An Introduction to Coding Theory
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Module 01
Lecture Number 04
Generator Matrix and Parity Check Matrix

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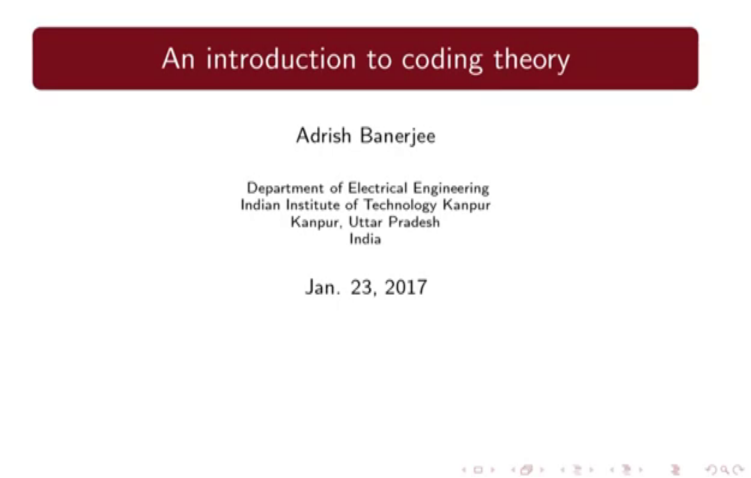

Welcome to the course on Coding Theory. Today in this lecture we are going to describe what we mean by generator matrix

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and parity check matrix So we will continue our discussion with introduction to linear block codes.

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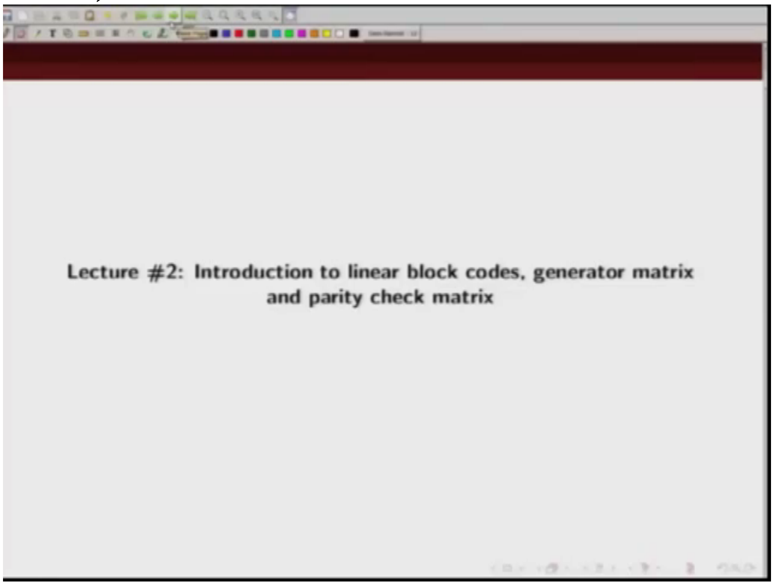

We will first describe what is a generator matrix and what is a parity check matrix and how are they related. So as we described in the last class,

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an encoder for a linear block codes, what it does it takes a block of k-bits and maps it to the, to n-bit. Now the matrix,

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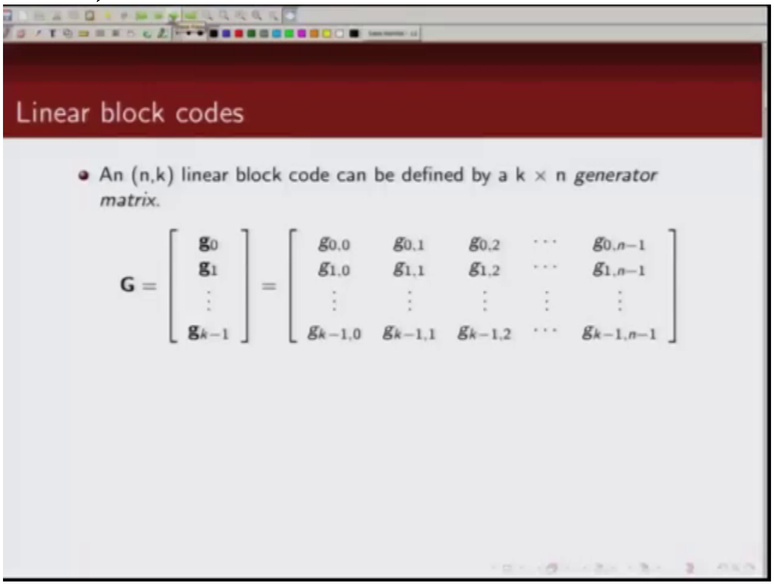

we can use a k cross n matrix to define this mapping from k information bits to n coded bits

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and this matrix is basically our

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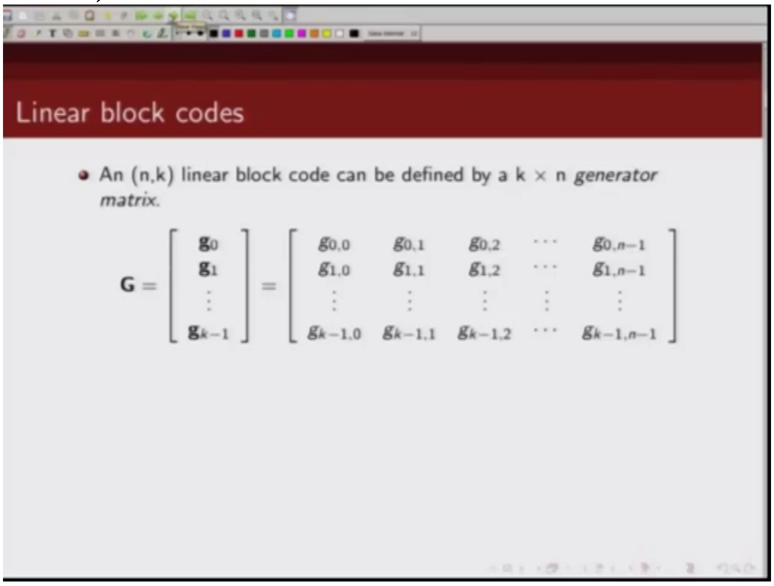

generator matrix for a, so for a n-k linear block code the mapping of k information bits to n-coded bits is defined by this generator matrix G which is of rank k so if we denote information bits by u

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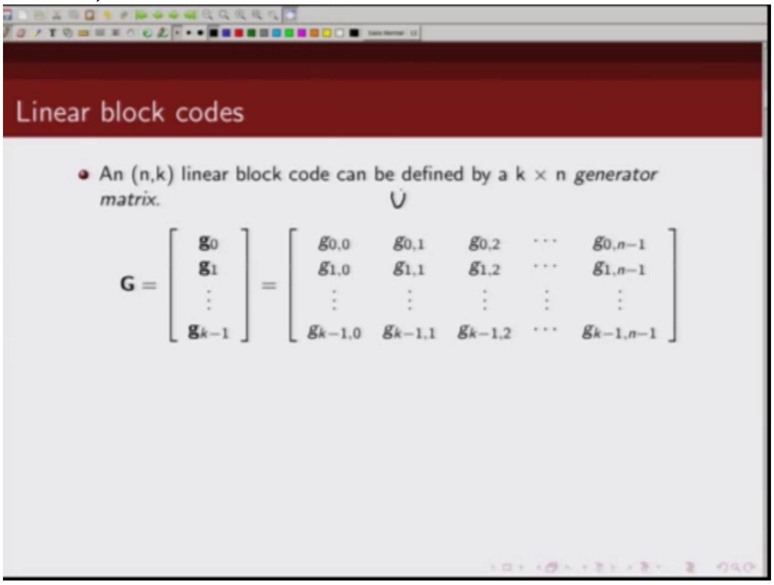

and we denote our coded bits by v,

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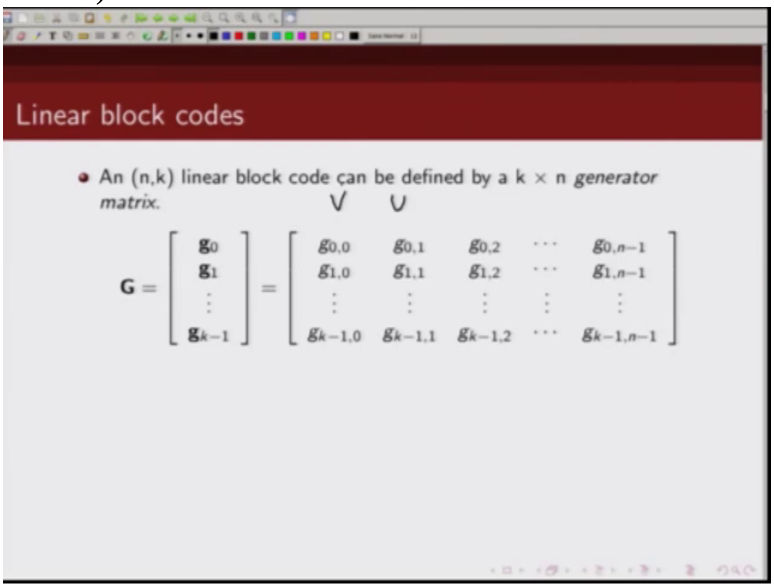

then we can write v as u times g

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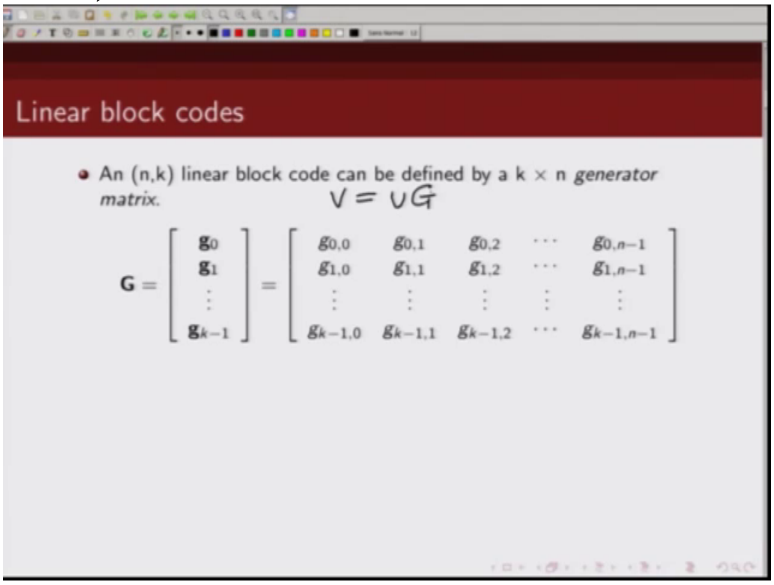

where our u is 1 cross k vector and this is,

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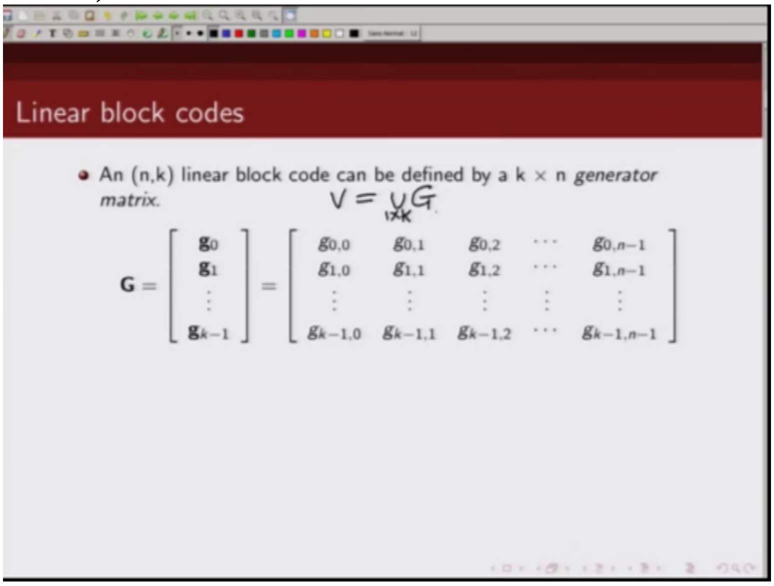

generator matrix is k cross n matrix and our output

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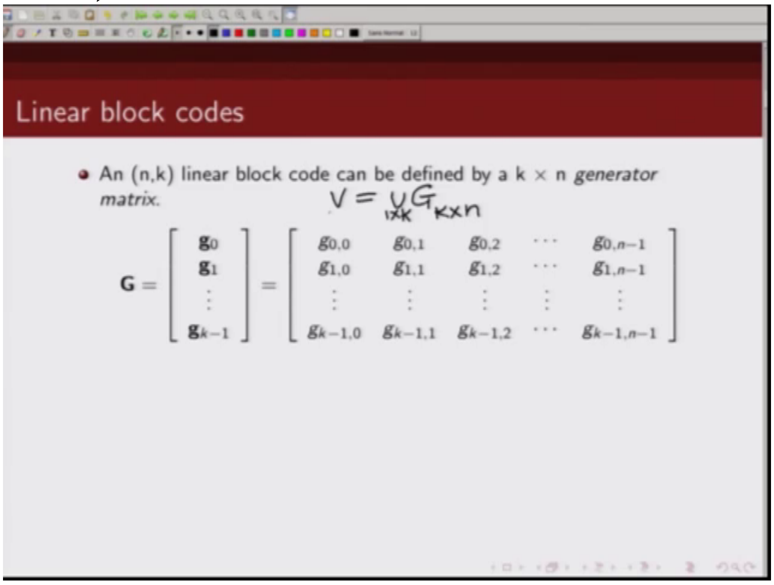

coded bit is 1 cross n vector.

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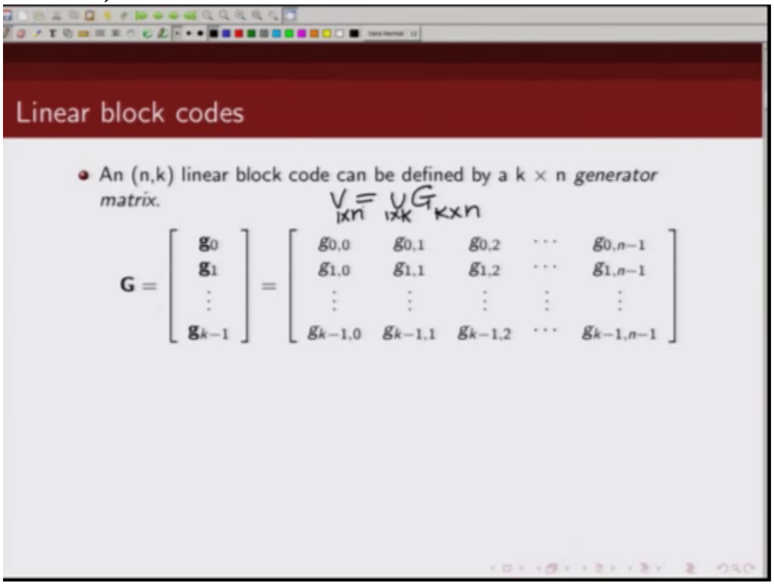

So as a name suggests, basically generator matrix is used to generate our codeword. So we generate our codewords using this generator matrix and this generator matrix gives the mapping between the information bits u to coded bits v. So how do we find codewords? We find codewords by taking linear combinations of rows of this generator matrix. In case of binary codes so then these entries in the generator matrix are either 0 or 1 depending upon which bits are used to generate a particular coded sequence. So we form

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a set of 2 k codewords by taking linear combinations

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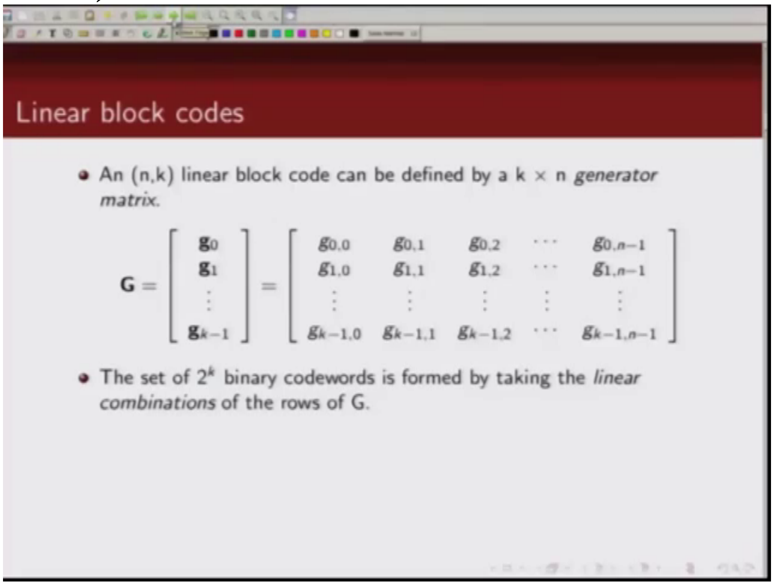

of rows of these generator matrix

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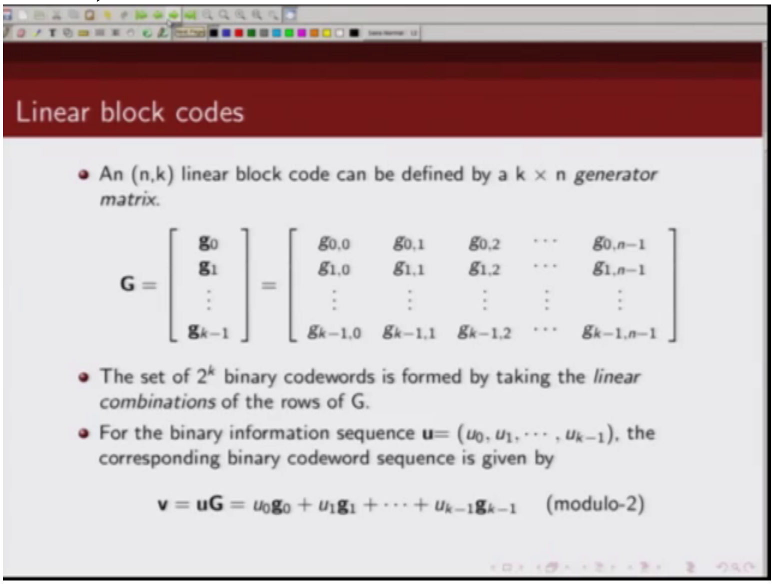

