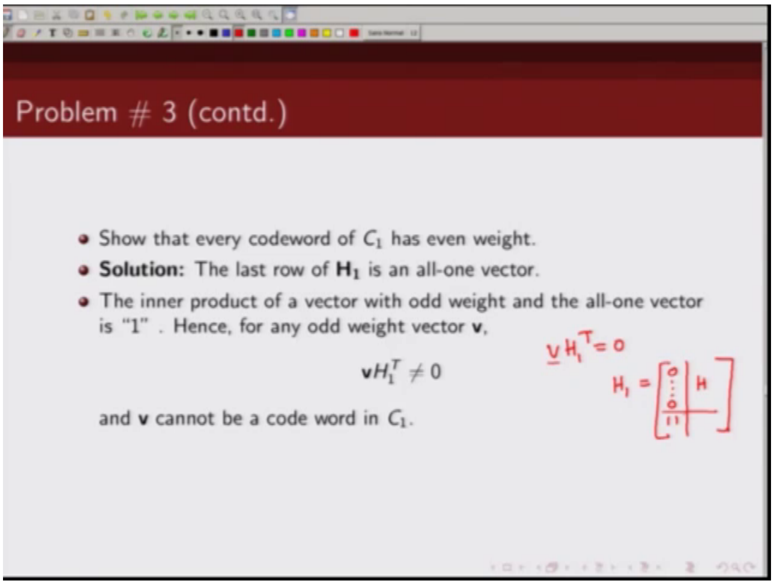

sorry you have your, this is 1, this is 1 and this is all 1 vector. Now when you do v H transpose

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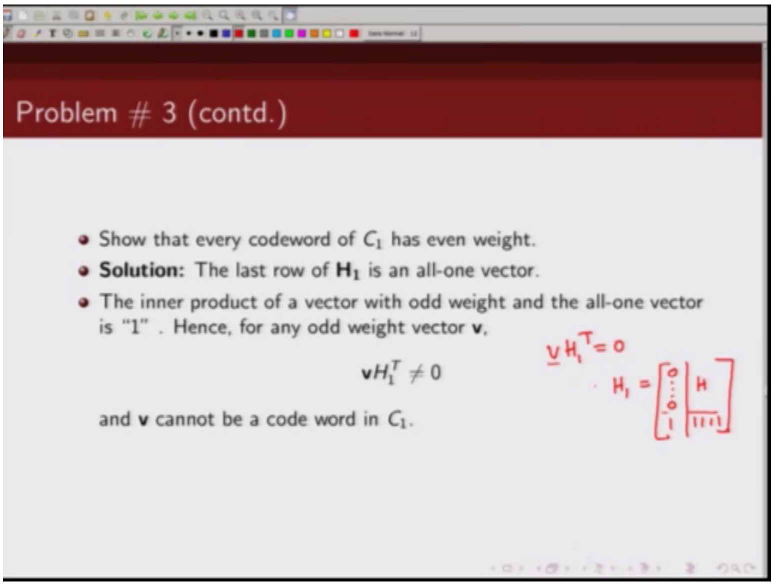

so let's say v is your v 0 to v n minus 1. When you do v H 1 transpose what you will get

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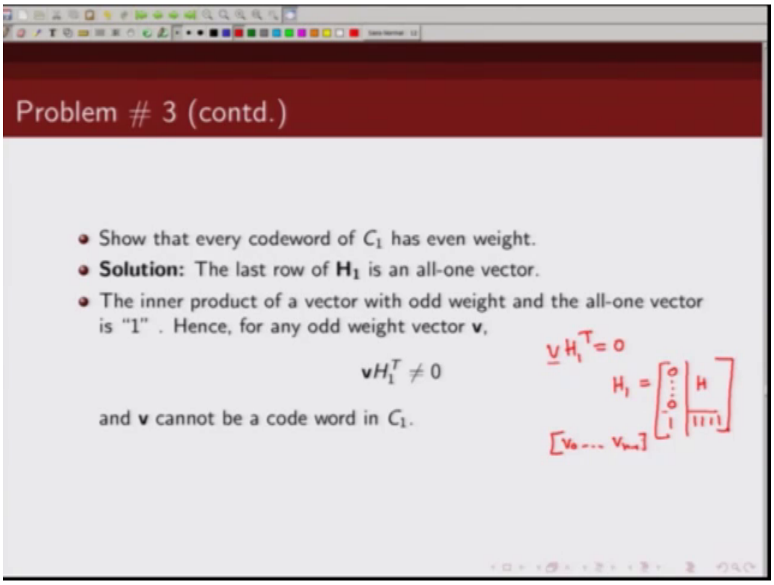

is v 0 plus v 1 plus v 2 up to v n minus 1 is going to

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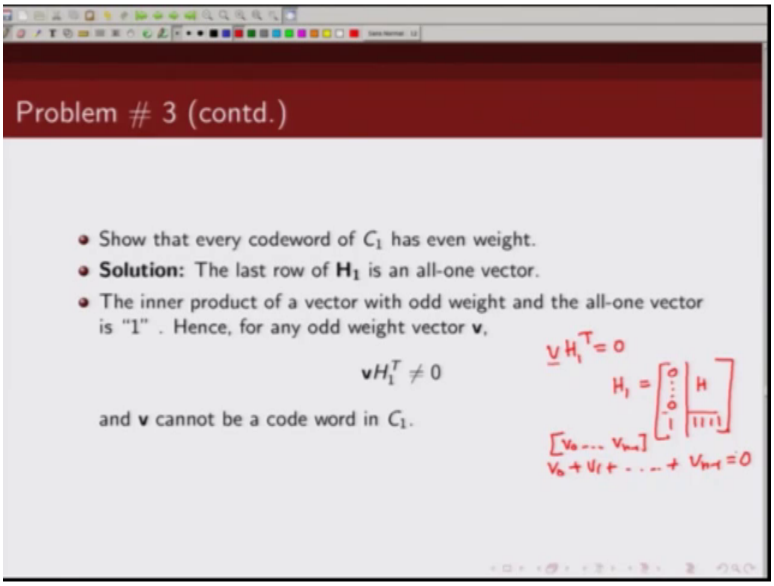

be 0. Now if this v has odd number of 1's this sum cannot be 0, right? Hence we prove that v has to have even number of 1's because we know if v is a valid codeword then v H 1 transpose should be 0. So if we do v H transpose because the last row of this parity check matrix H 1 is all 1, the condition that we will get is the individual components of this parity, code vector v, v 0 plus v 1 plus v 2 plus v 3 up to v n minus 1, basically v n minus 1 they should all add up to 0. Hence we cannot have an odd weight vector which will give