So we can, as I said, we can write our coded sequence as q times p which is basically linear combination of rows of the generator matrix. So these are basically linearly independent k rows and the rank of this generator matrix is k. Since we are without loss of generality, since

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we are talking about binary linear block codes, so we will be doing this addition

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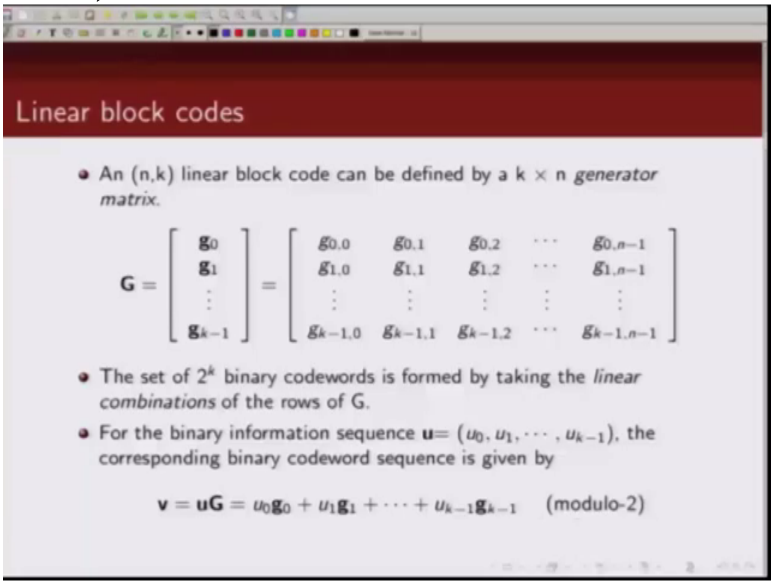

modulus 2 So what are the

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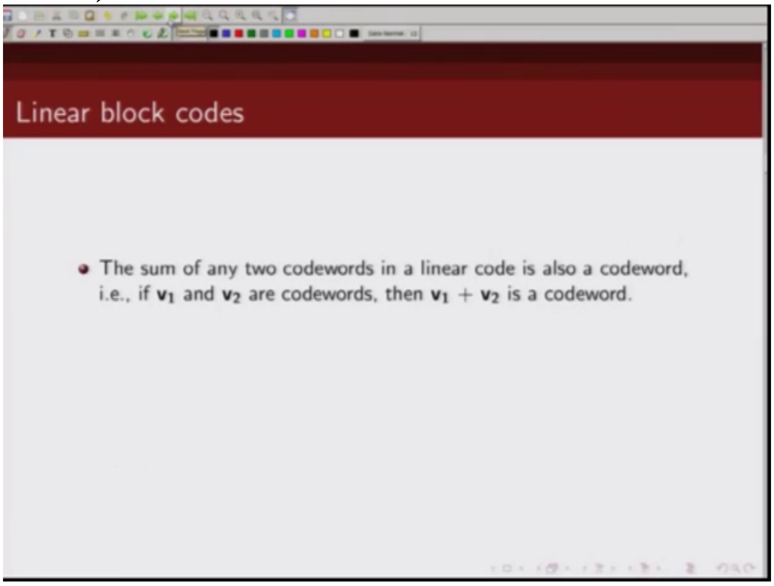

properties of linear block code? Sum of any two codewords in a linear code is also a valid codeword. So if v 1 and

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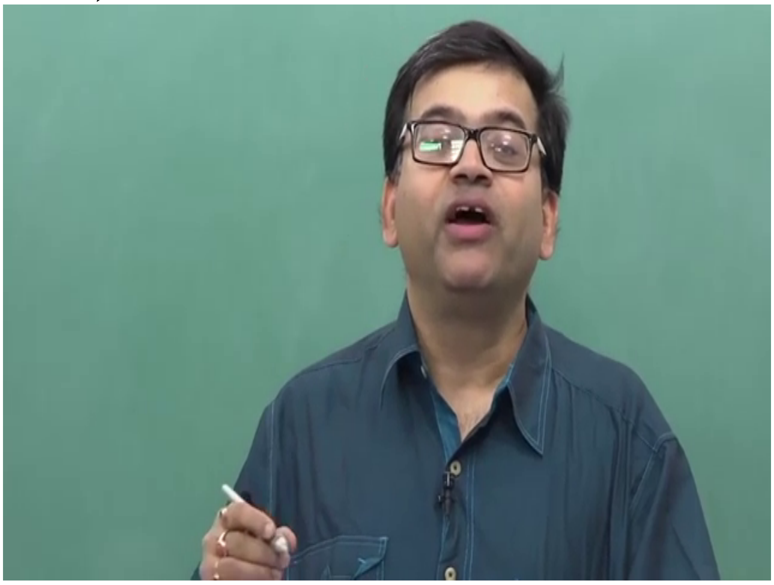

v 2 are valid codewords then v 1 plus v 2 will also be a valid codeword

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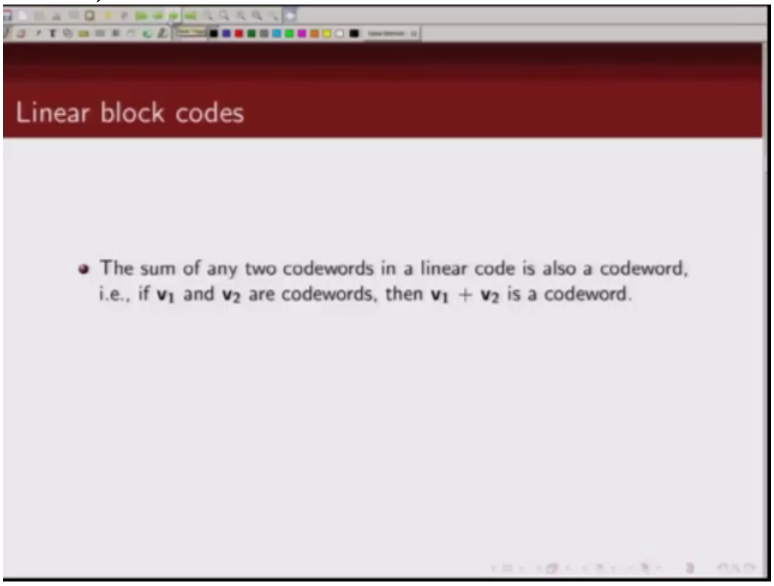

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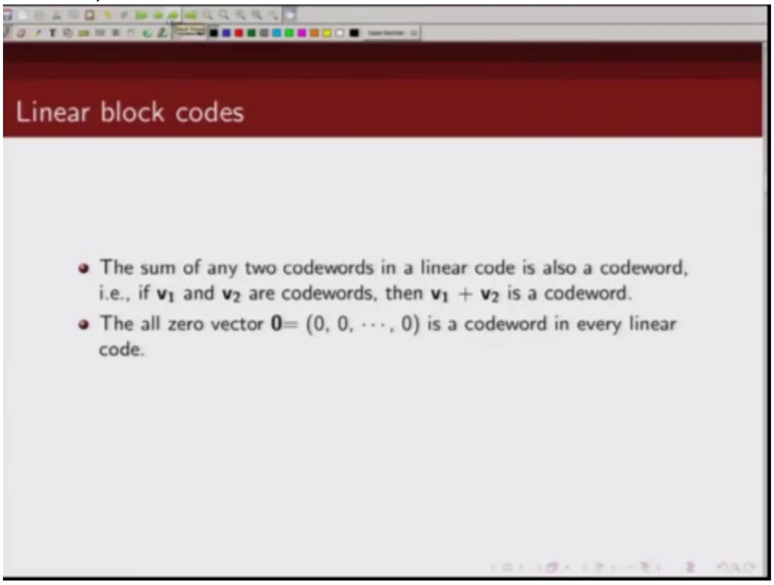

Also, an all zero codeword is a valid codeword in any linear block codes.

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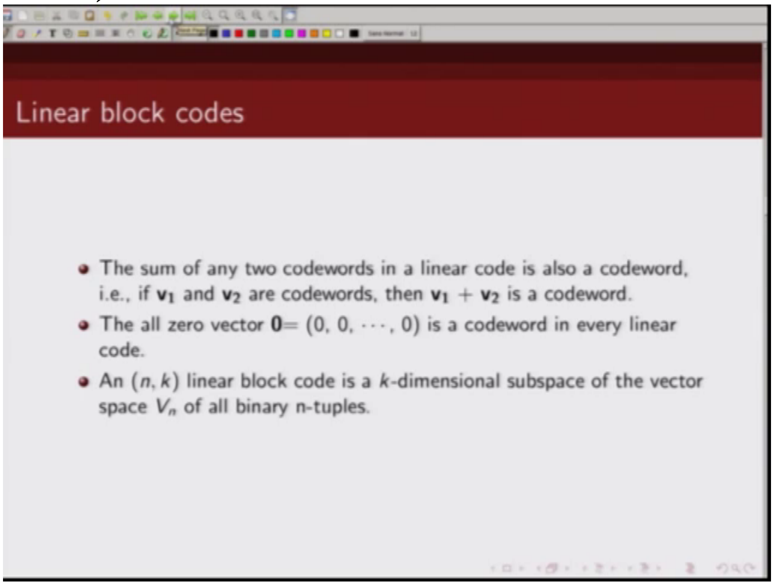

So we can define a linear block code, n k linear block code as k dimensional space of vector space v n of all binary n-tuples; so we can define a linear binary block codes as a k dimensional subspace of vector space v n of all binary n tuples.

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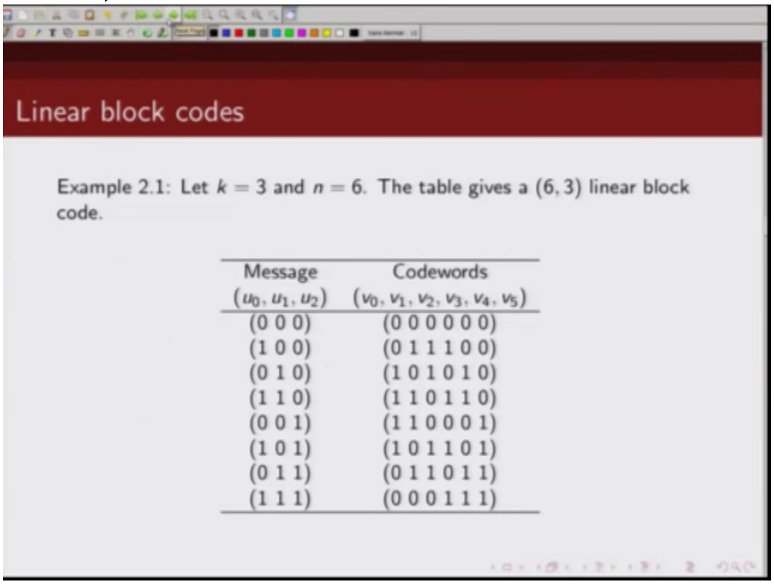

Message (u_0, u_1, u_2)	Codewords $(v_0, v_1, v_2, v_3, v_4, v_5)$
(0 0 0)	(0 0 0 0 0 0)
(1 0 0)	(0 1 1 1 0 0)
(0 1 0)	(1 0 1 0 1 0)
(1 1 0)	(1 1 0 1 1 0)
(0 0 1)	(1 1 0 0 0 1)
(1 0 1)	(1 0 1 1 0 1)
(0 1 1)	(0 1 1 0 1 1)
(1 1 1)	(0 0 0 1 1 1)

Now let us take an example to illustrate what is a generator matrix. So in this example, we have considered 3

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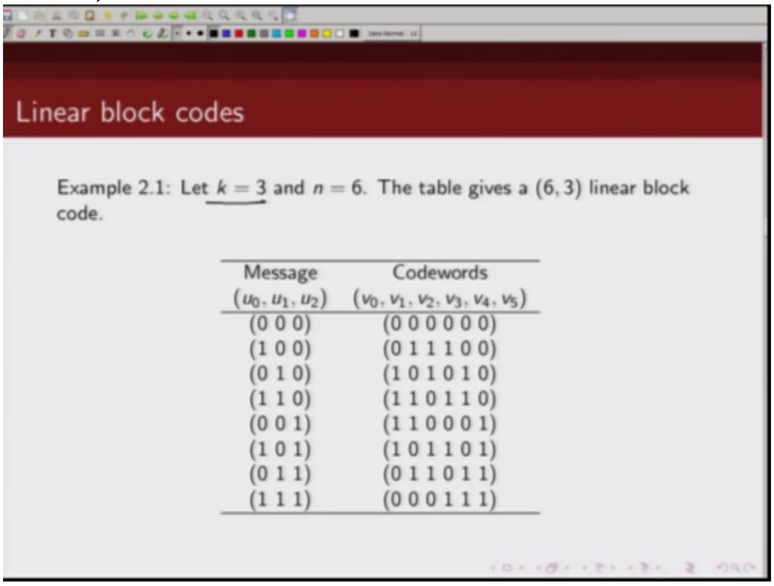

Message (u_0, u_1, u_2)	Codewords $(v_0, v_1, v_2, v_3, v_4, v_5)$
(0 0 0)	(0 0 0 0 0 0)
(1 0 0)	(0 1 1 1 0 0)
(0 1 0)	(1 0 1 0 1 0)
(1 1 0)	(1 1 0 1 1 0)
(0 0 1)	(1 1 0 0 0 1)
(1 0 1)	(1 0 1 1 0 1)
(0 1 1)	(0 1 1 0 1 1)
(1 1 1)	(0 0 0 1 1 1)

information bits and 6 coded bits. And in this table I have given you the set of 8 information sequences and their corresponding codewords. So how do we find the generator matrix in this case? So we will have to look at each of this code bits and see how are we generating these code bits in terms of

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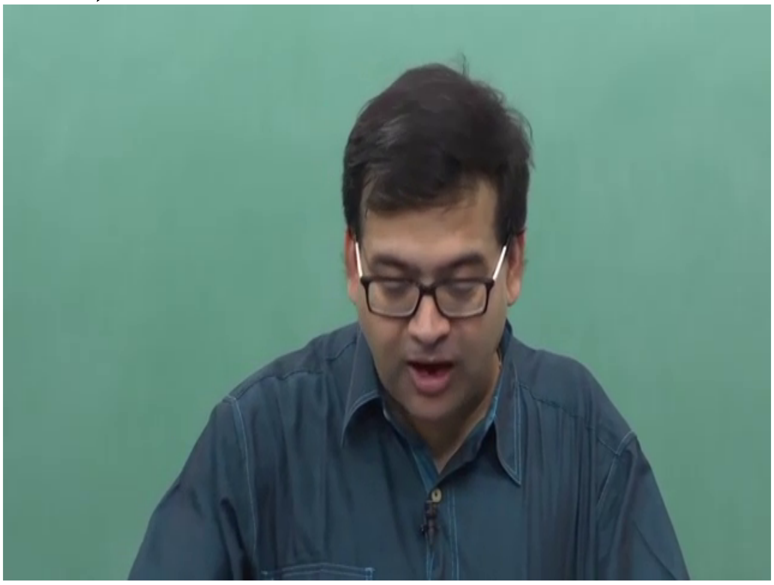

message bits u 0, u 1 and u 2. So first thing we are going to do is

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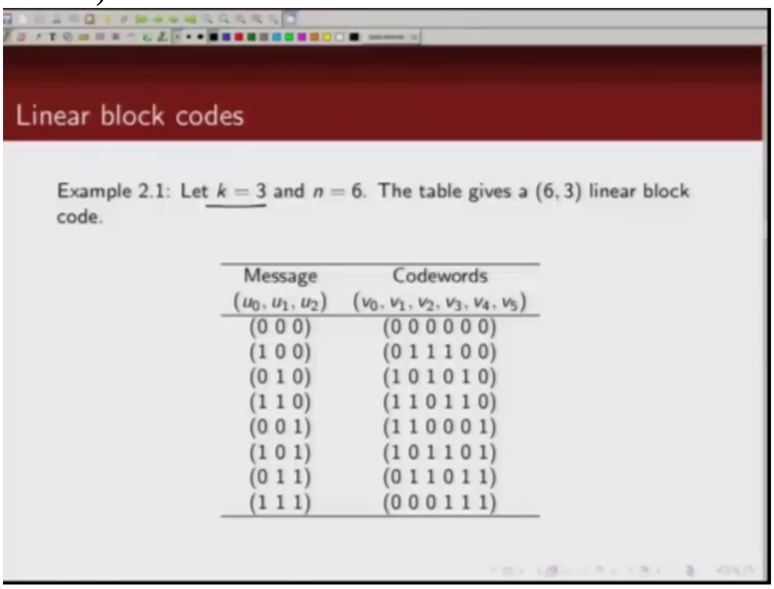

look at each of these code bits, v 0, v 1, v 2, v 3, v 4, v 5 and write them in terms of u 0, u 1, u 2, Ok. So let's look at each of these. So v 0 is u 1 plus u 2, we can see easily v 0 is this column and we can see this is same as u 1 plus u 2. So u 1 plus u 2 in this case is 0, u 1 plus u 2 is 0, 1 plus 0 is 1, 1 plus 0 plus 1 is 1, 1 plus 1 is 0 modular 2 and 1 plus 1 is 0 modular 2. So this v 0 is basically given by

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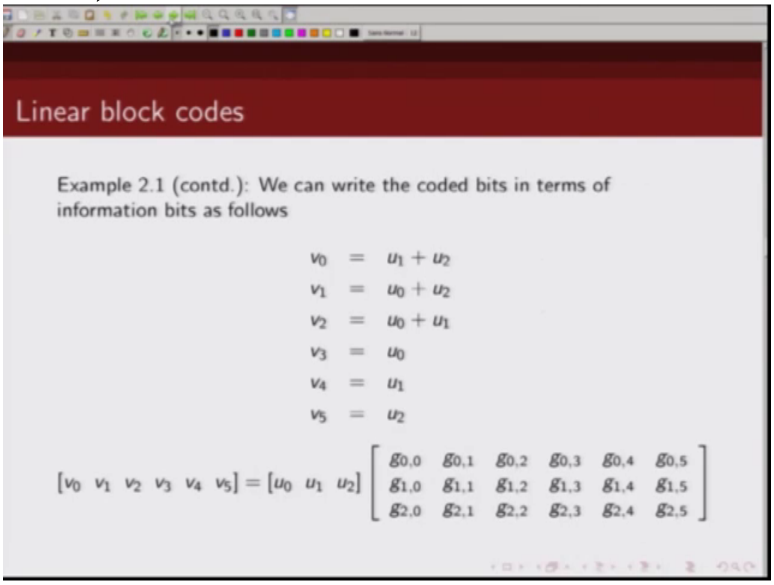

u 1 plus u 2 Similarly we can see v 1 is given by u 0 plus u 2 and v 2 is given by u 0 plus u 1. So let's just

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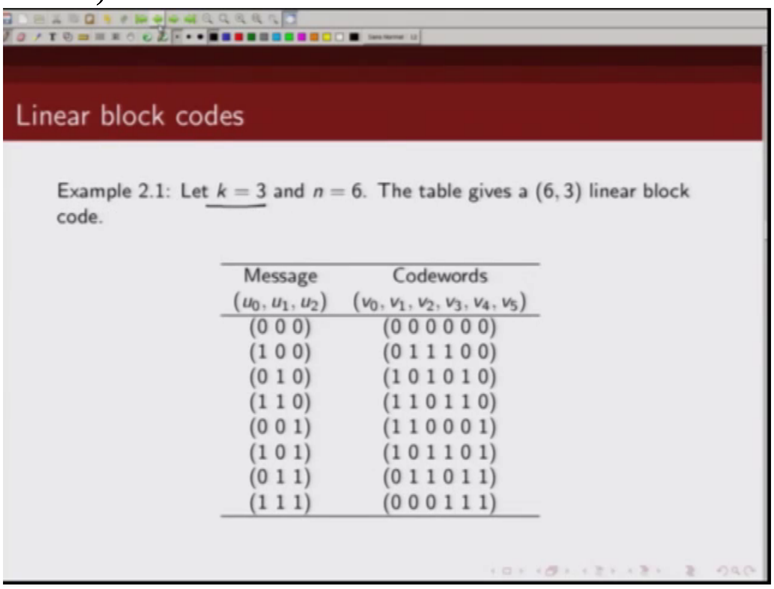

check, let's say v 2. v 2 you can see, is given by, v 2 is

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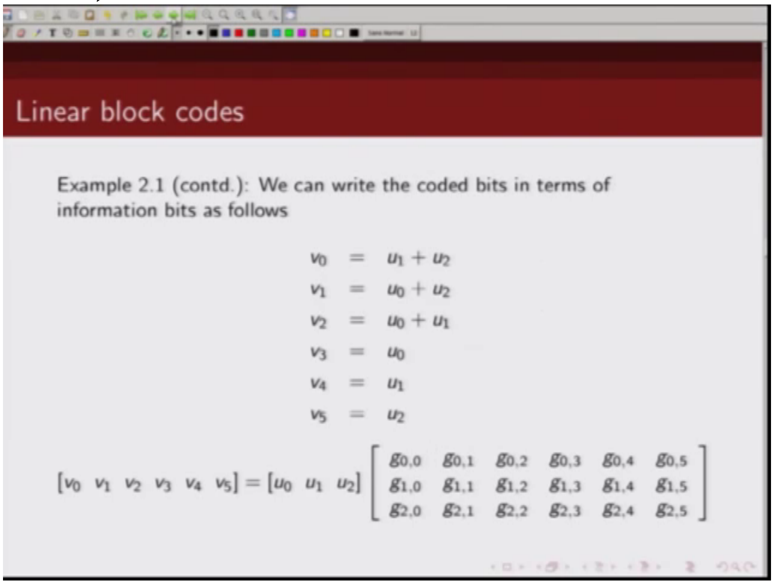

given by u 0 plus u 1 You can check v 2

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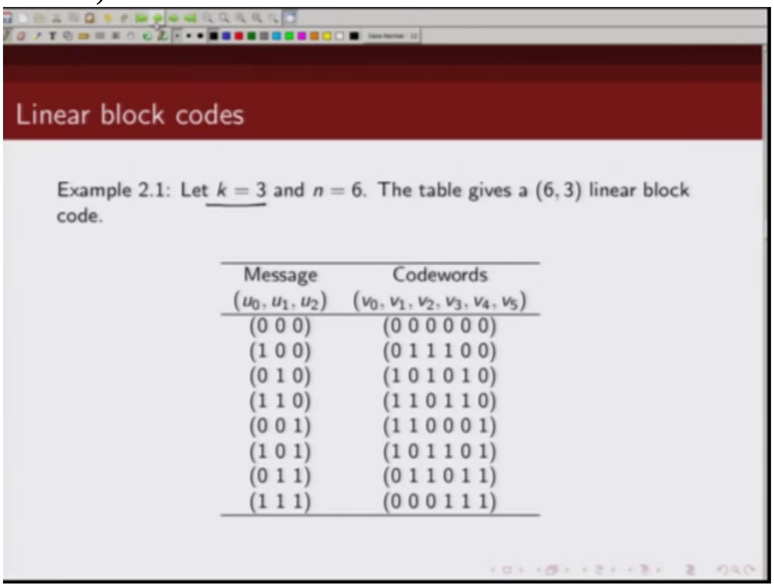

is given by, so u 0 plus u 1, 0 plus 0 is 0, 1 plus 0 is 1, 0 plus 1 is 1, 1 plus 1 is 0, 0 plus 0 is 0, 1 plus 0 is 1, 0 plus 1 is 1, and 1 plus 1 is 0. Similarly

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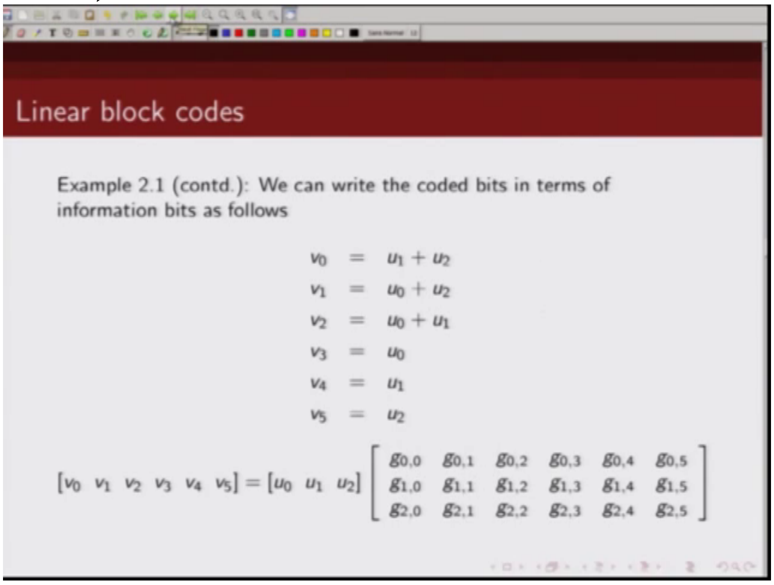

we notice that v 3, v 4, v 5 are nothing but information bits u 0, u 1 and u 2 respectively. So let's go back.

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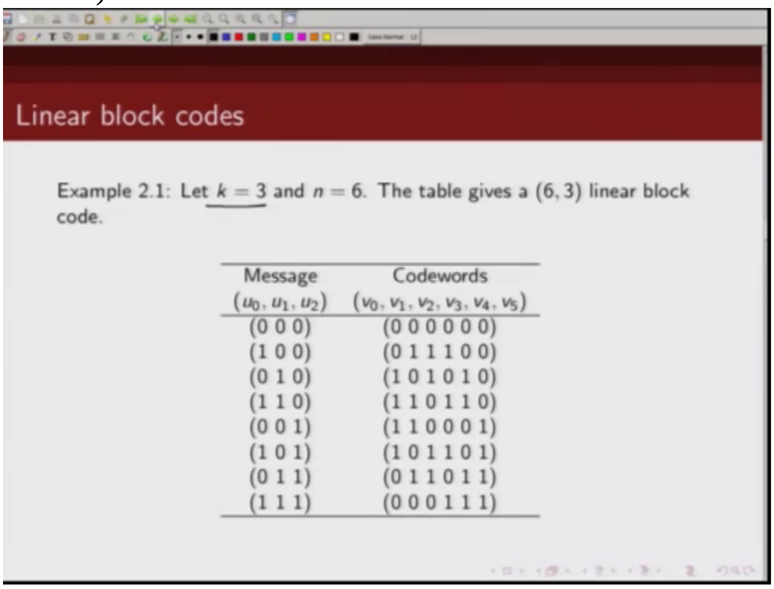

v 3 is this column and we can see this is same as u 0. 0 1 0 1 0 1 0 1, similarly v 4 is equal to u 1 and v 5 is same as u 2. So now

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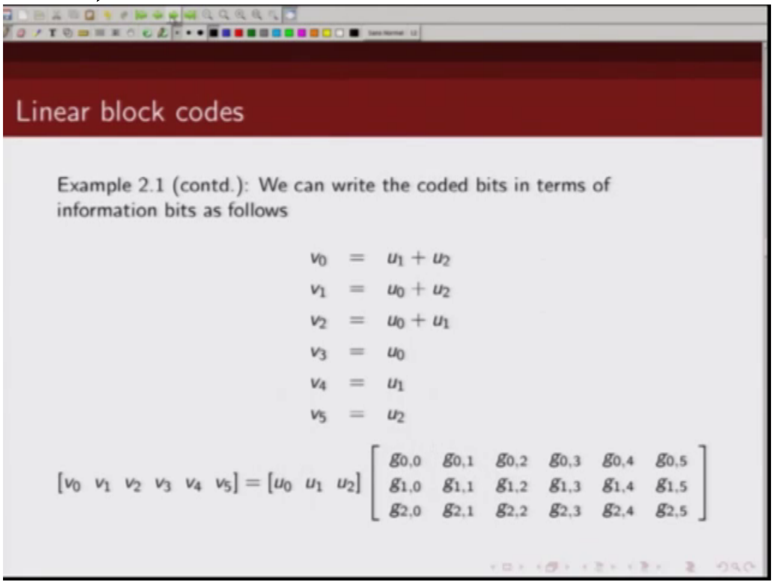

we have written our coded bits in terms of our information bits. This set of 6 equations I can write it in a matrix form. So I can write my coded bits in terms of information bits and this matrix G which is our generator matrix will tell us how are we generating each of these coded bits as a linear combination of these information bits. So if we compare each equation, let's look at v 0. So what is v 0? v 0 is u 0 g 0 0 plus u 1 g 1 0 plus u 2 g 2 0 and what do we see here? v 0 is u 1 plus u 2. So that means g 0 0 is 0 because there is no u 0 term here. g 1 0 is 1 because there is a u 1 term here and g 2 0 is 1 because there is a u 2 term here. So this will be 0 1 1. Similarly look at v 1. v 1 is u 0 g 0 1 plus u 1 g 1 1 plus u 2 g 2 1. And if we compare it with v 1 here we see v 1 is u 0 plus u 2 that means this g 0 1 should be 1, g 1 1 should be 0, and this should be 1. Likewise we build up the other columns of the matrix. So if we do that

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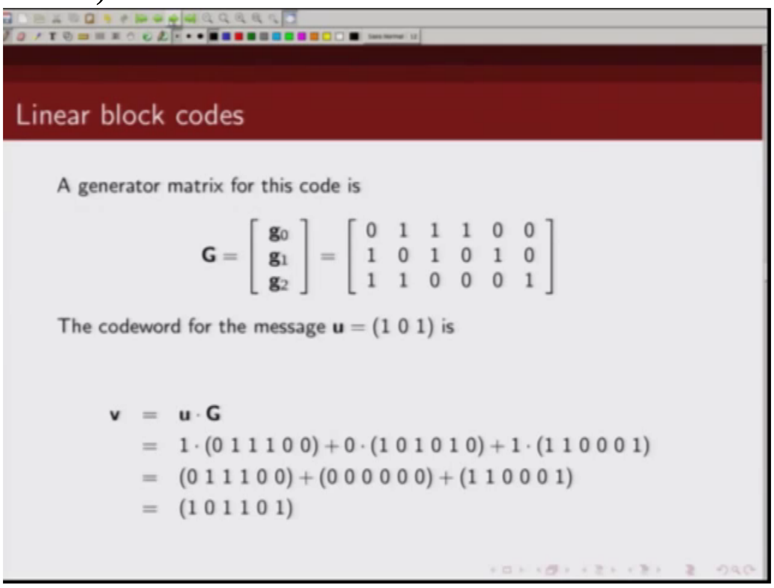

what we get is something like this. We can verify basically. Let's take second last column,

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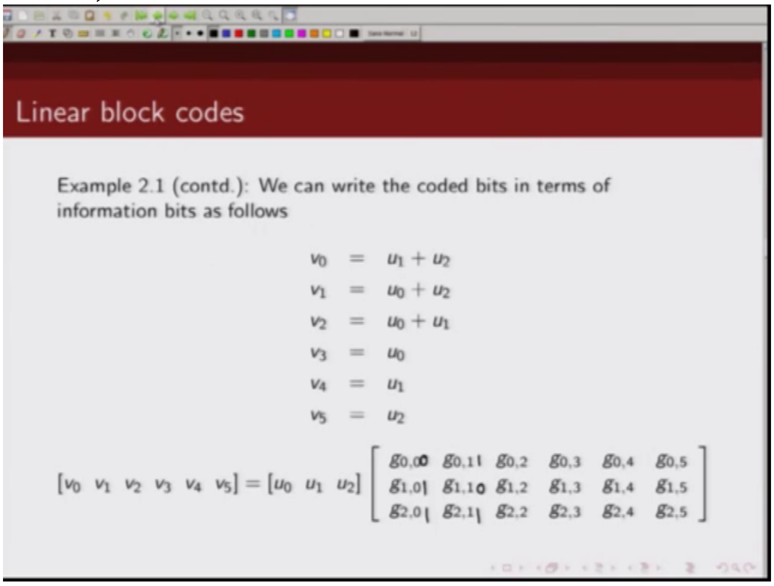

second last column so what is v 4. v 4 is u 0 times g 0 4 plus u 1 times g 1 4 plus u 2 times g 2 4 and what is v 4, v 4 is u 1 so then this should be 1 and this should be 0 and this should be 0 and this is what we have,

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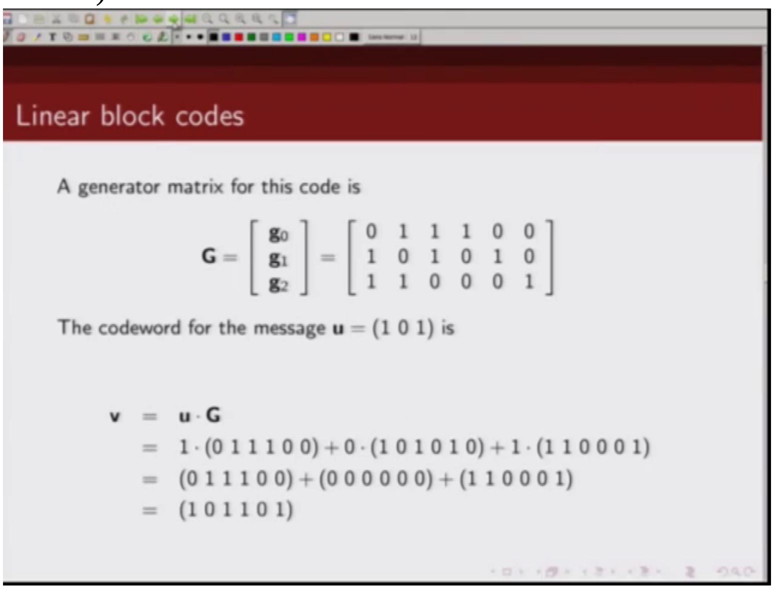

0 1 0 So now we can basically find out

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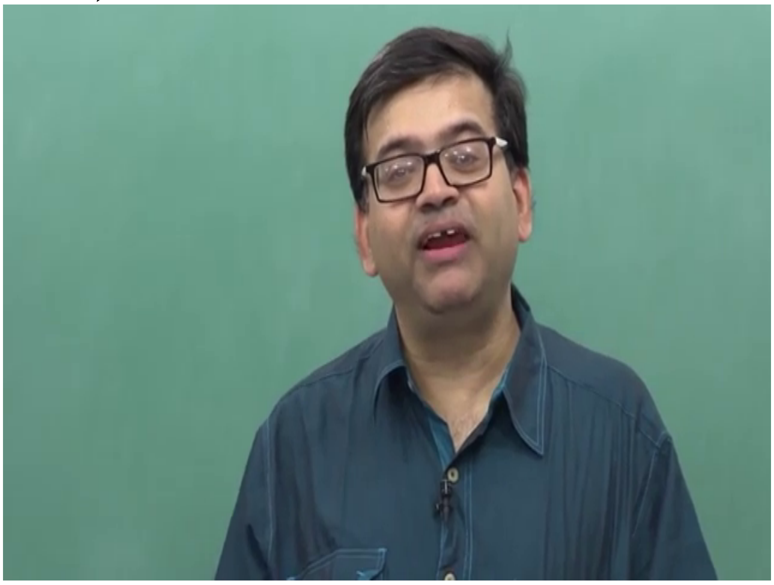

the generator matrix; so linear block code is completely described by its generator matrix And we said we can use the generator matrix to generate our codewords.