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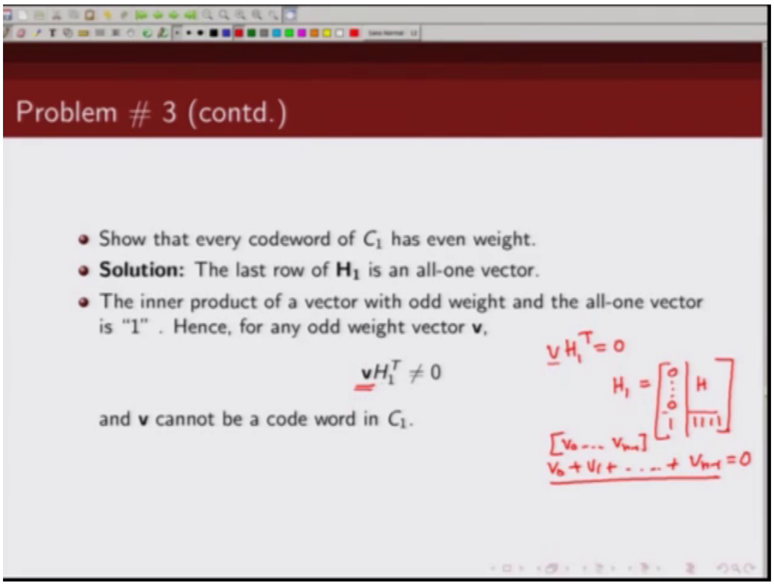

v H transpose, v H 1 transpose to be 0. Hence every codeword

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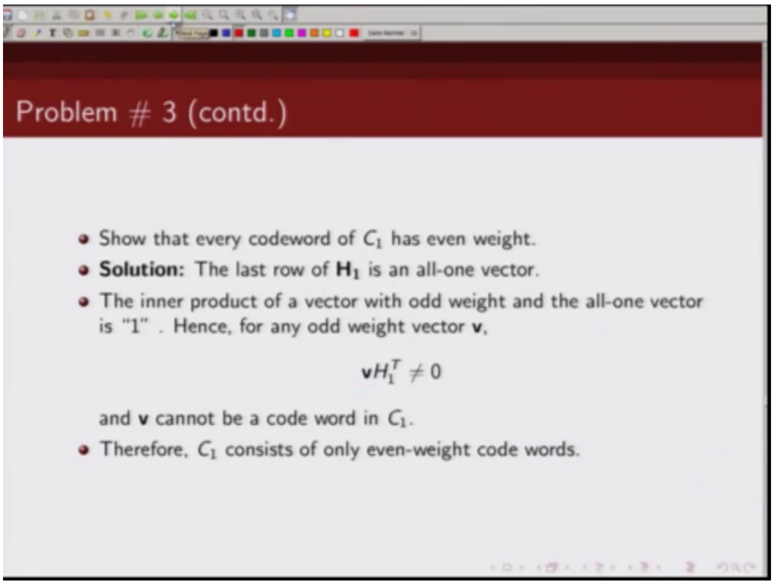

in C 1 has even weight.

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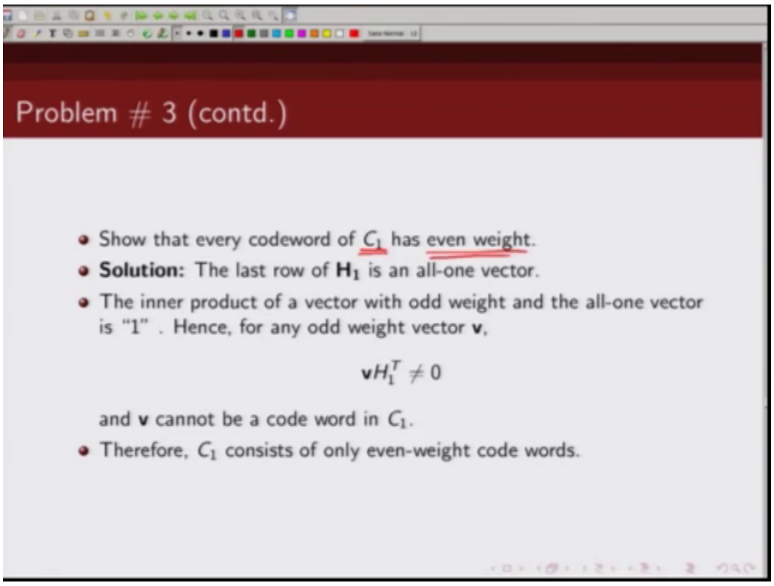

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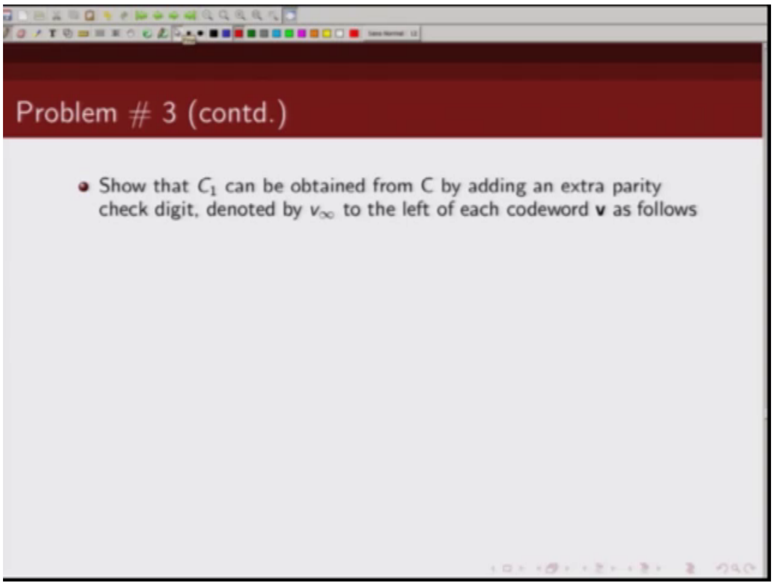

Next we are going to show, prove how we can generate this new

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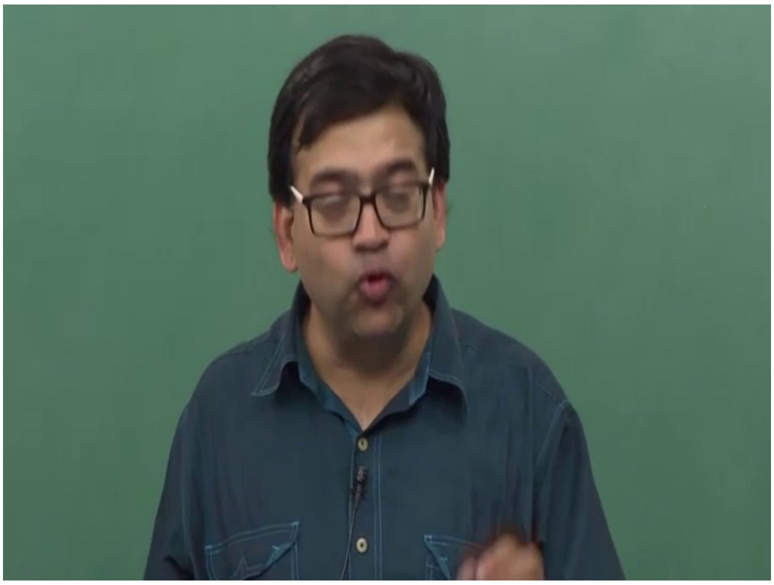

code C 1 from the original code C and what did we mention?

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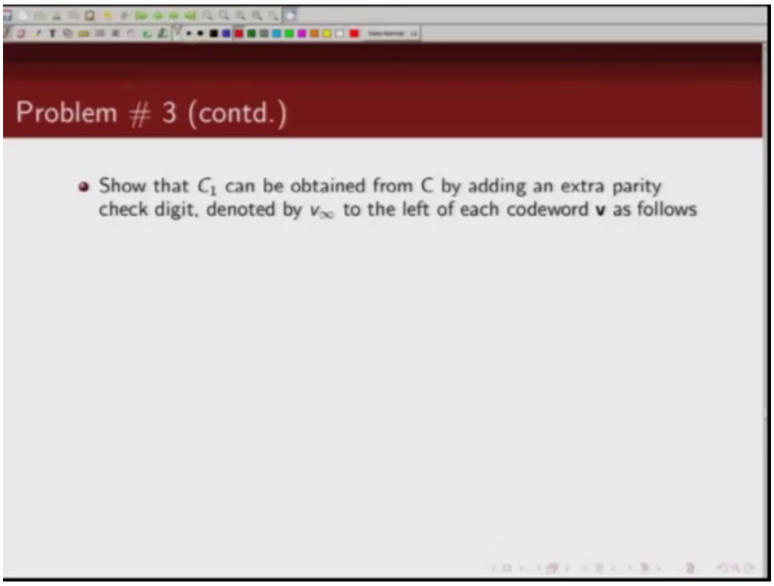

We mentioned that this new code C 1 can be

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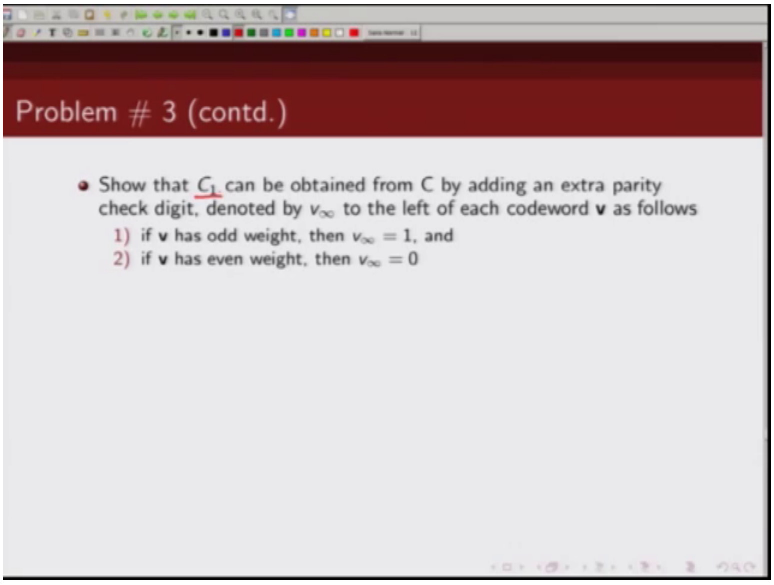

obtained from the original code C by adding an extra parity bit which we are denoting by v infinity to the left of the original codeword v in this fashion.

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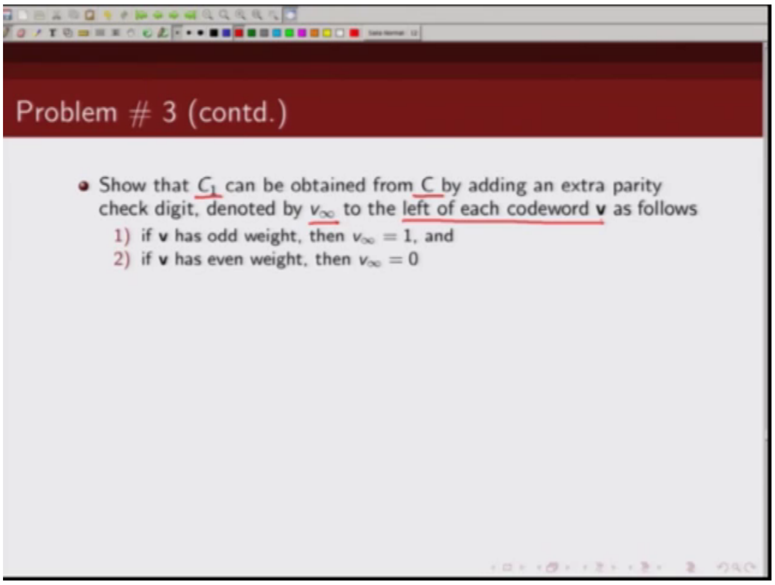

If v has odd weight then v infinity is odd parity and if v has even weight then v infinity is zero parity, is zero, is even parity.

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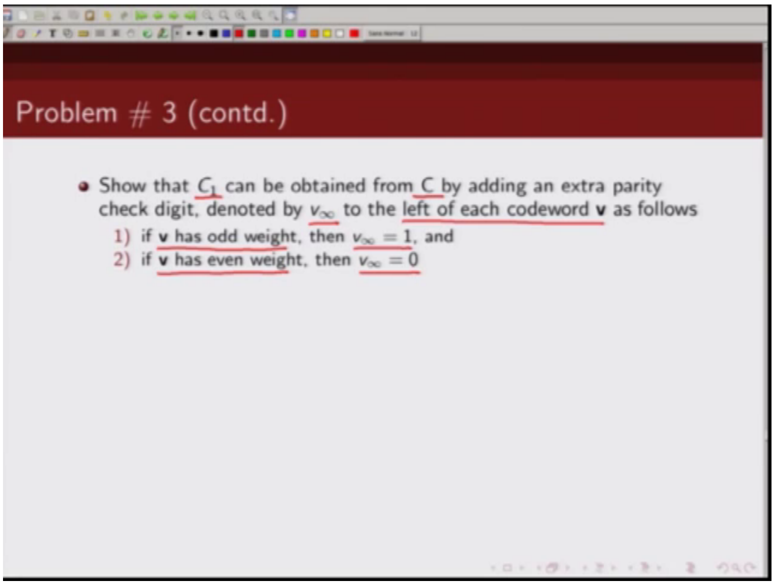