For example, if my information sequence is

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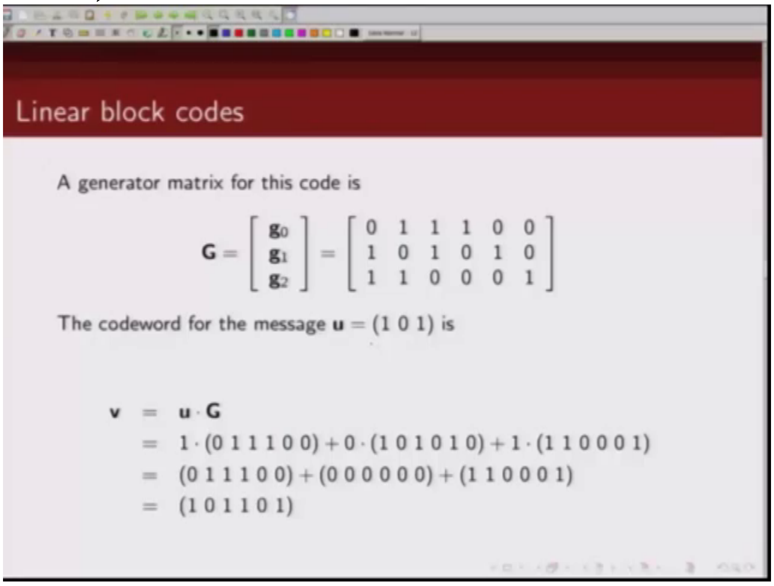

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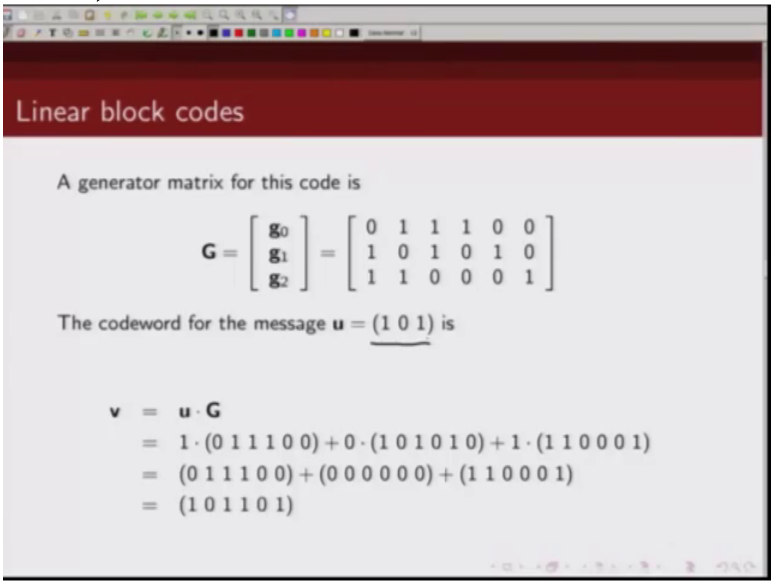

1 0 1, what should be my corresponding coded bits for the information sequence 1 0 1? How do I find that? So as I know, my output codeword is basically u times G

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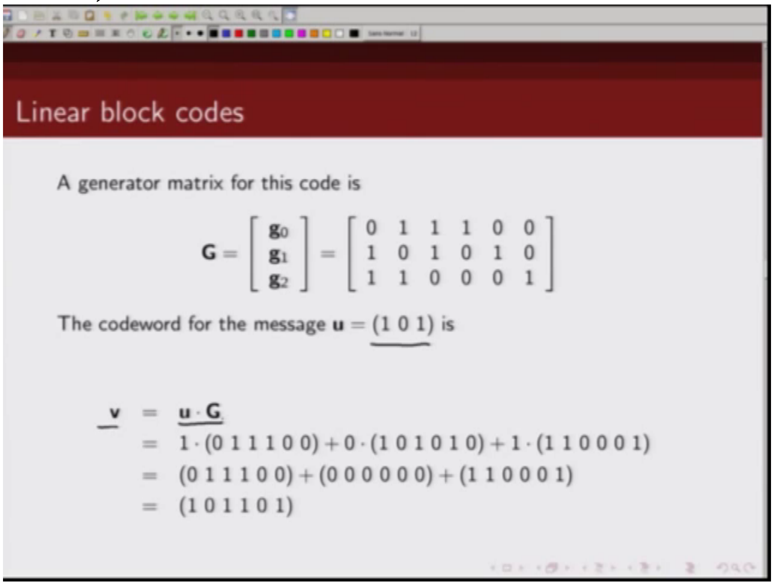

so I will take linear combinations of rows of my generator matrix. What are the rows of my generator matrix? These are the 3 rows of my generator matrix. So my coded bit corresponding to this information sequence would be 1 times G 0 plus 0 times G 1 plus 1 times G 2. So that's what I have written here, 1 times G 0, 0 times G 1 plus 1 times G 2. So this is basically 0. So what I have

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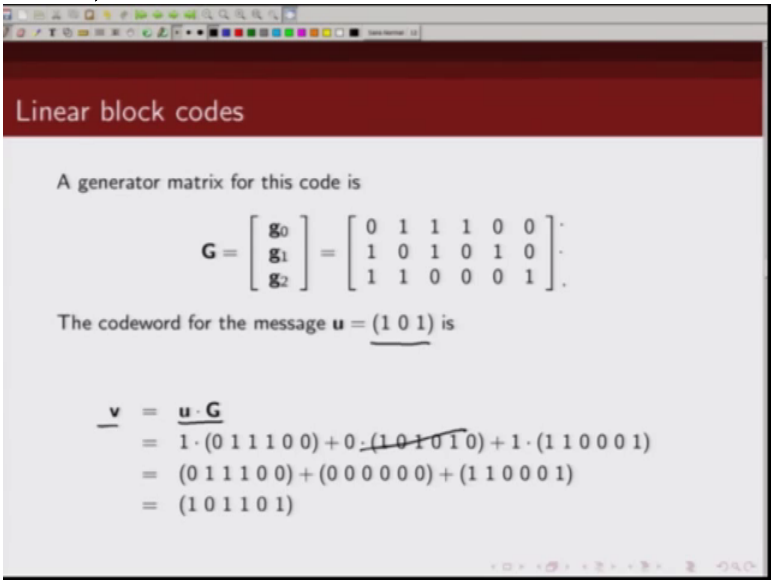

is then, this plus this right? So let's look at 0 plus 1 would be 1, 1 plus 1 would be 0, 1 plus 0 is 1, 1 plus 0 is 1, 0 plus 0 is 0 and 0 plus 1 is 1. So my codeword corresponding to this information

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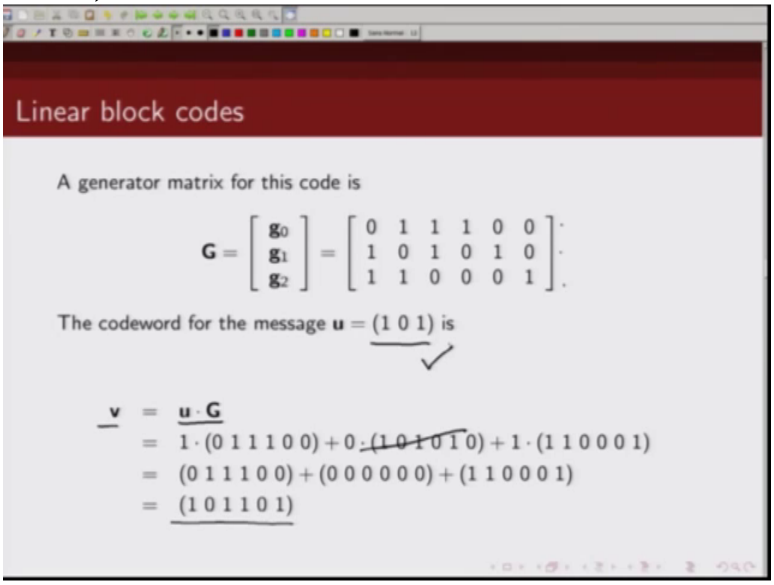

message bits, this information bit is given by 1 0 1 1 0 1, Ok?

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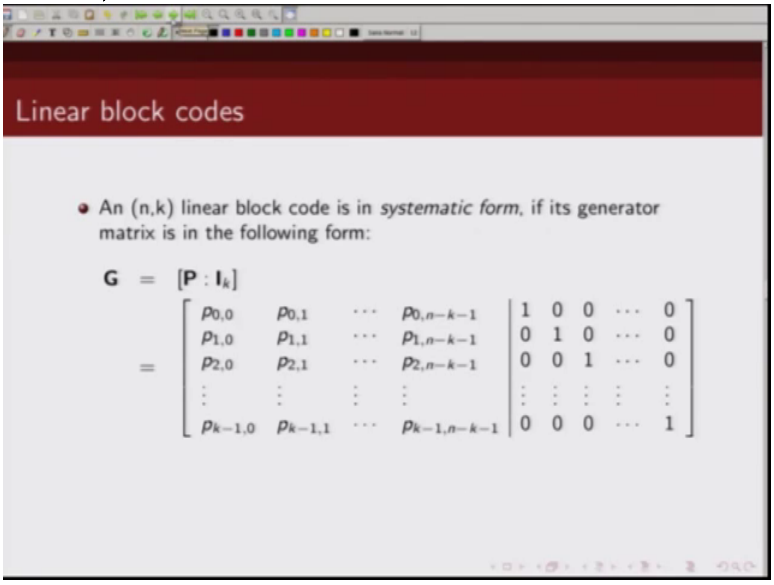

Now what do we mean by a linear code in systematic form? Now if we are able to,

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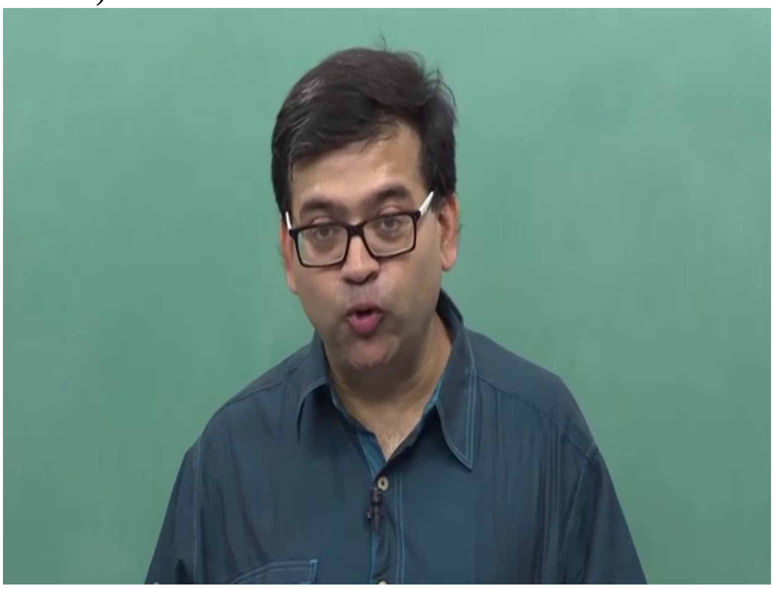

among the coded bits if we are able to separate them out into, if the message bits appear directly in the coded bit sequence then we can separate out the message bits from the parity bits. For example, go back to this example.

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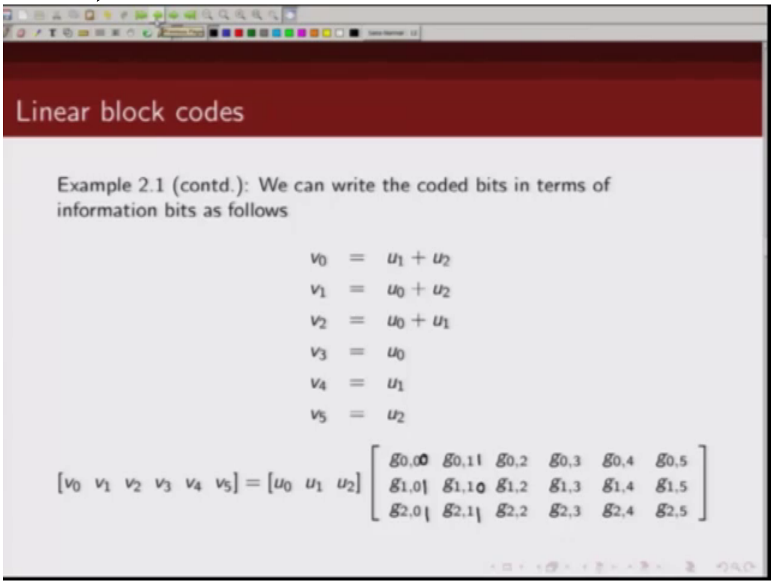

What do we have here? We have 3 of these coded bits

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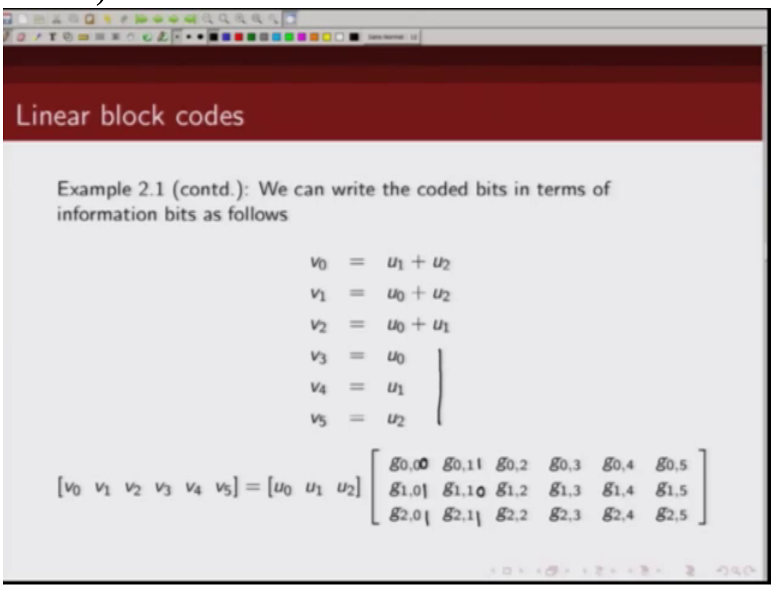

exactly same as information bits, and the other 3 bits,

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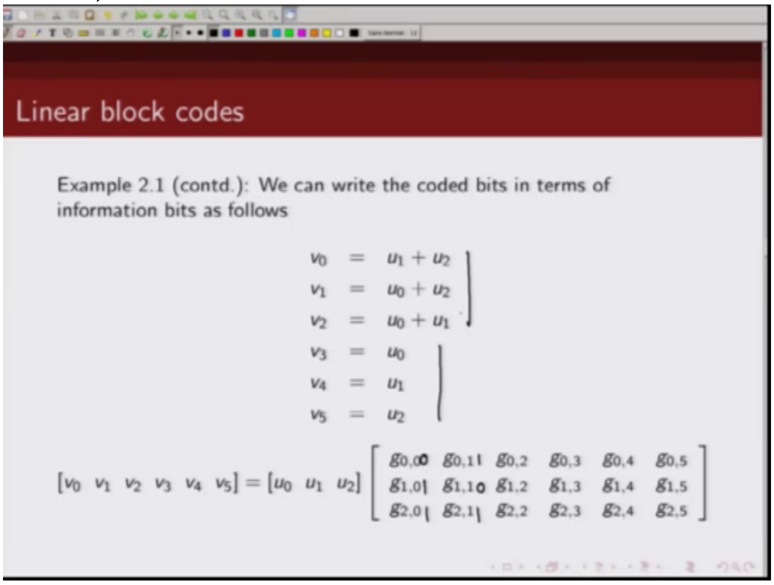

parity bits are linear combination of this message bits. So from the output codeword we, we can clearly separate out the information sequence which is in this case given by v 3, v 4 and v 5. So in this case, v 1 v 2 v 3

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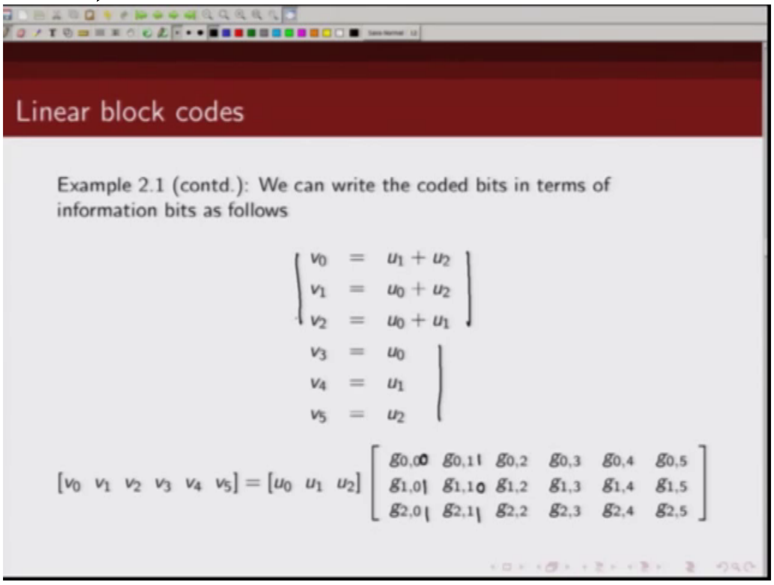

are these n minus k parity bits

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and v 3 v 4 v 5 are my

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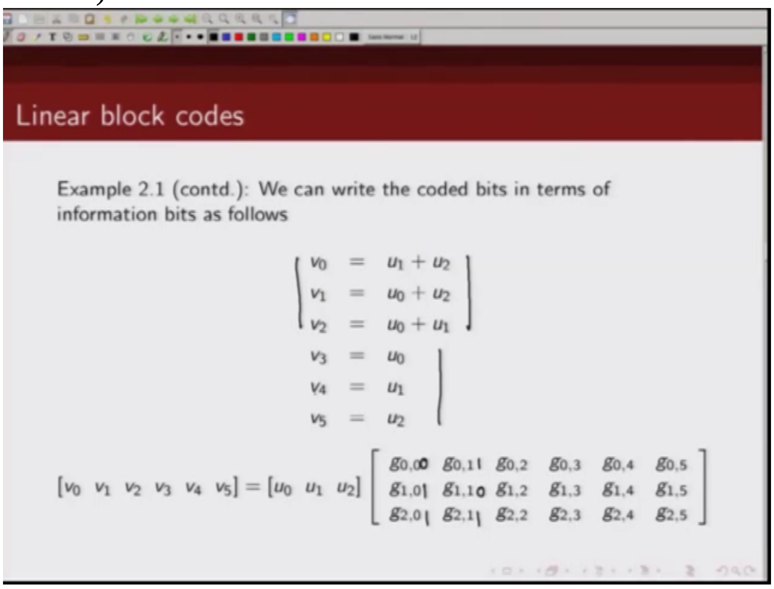

information bits So in this particular example, we can see that

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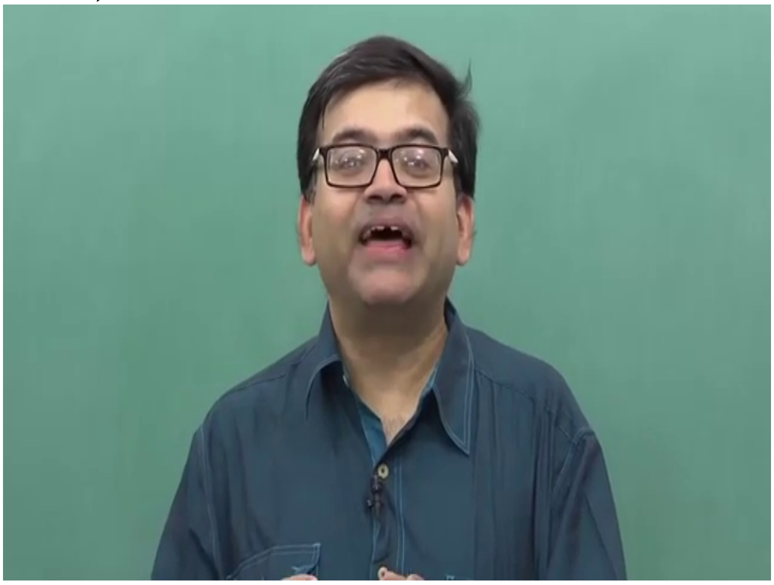

that we are able to separate out information bits directly from the coded bits. So in a systematic,