So let's prove this. So let's see,

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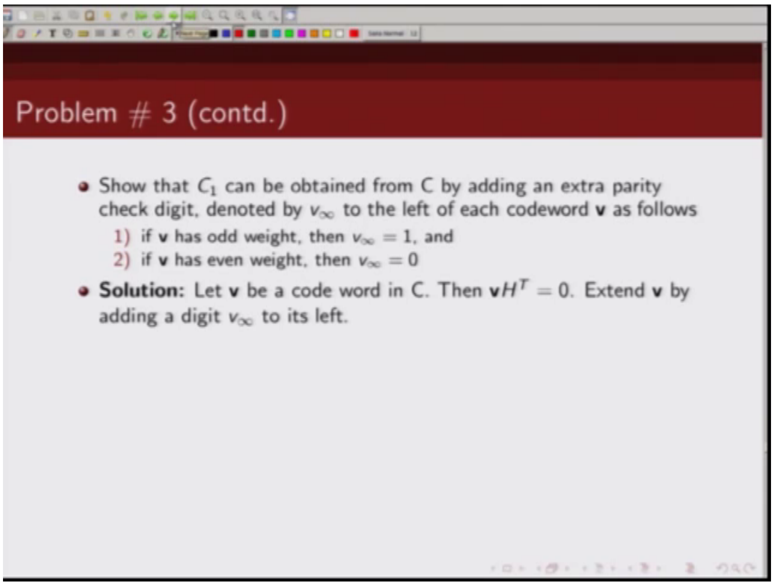

let v be a codeword in C. Then v H transpose will be 0.

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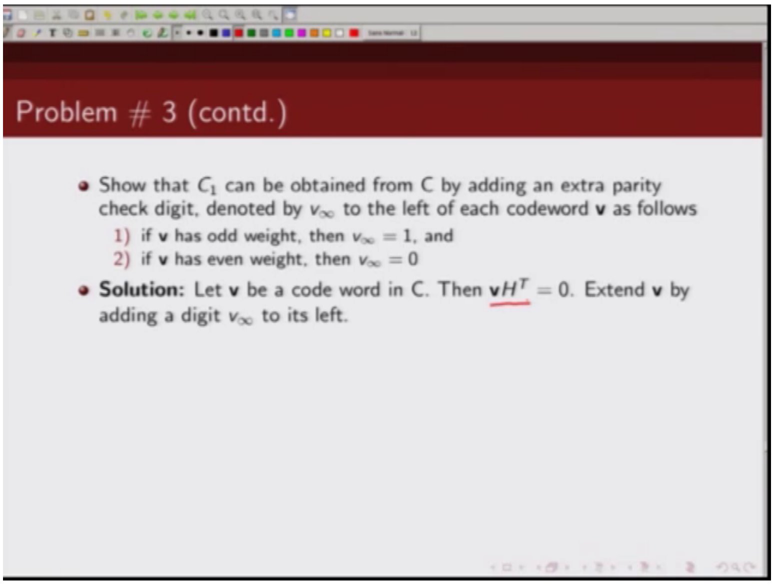

Now we are extending this original code v by adding a bit

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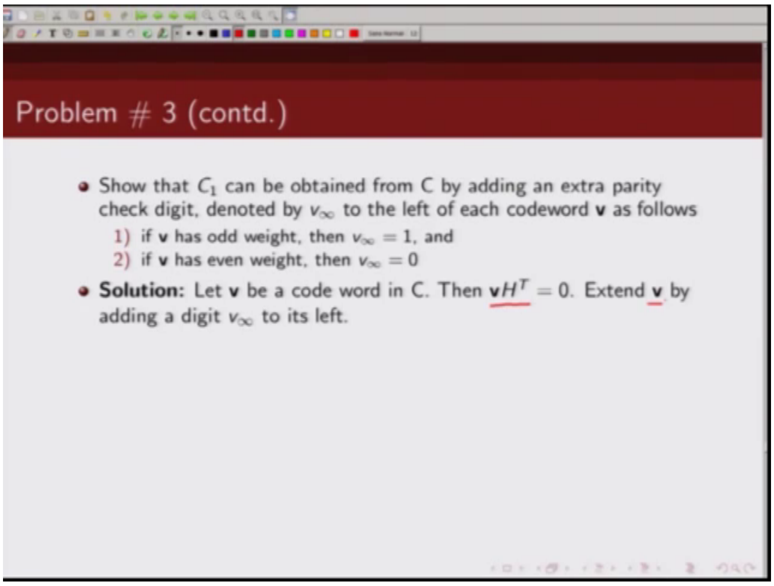

v infinity to its left. So we are defining a new codeword

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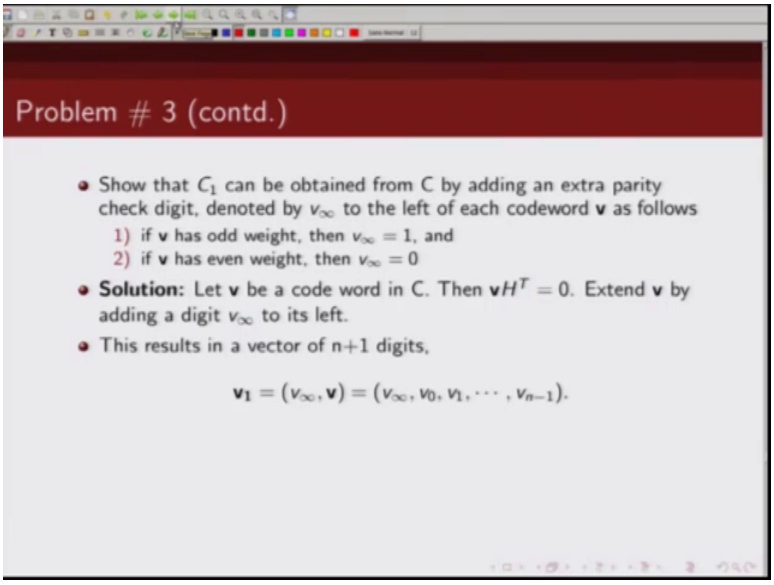

of length n plus 1 which is defined as follows. So this is your original codeword v which is basically

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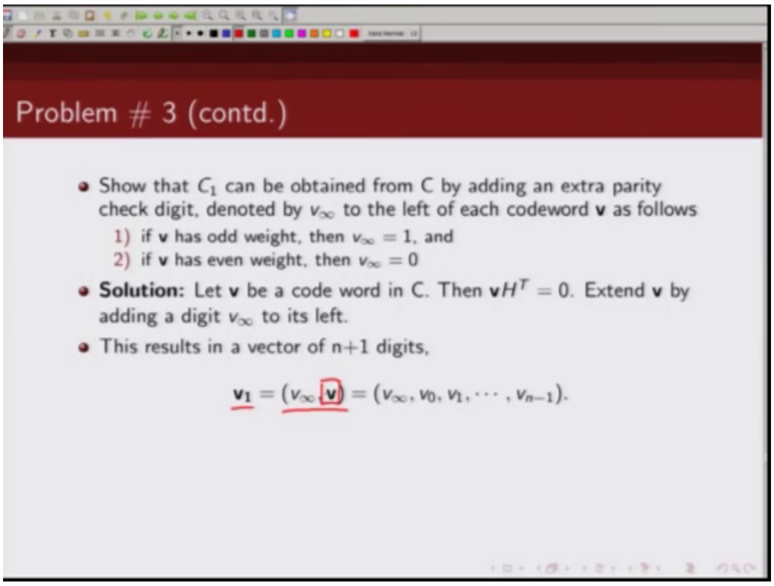

v 0 to v n minus 1 and then this is the additional parity bit

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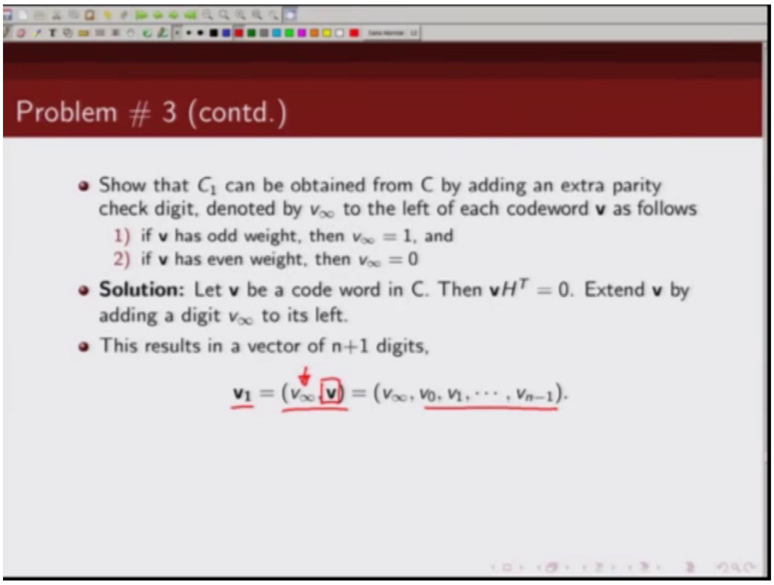

that you added to the left. Now if

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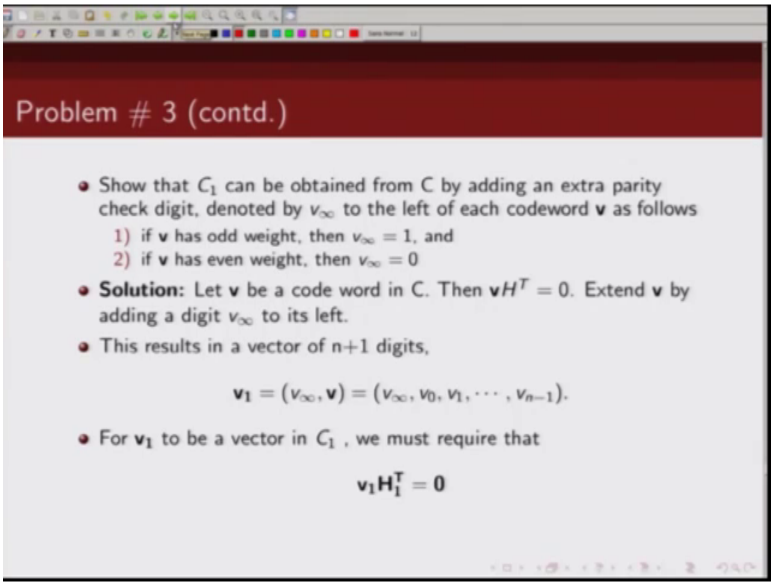

v 1 is a valid codeword, then it should satisfy the property that v 1 H 1 transpose should be 0. And what is our H 1? Again please recall, our H 1 is of form like this. So the first column here is 0, then you have here the original H matrix H, and this is all 1 vector. So when we do

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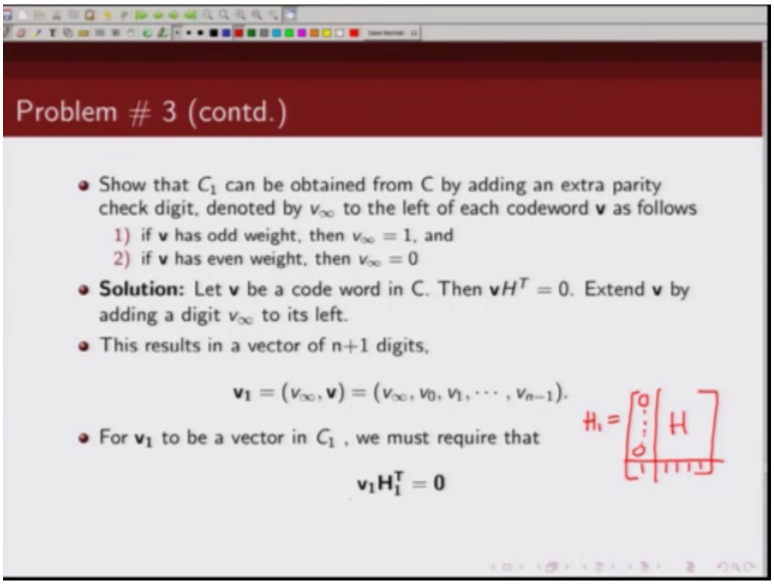

v H transpose, so when v 1 will be multiplied by this last row what we will get is condition of

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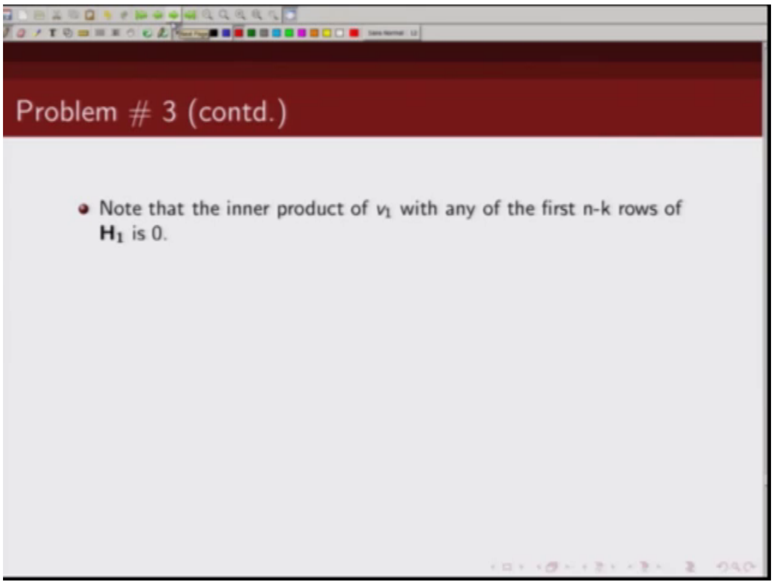