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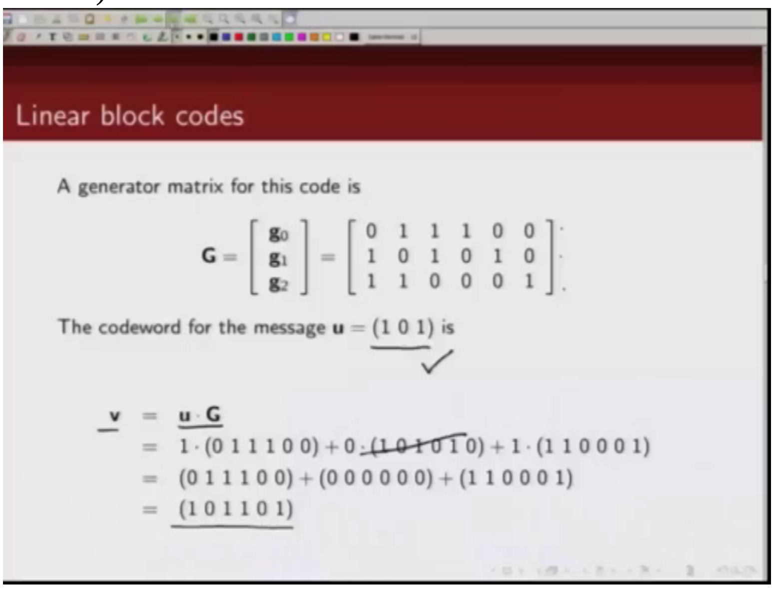

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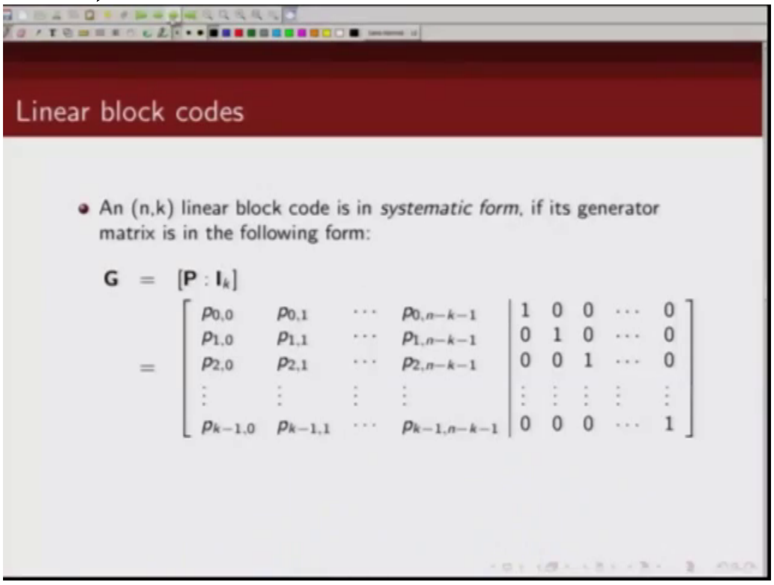

a block code in a systematic form, we are able to separate out

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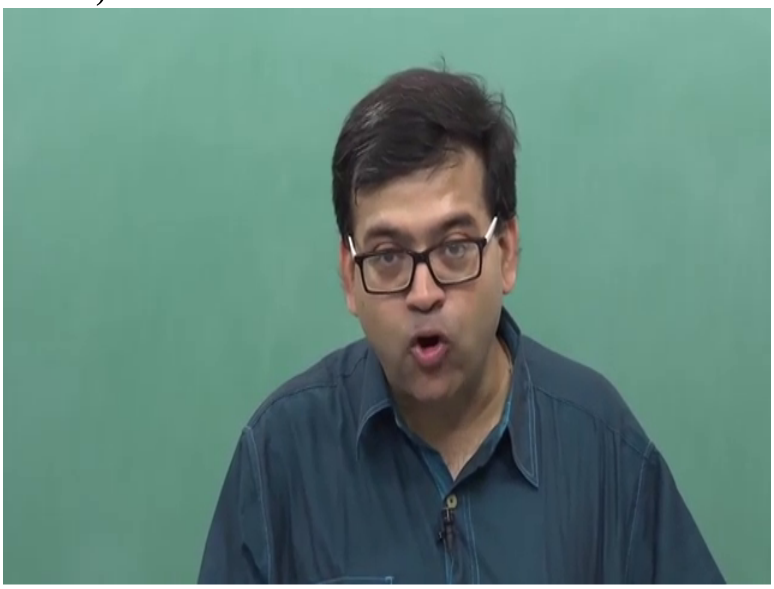

out the information bit part from the coded bits So a

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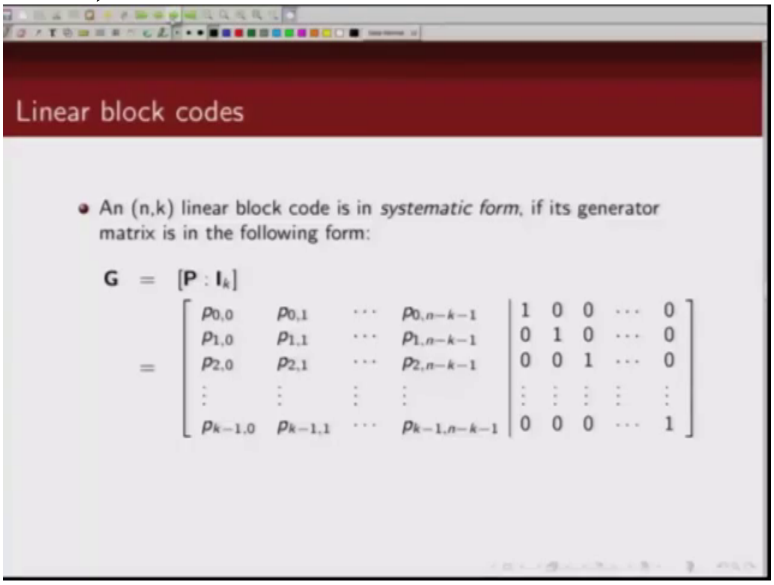

generator matrix for a linear block code in systematic form will be of the form like this or it would be basically i times i k times some some,

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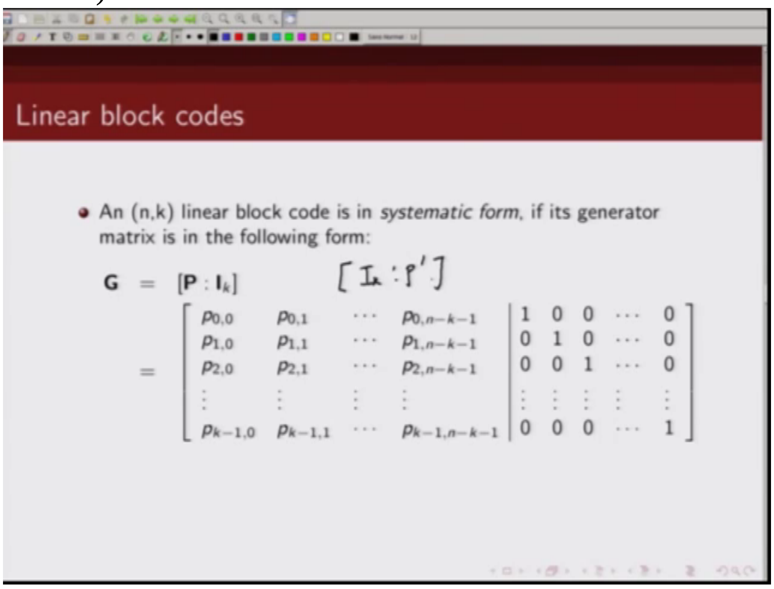

either of this form. Now why do we say that? So only when we have our, part of our generator matrix of the form of identity, then what is going to happen? When we multiply our information sequence

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with this sort of generator matrix you will see part of

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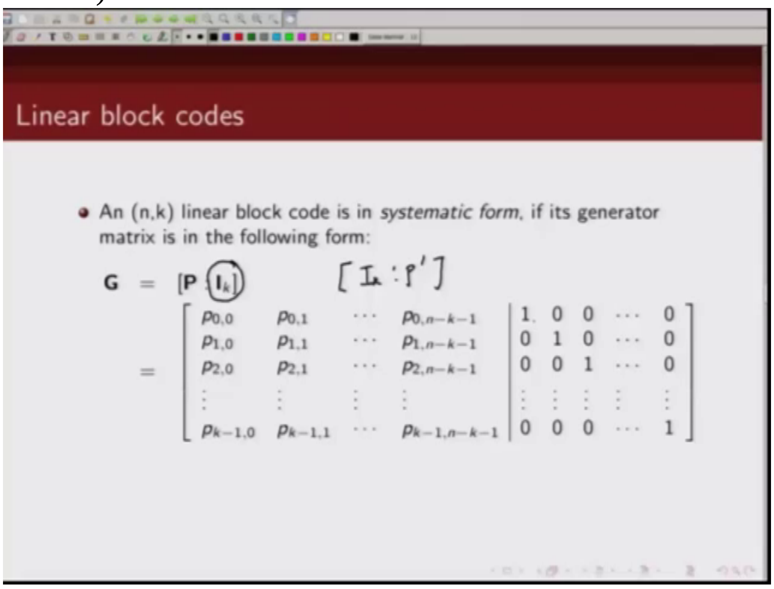

my coded bits will just depend on one particular information bit sequence. So if I have write down the corresponding equations for coded sequence

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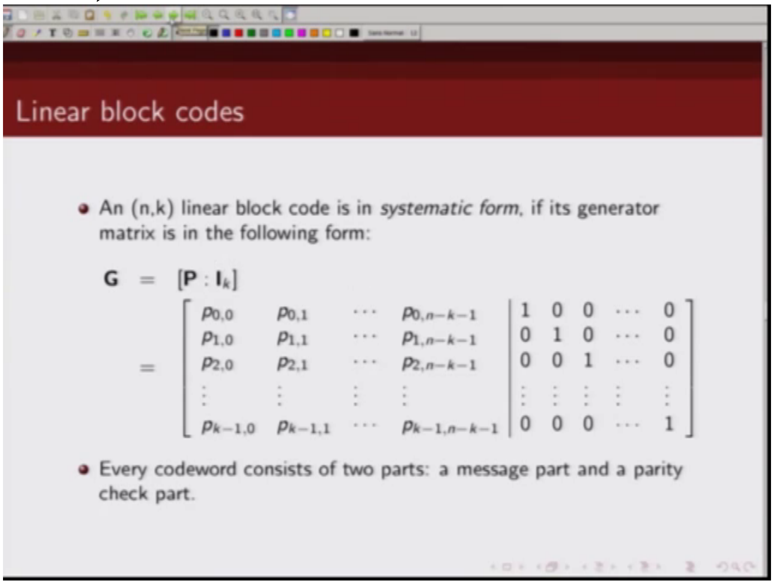

what you will see that some coded bits directly depend on the

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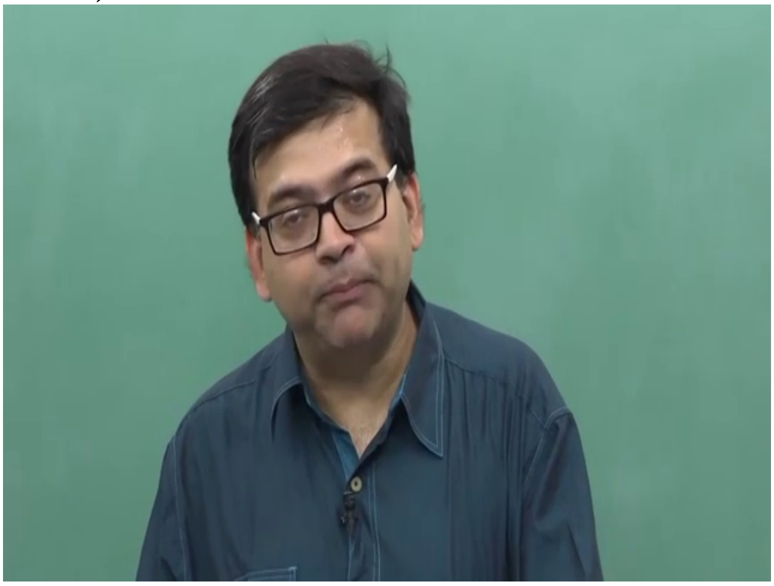

message bits and then rest are, which are parity bits are linear combination of these message bits. So in a systematic form basically we can separate out the message part from the parity bit part. So

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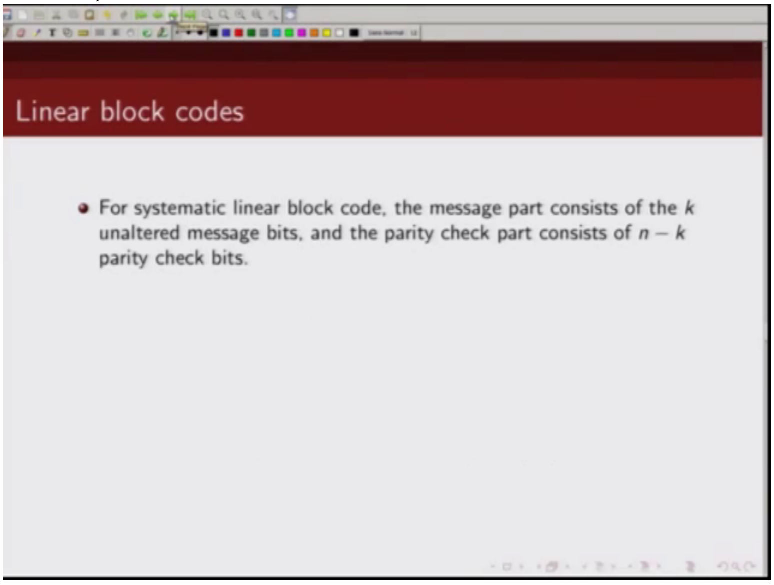

as I said, for a systematic linear block codes, the message part will consist of the k information

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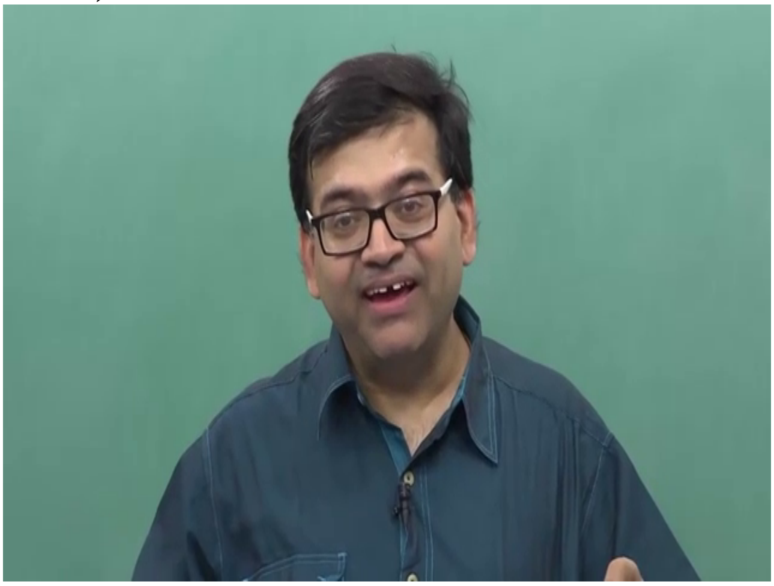

bits and the remaining n minus k bits which are the parity bits,

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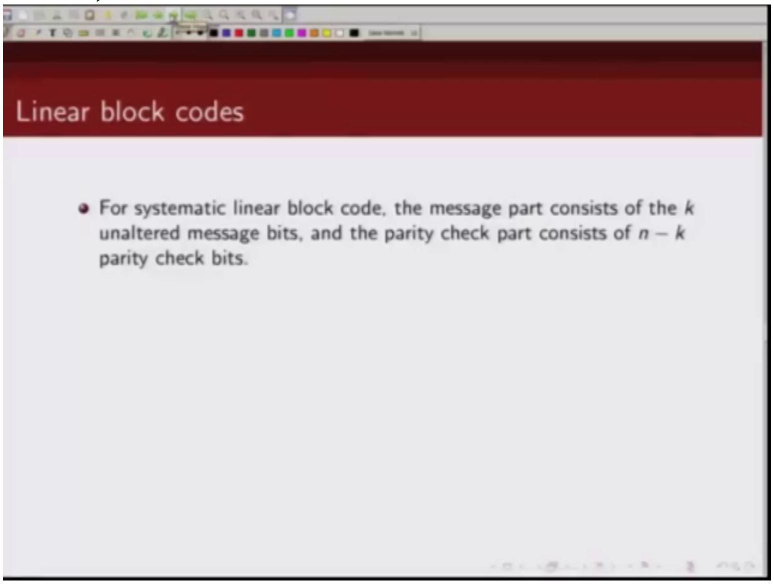

basically they will be linear combination of these message bits. So we

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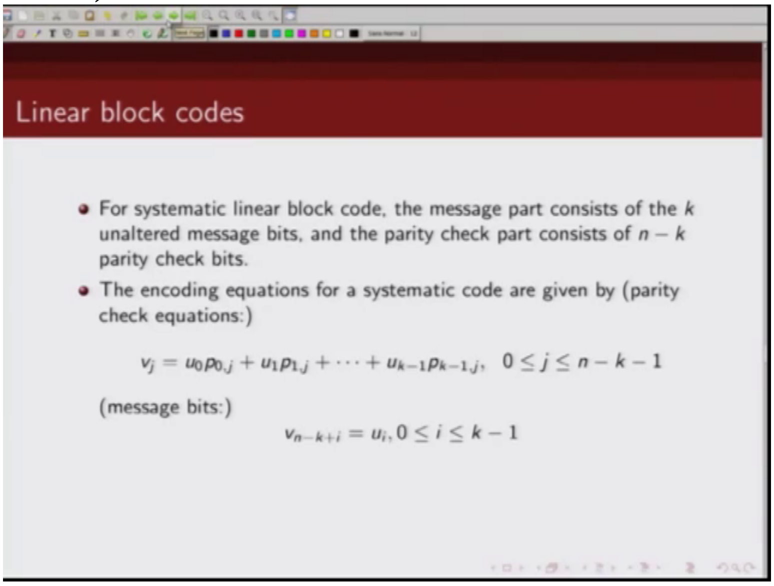

can write down the encoding equations for these matrices, for these, for systematic code, so if you look at