this form

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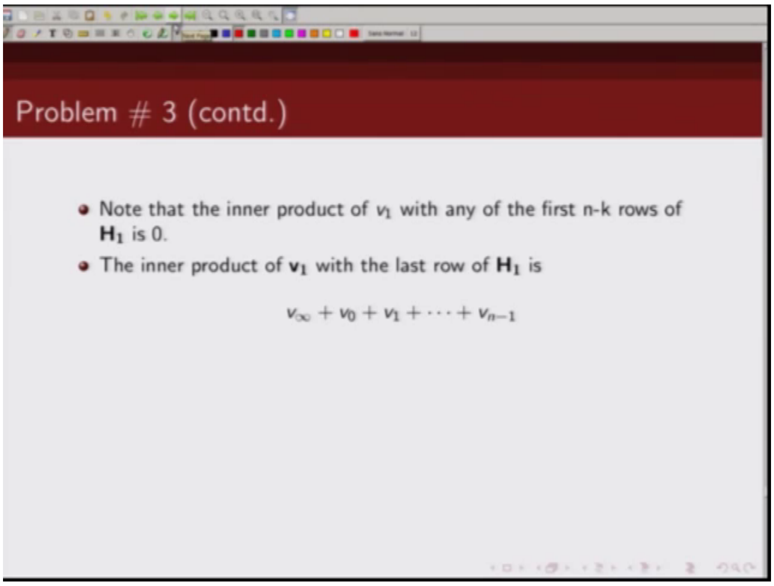

v infinity plus v 0 plus v 1 plus v 2 plus v n minus 1, that's basically should be equal to 0, Ok. Now how are we getting this condition again? We are making

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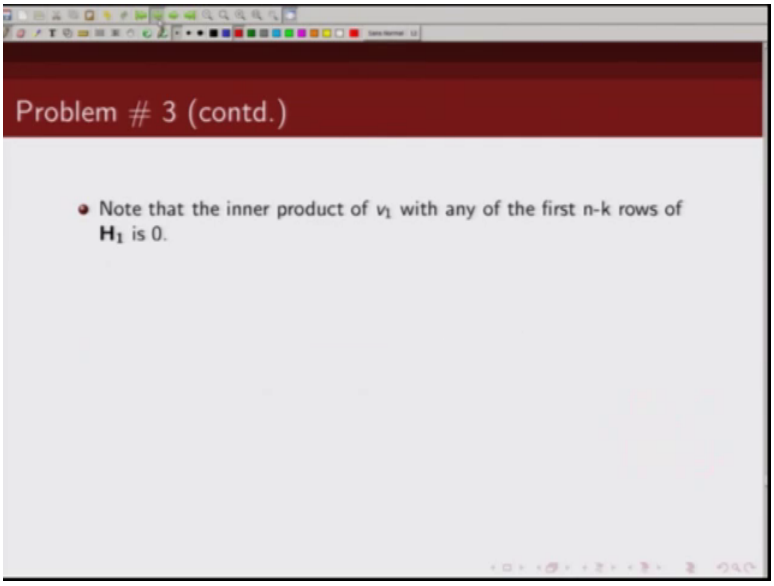

use of the

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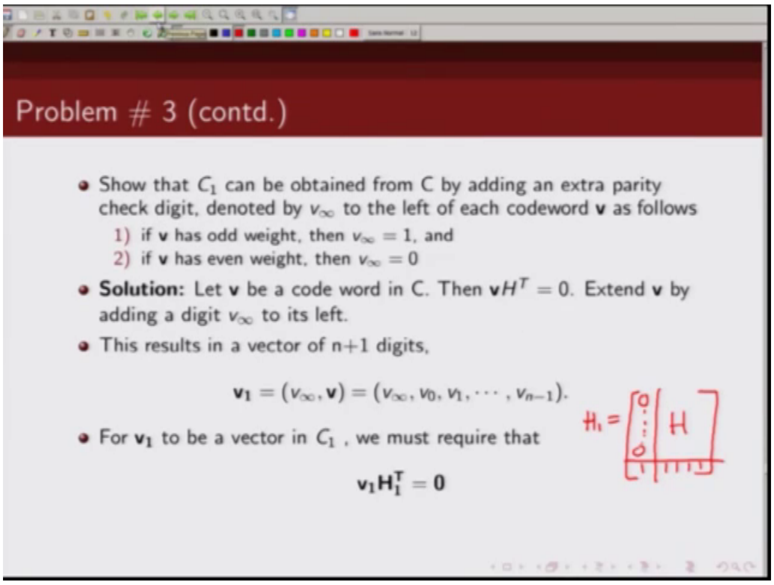

fact that v 1 H 1

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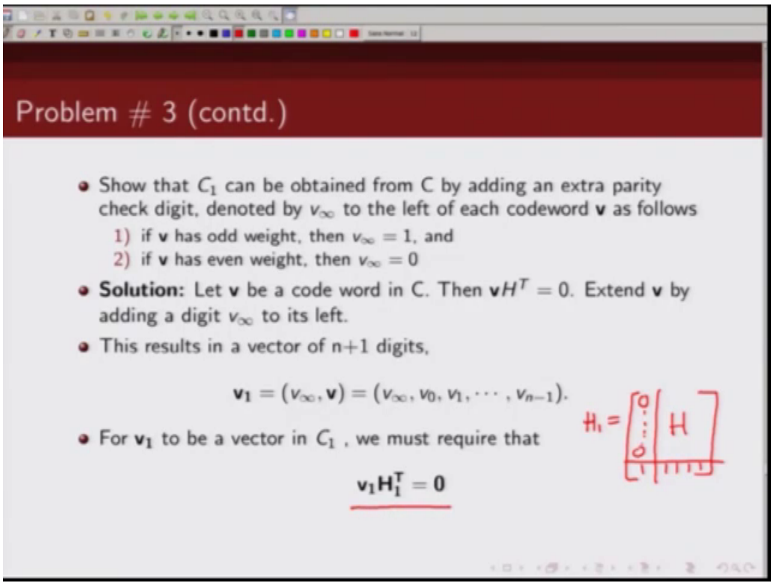

transpose is 0 and H 1 is of form like this. So when we do v 1 H 1 transpose, the last row which will be H transpose will be last column. If we multiply v with that H 1 transpose column what we would get is something of this

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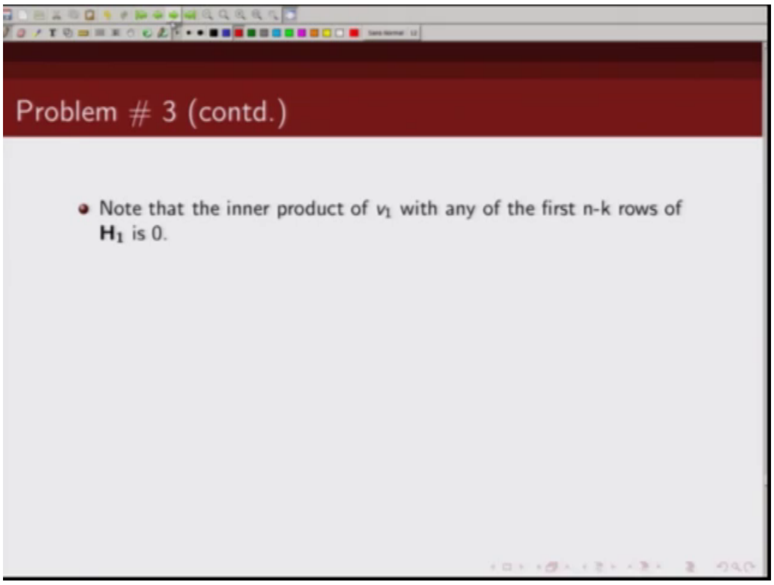

form.

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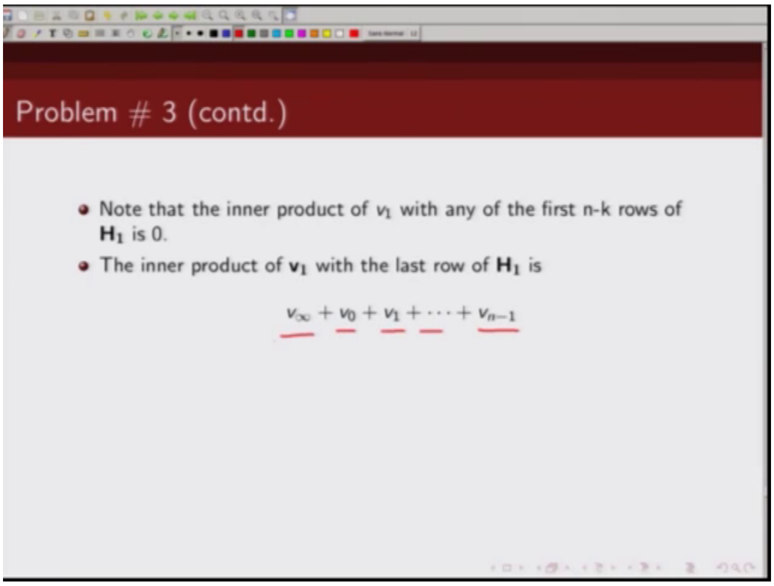

Now this should be equal to 0 if v 1 is a valid codeword, right? So

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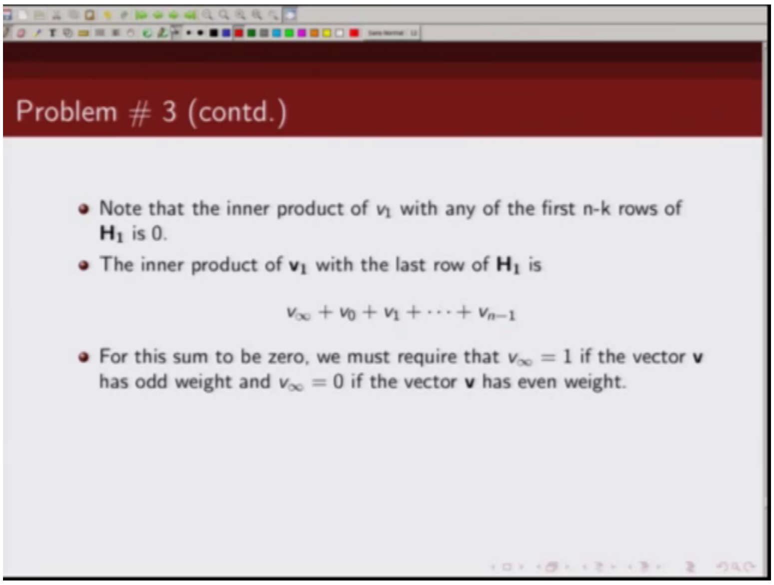

if the sum has to be 0, what do we need? See the original codeword is odd weight codeword. We need v infinity to be 1. And if the original codeword is even parity, then this new parity bit should be 0 and that's basically the proof how we can extend the original code to construct a new code. And this is basically,

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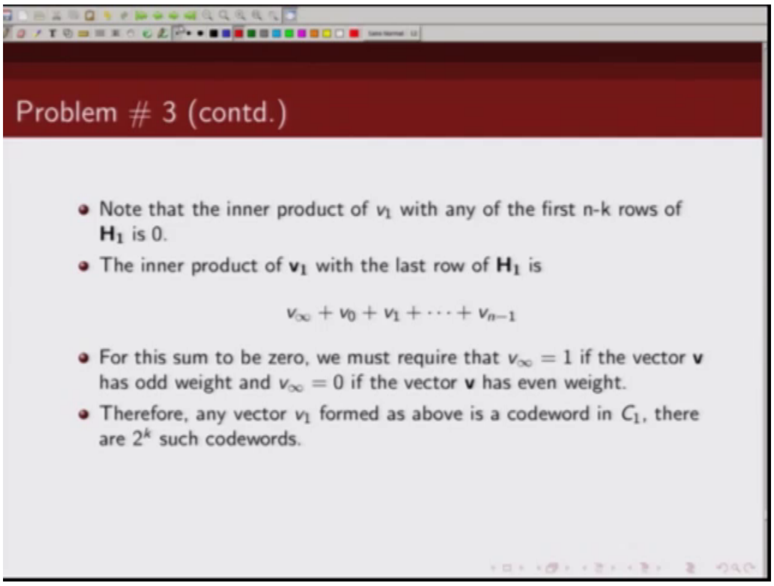

if this is equal to 0, we know that v H transpose, v 1 H 1 transpose is 0, so v 1 is the valid codeword in C 1.