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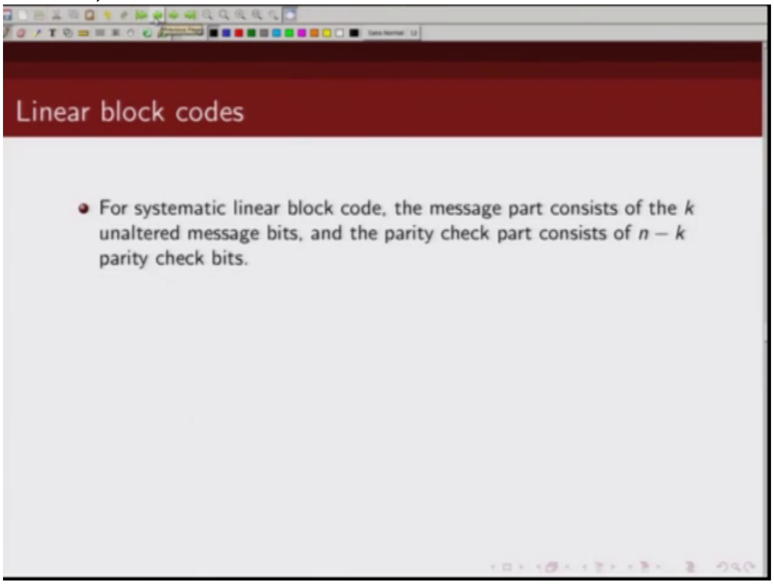

what is our

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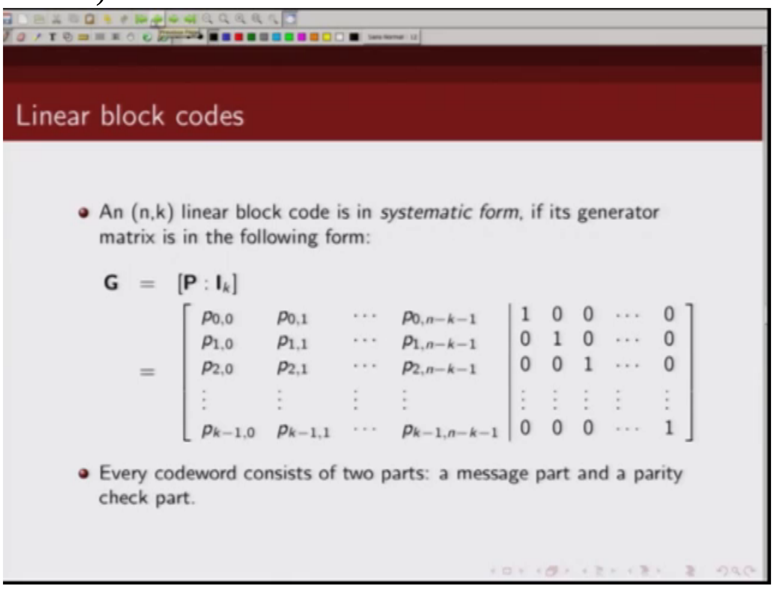

encoding equation? Our v is u times G where G is of form like this, Ok? So if we write u which is basically u 0, u 1 to u k minus 1 times this G matrix, what we will get is

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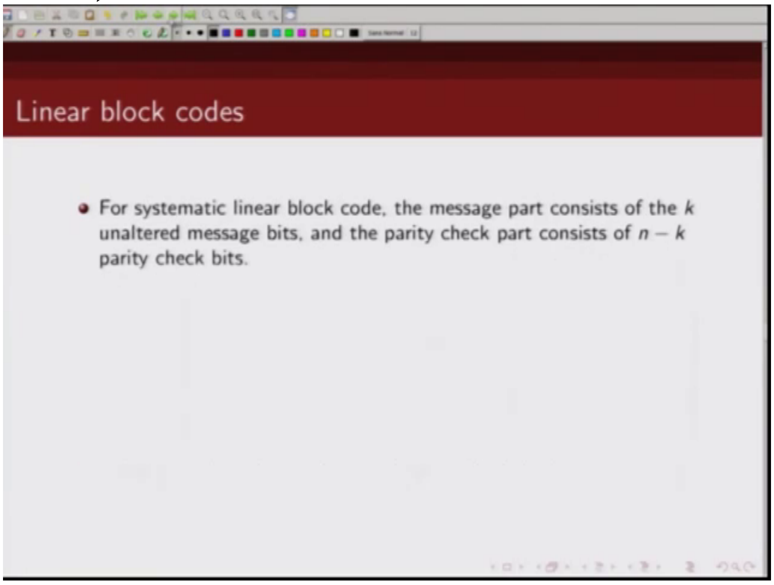

a form

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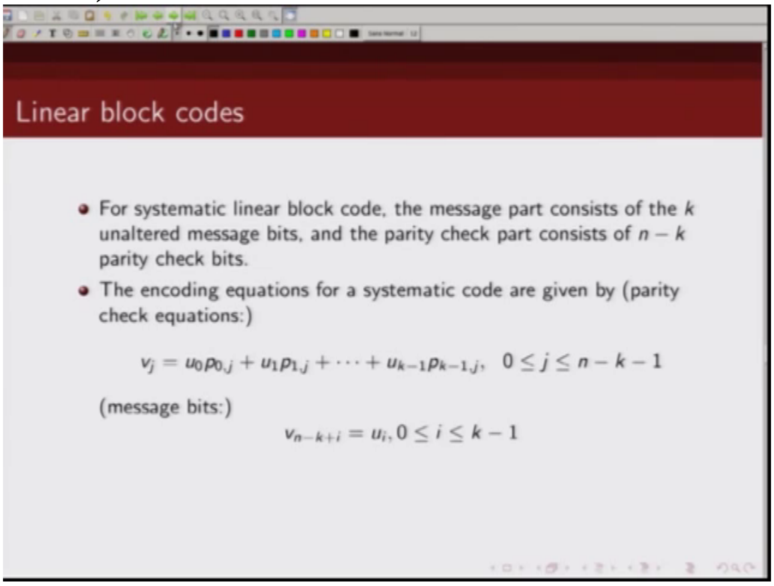

like this. So you will have n minus k parity equations, parity check equations which are given by this expression and then you will have remaining

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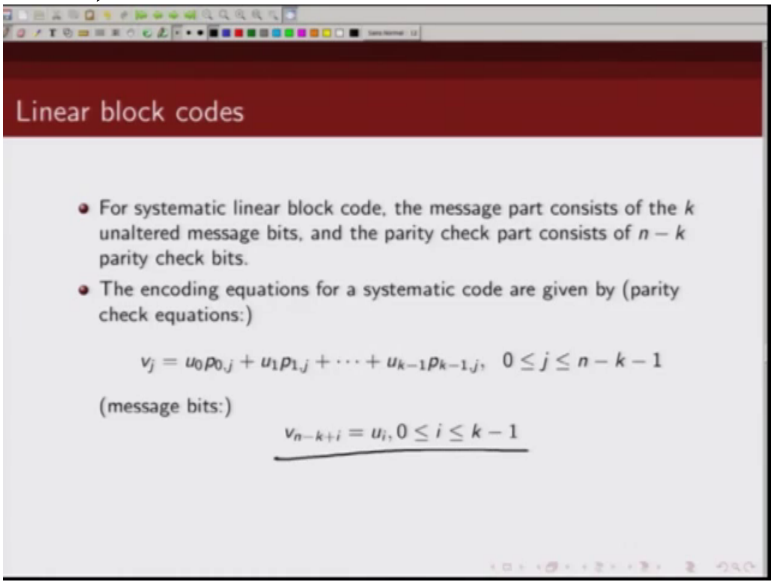

k unaltered message bits. So for a linear block code in a systematic form the encoding equations will be of this form. And as

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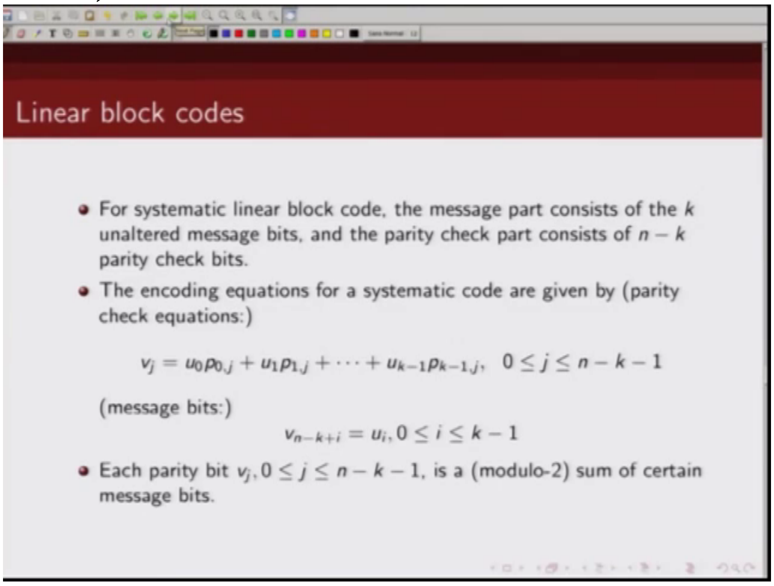

as I said since we are restricting ourselves without any loss of generality

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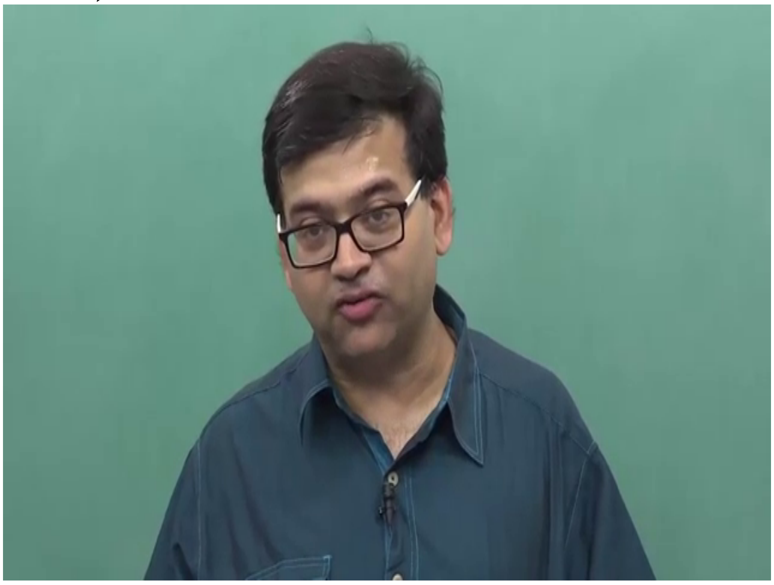

to binary codewords, this addition is basically done

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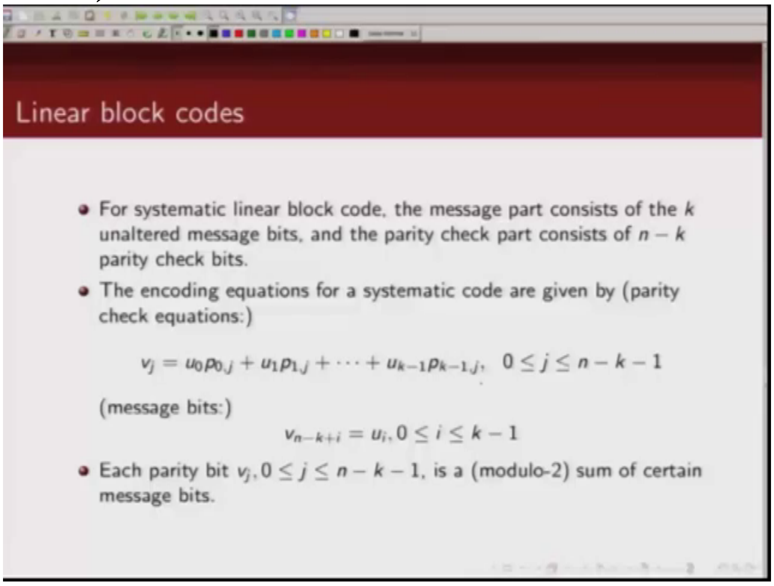

modulo 2

So what we have seen so far is we can describe

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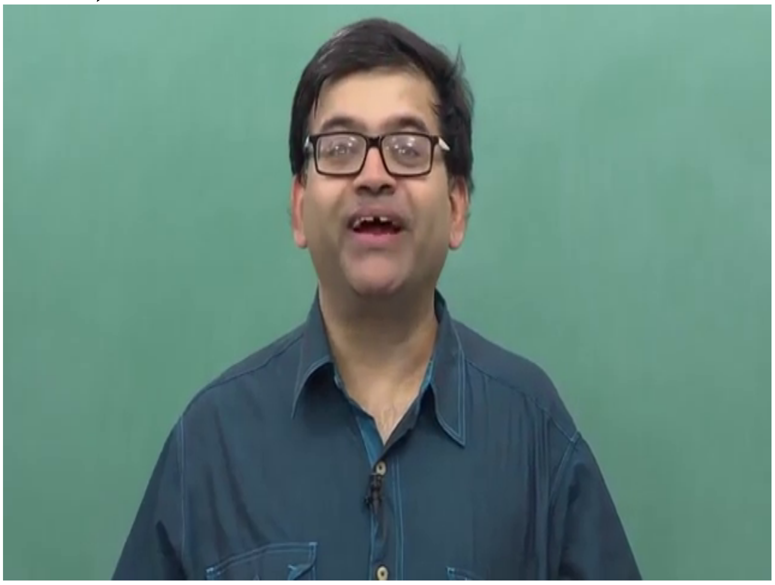

a linear block code by its generator matrix which is a k cross n matrix and we can use this generator matrix to generate our set of codewords. Now there is another matrix which we call parity check matrix which is related to our generator matrix, we will show, which can also be used to completely describe a linear block code. So for a n k linear block code

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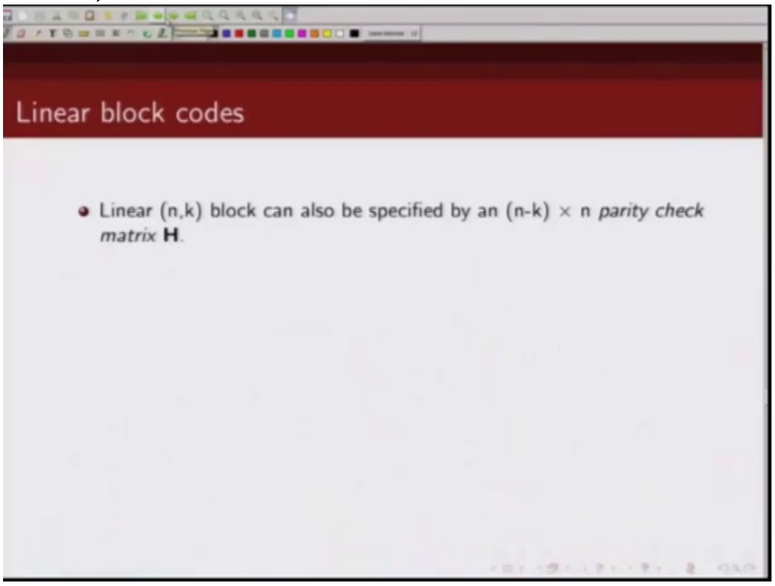

can be specified by a n minus k cross n

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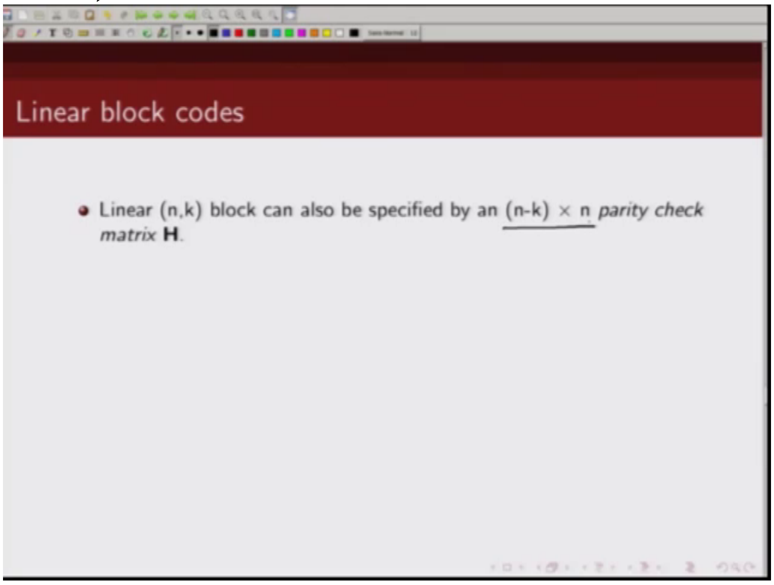

parity check matrix which we denote by G, H, the generator matrix we denote by G and the parity

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check matrix we denote by H. Now this parity check

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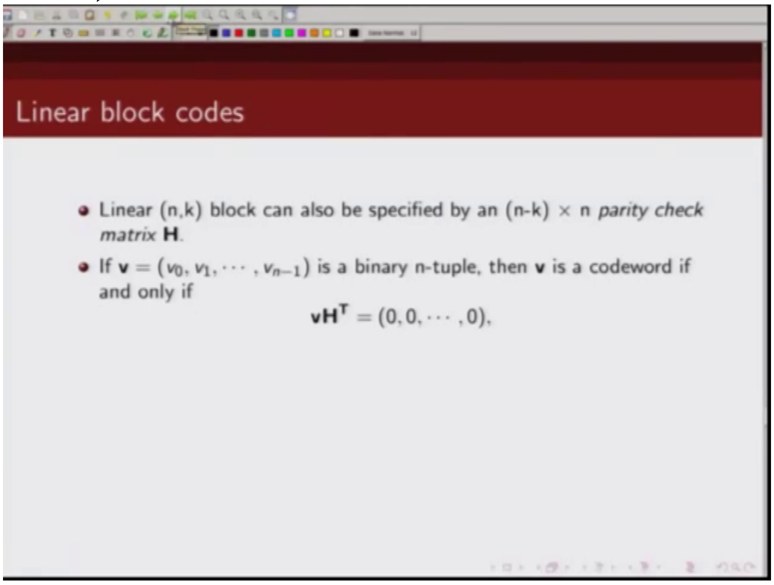

matrix has this property that if v is your valid codeword, if and only if v H transpose is going to be 0. So if v is a valid codeword v H transpose will be 0. So let us see how we can derive our parity check matrix from a