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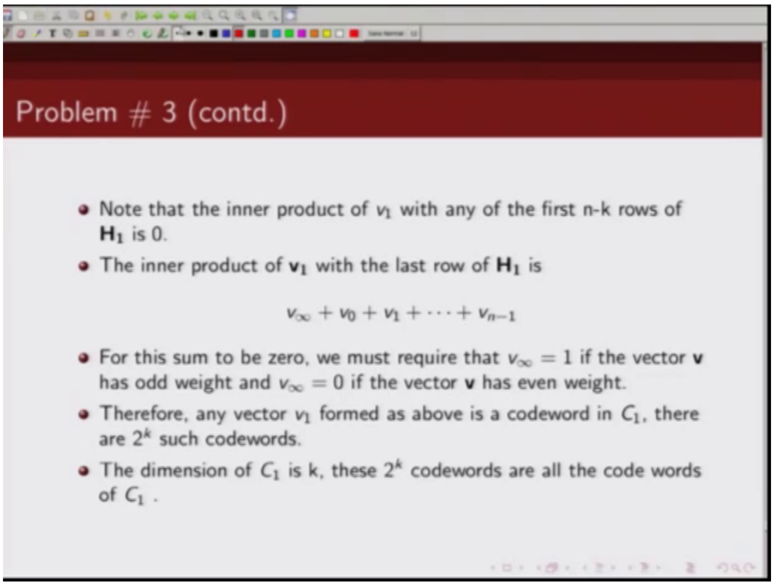

And total there are 2 k codewords. This we have already proved in the first part

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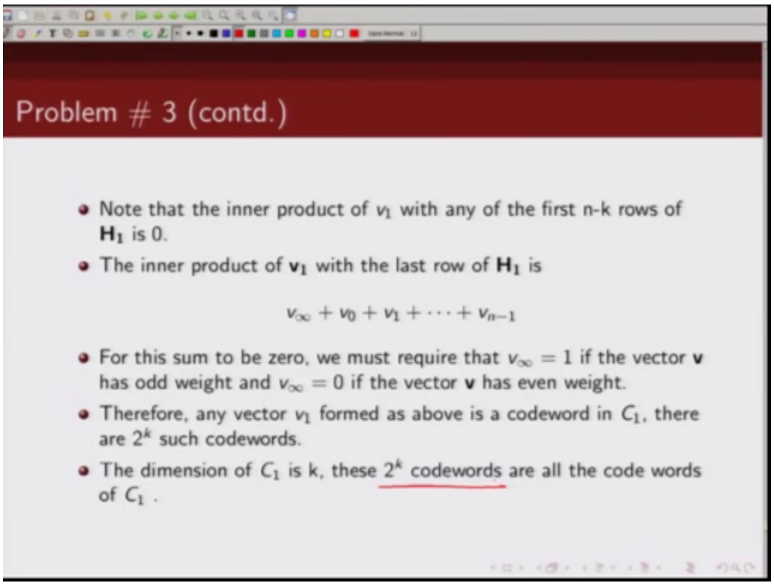

that there are total

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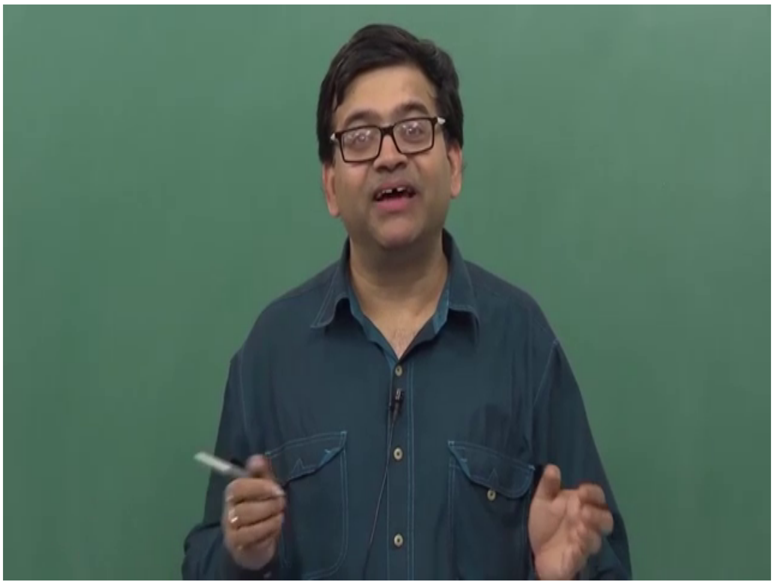

2 k codewords of length n plus 1,

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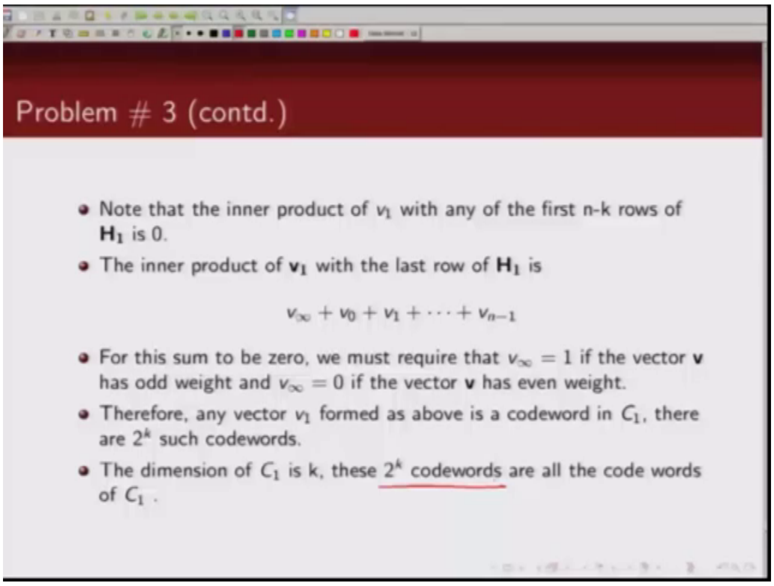

Ok. So with this,

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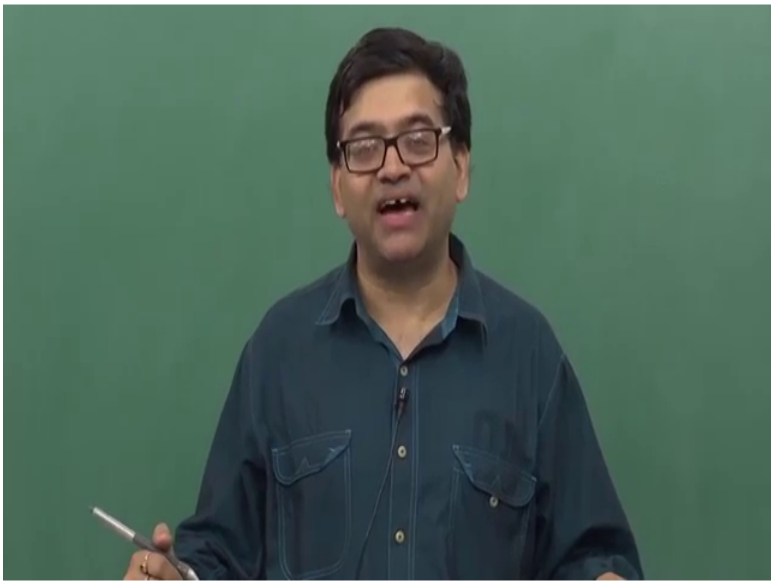

I will conclude this lecture. Thank you