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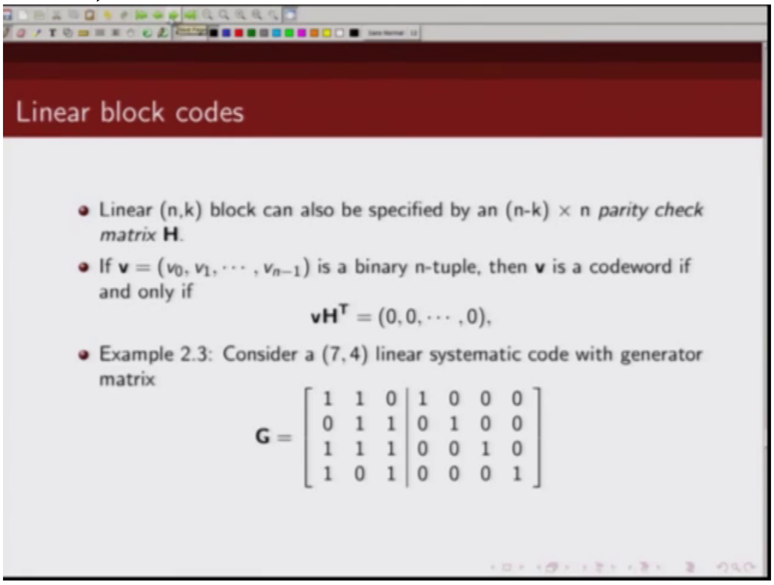

generator matrix and what's a relation of the generation matrix with the parity check matrix. So we will take an example of a 7 4 systematic linear block code whose generator matrix is given by this. So since this is a systematic code we can write it of the form p times this i k. This generator matrix can be written of this form,

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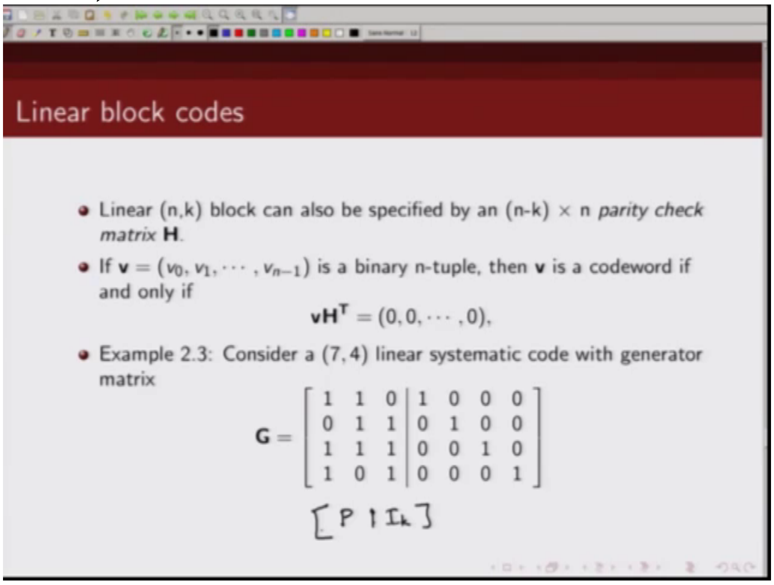

Ok. Now from this generator equation we can write our coded bits

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in terms of our message bits So let's do that. So

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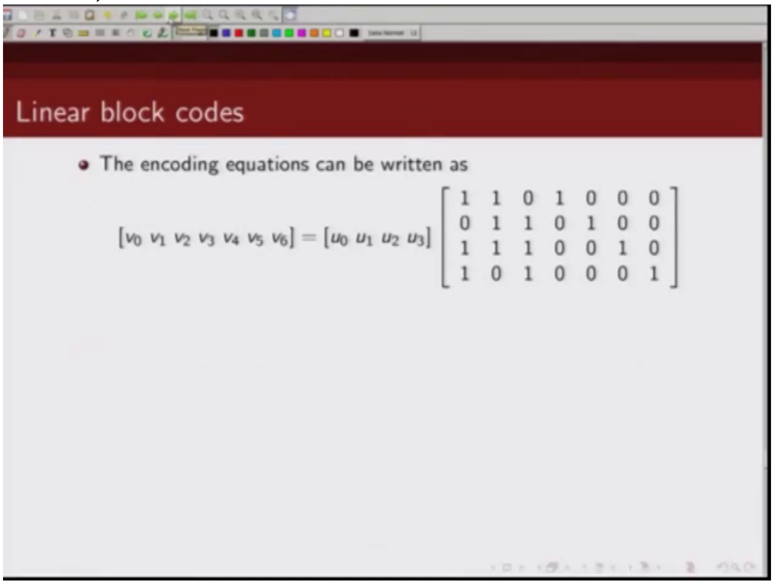

the encoding equation; this was our generator matrix G,

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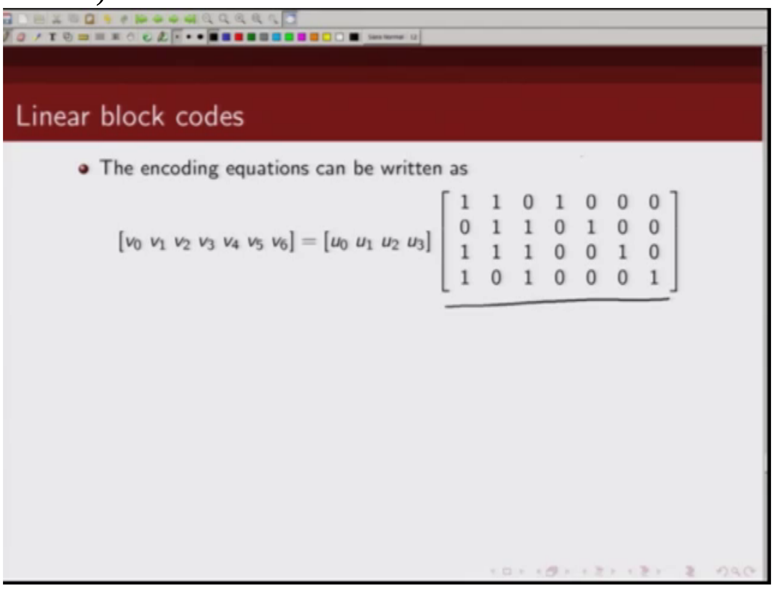

this is

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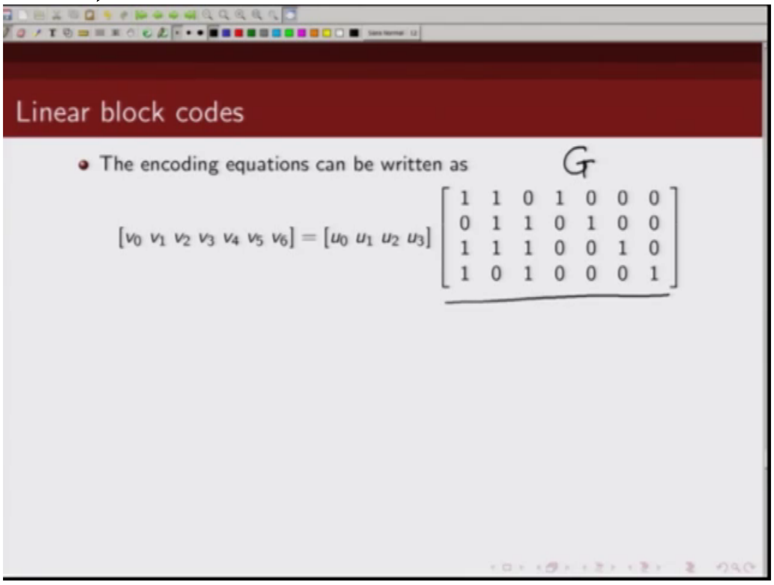

our information bits message

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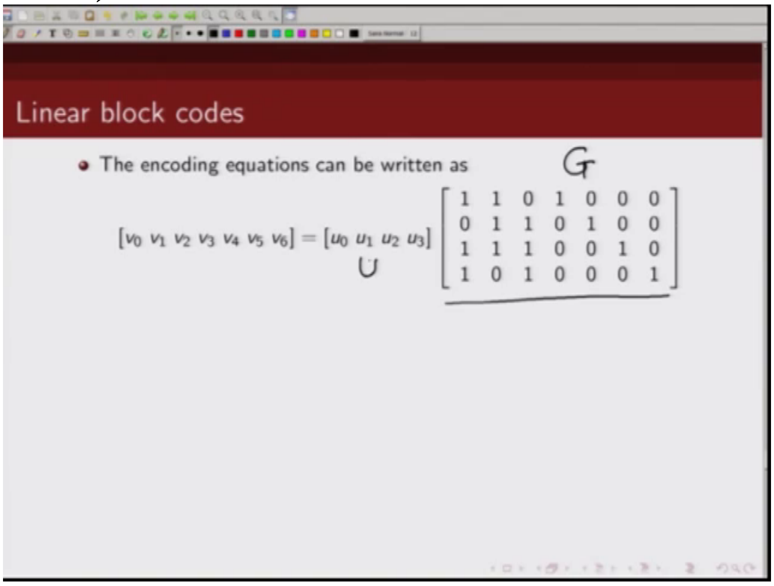

bits u and this is our coded bits v. So we can write v

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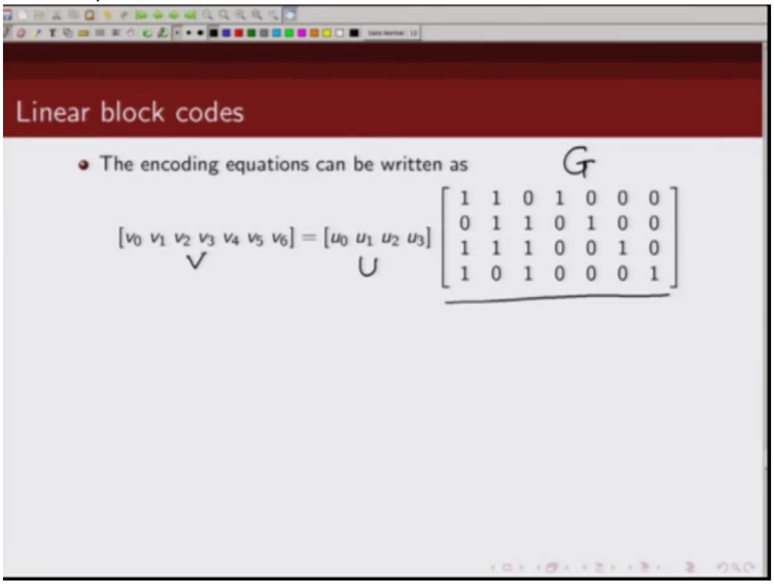

as u times G So it's a 7 4 code. So there are 4 information bits. I denote them by u 0 u 1 u 2 u 3. And there are 7 coded bits. I denote them by v 0 v 1 v 2 to v 6. And this generator matrix I have already given you this. So I can write down the encoding equations. We do that.

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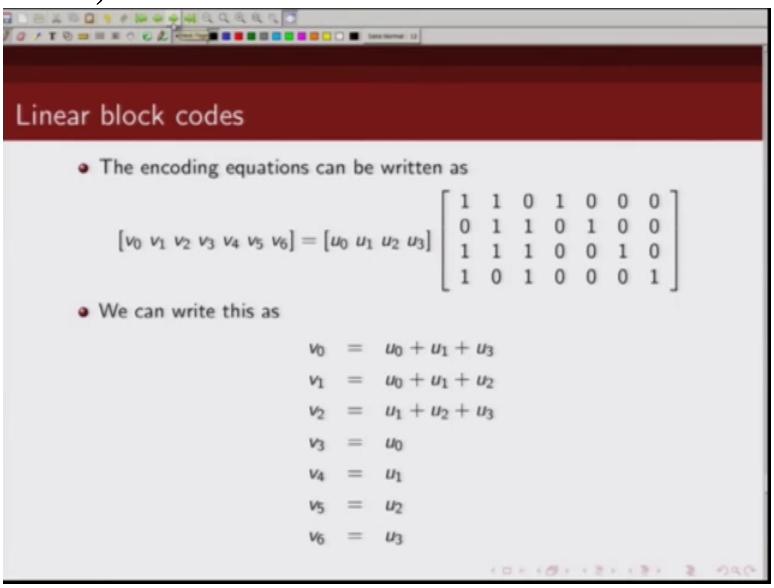

You can see from this what is v 0. It is u 0 plus u 2 plus u 3. There is a typing mistake here. This should have been u 2. So v 0 is u 0 times 1 plus 0 times u 1 plus 1 times u 2 plus 1 times u 3. So this v 0 is given by u 0 plus u 2 plus u 3. Similarly what is v 1? v 1 is given by u 0 plus u 1 plus u 2. v 2 is given by u 1, u 0 into 0, u 1 into 1, u 2 plus u 2 into 1 plus u 3 into 1. So v 2 is given by u 1 plus u 2 plus u 3. So that's what I have here. What is v 3? v 3 is given by u 0 into 1 and rest are all 0s So v 3 is nothing but u 0. Similarly v 4 is u 1, v 5 is u 2 and v 6 is u 3.

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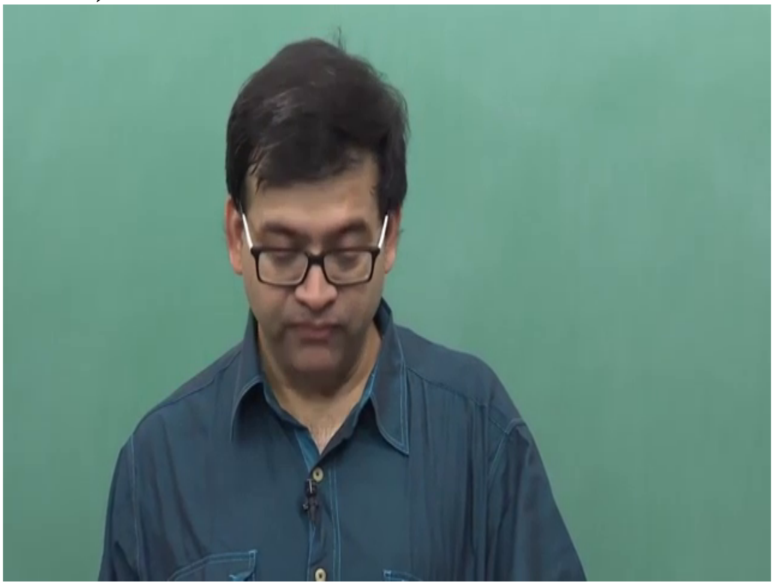

So I now have set of 7 coded bits and this shows the relation between the coded bits and information bits. Now we are, since we are restricting ourselves to binary codes we can

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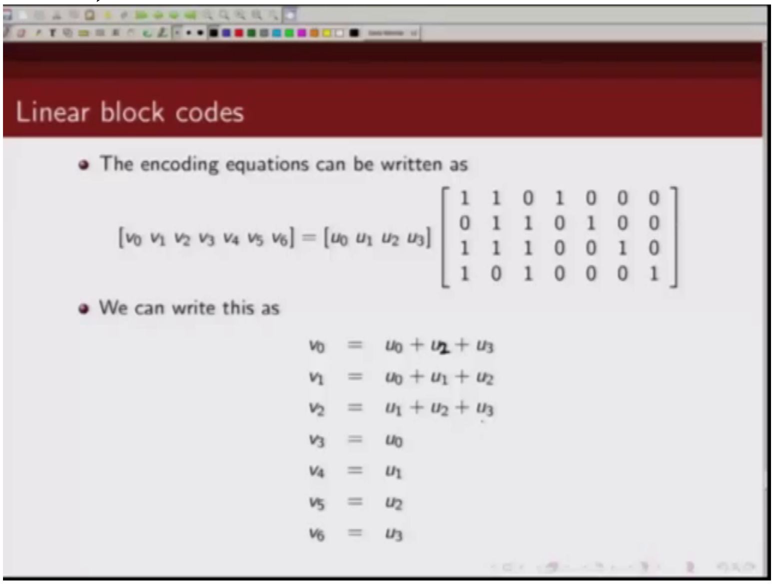

even write this equation like this, v 0 plus u 0 plus u 2 plus u 3 is equal to 0, correct?

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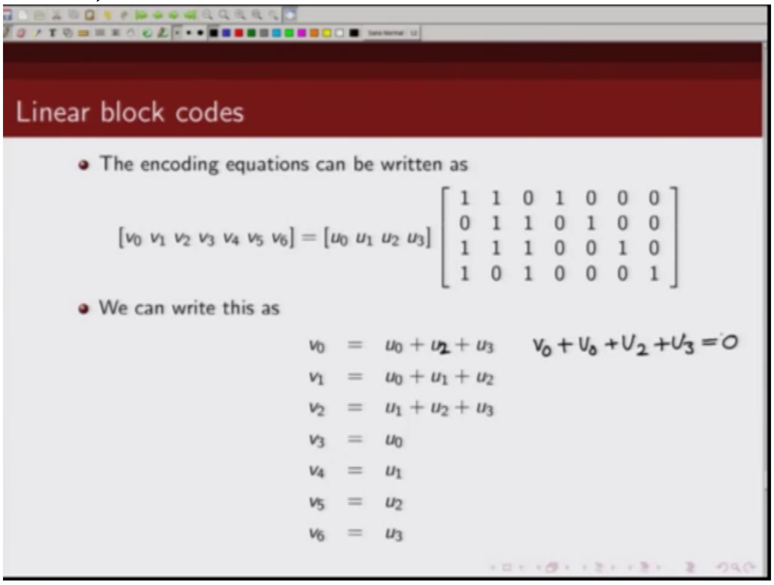

Because this v 0 v 1 is nothing but parity bit which is basically nothing but like 1 or 0. So we add this to this. Modulo 2 sum will be 0. So this similarly we can write as v 1 plus u 0 plus u 1 plus u 2 equal to 0

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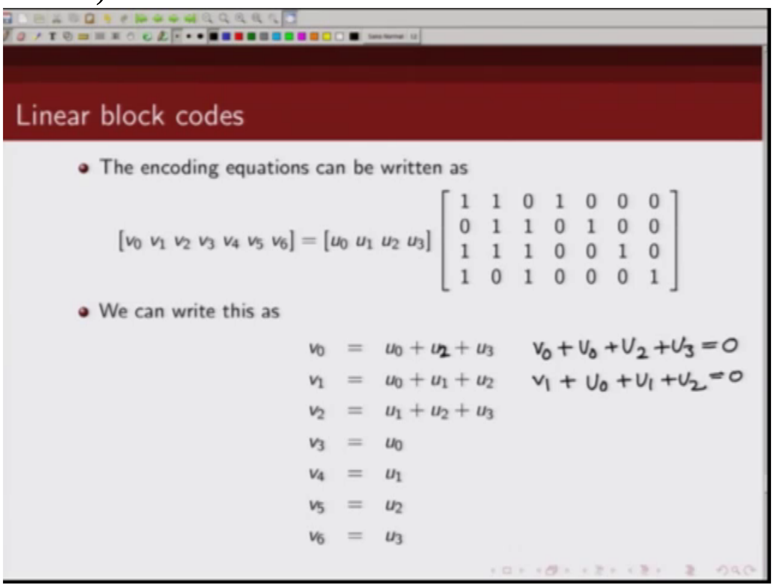

and this can be written as v 2 plus u 1 plus u 2 plus u 3 is equal to 0.

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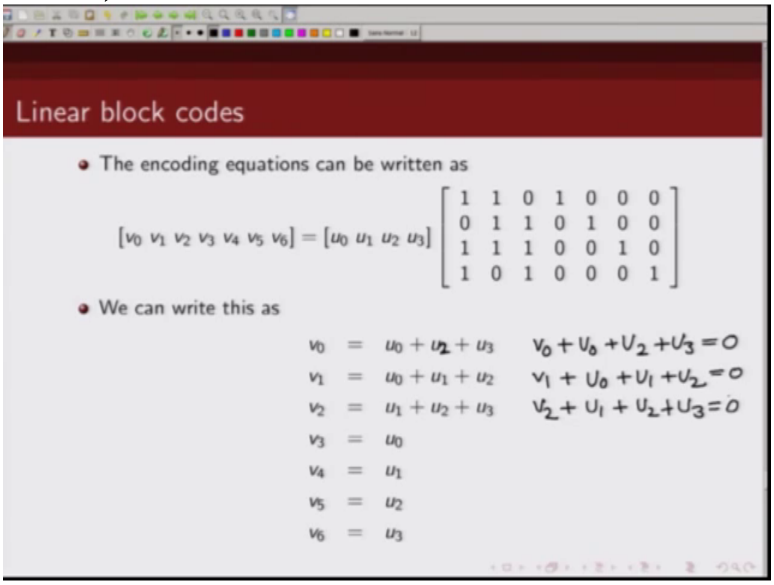

The next, what we would try to do is we would try to write these parity check equations in terms of other coded bits. So we can see here, u 0 is nothing but v 3. So wherever u 0 appears, we can replace it by v 3. Similarly u 1 is equal to v 4. So wherever u 1 appears we can replace it by v 4. u 2 is equal to v 5, so we can replace u 2 by v 5. And u 3 is equal to v 6. We can replace u 3 in terms of v 6. By doing this, what we will get is set of

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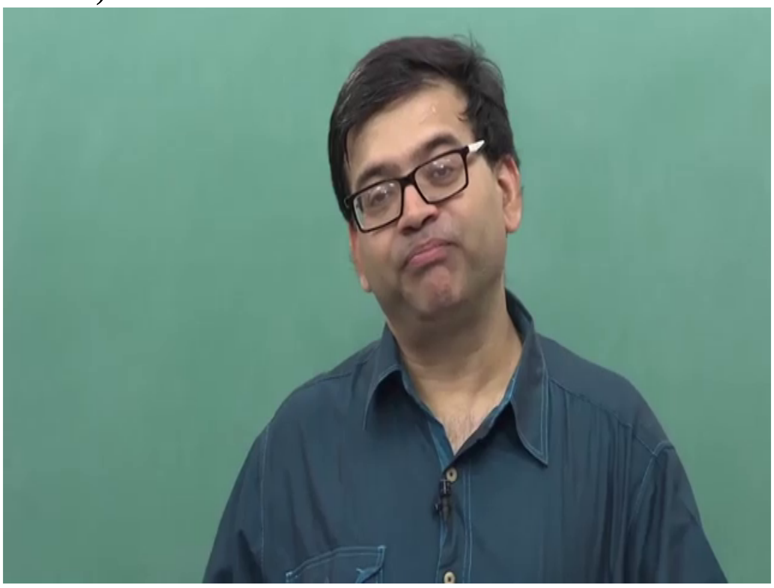

equations which basically are dependent on these coded bits. If we do that, what we get

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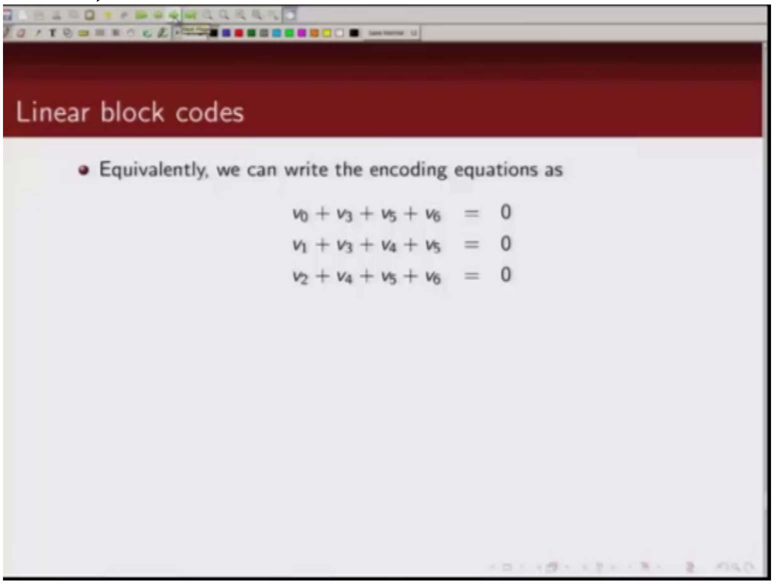

is something like this The first expression basically which was,

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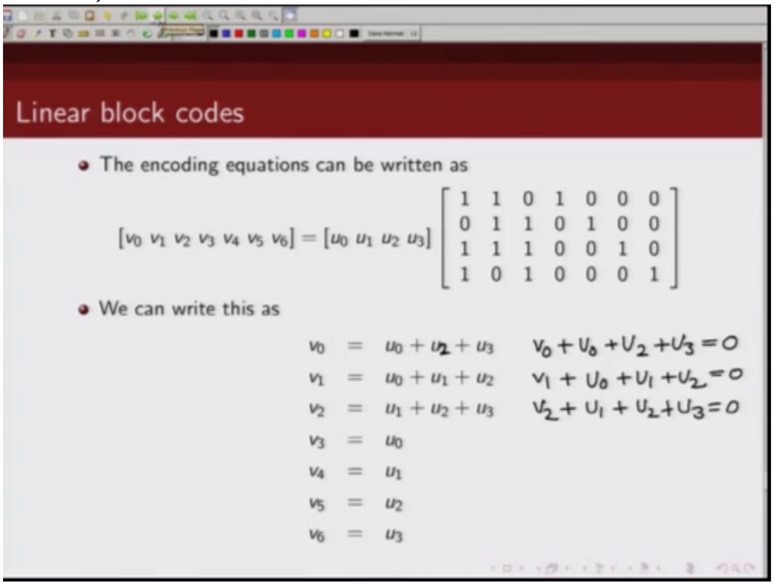

v 0 plus u 0 the u 2 plus u 3 Now this can be re-written as v 0 plus what is u 0, u 0 is v 3, v 3 plus what is u 2, u 2 is v 5, v 5 plus what is u 3, it is v 6. So

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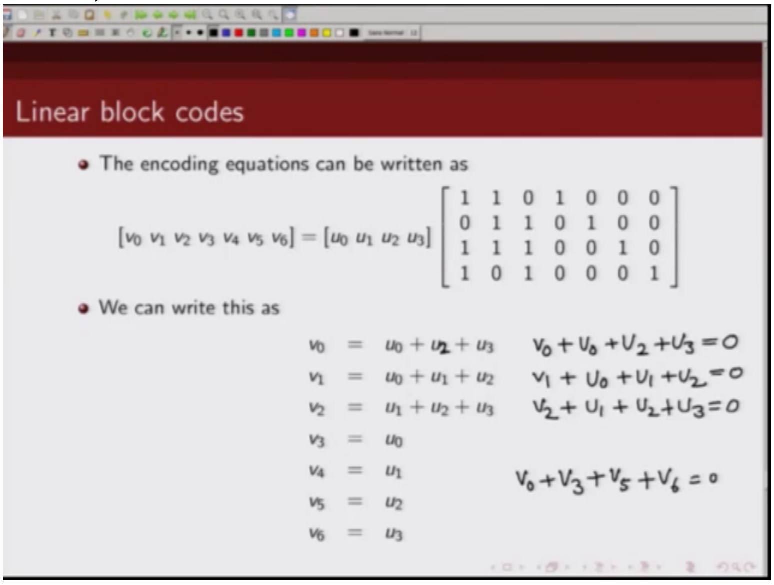

v 0 plus v 3 plus v 5 plus v 6 is 0, and that is what we have here.

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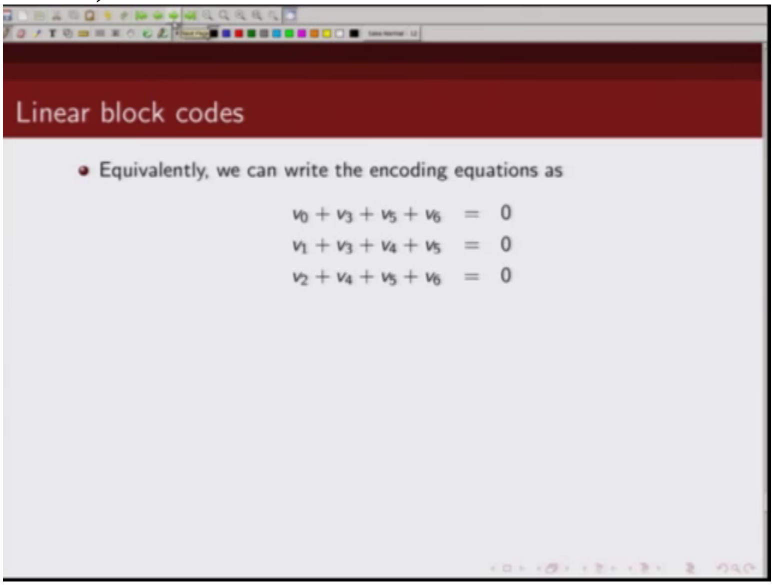

v 0 plus v 3 plus v 5 plus v 6 is equal to 0. Similarly we can write

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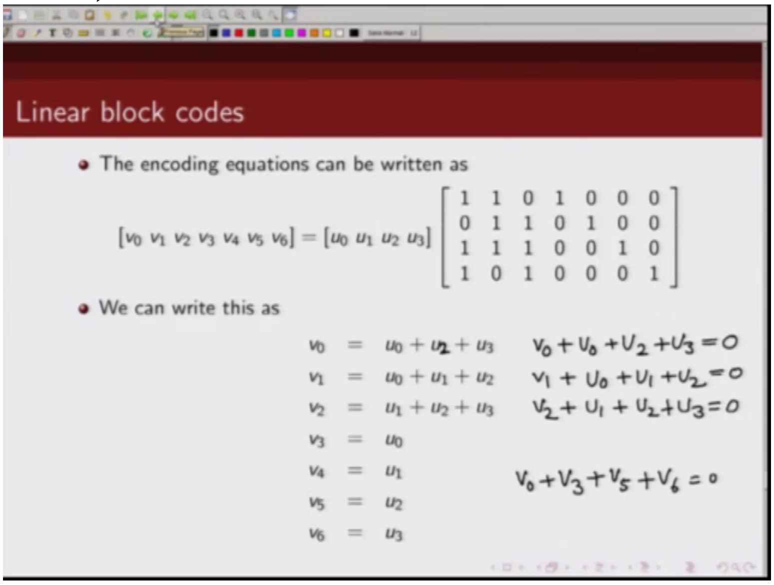

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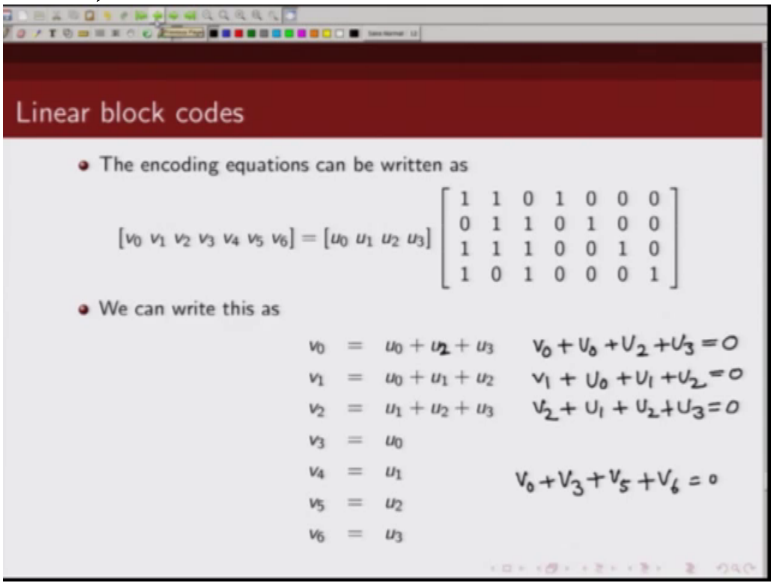

the other equations as well Here also we will replace u 0, u 1, u 2 by v 3, v 4, v 5 and what we will get

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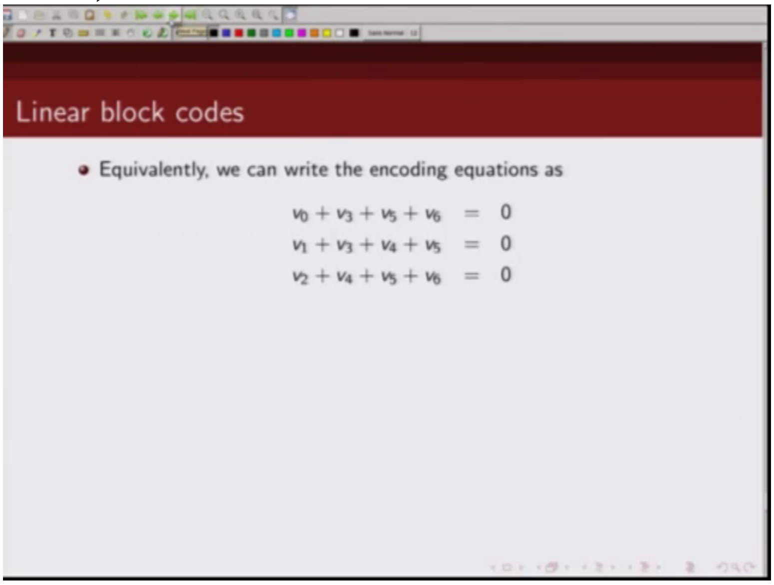

is v 1 plus v 3 plus v 4 plus v 5 is equal to 0 and similarly the last parity check equation can be written as v 2 plus, v 2 plus u 1 is

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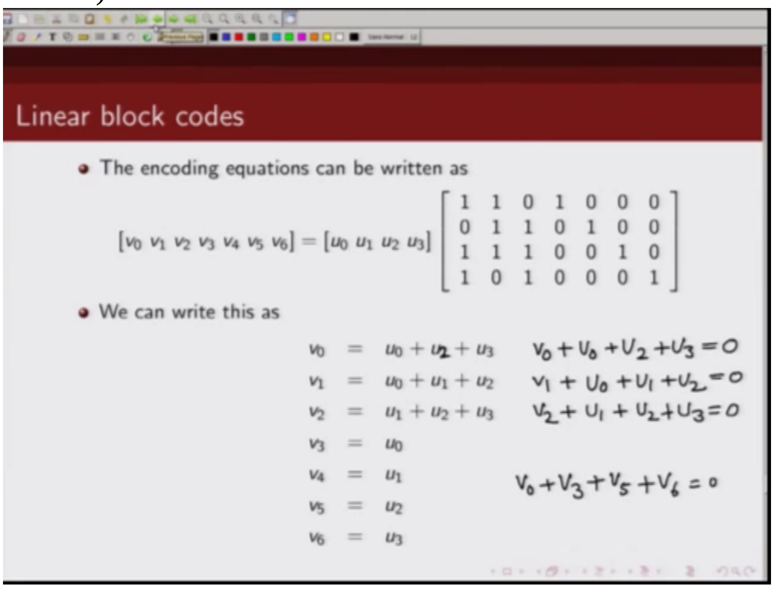

v 4, u 2 is v 5 plus u 3 is v 6. So that's what we have here. u 2, v 2 plus v 4 plus v 5 plus v 6 is equal to 0. So now we have set of encode equations in terms of coded bits. Next,

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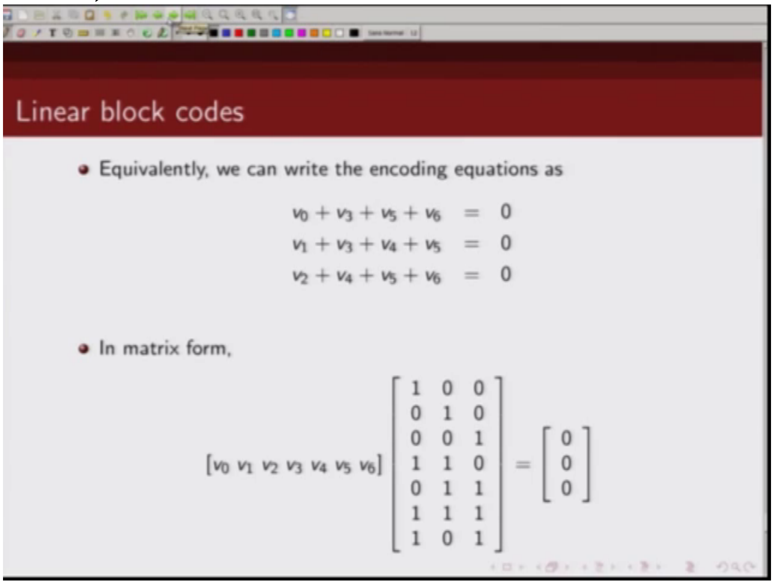

the same thing we can write it in a matrix form. So I have my

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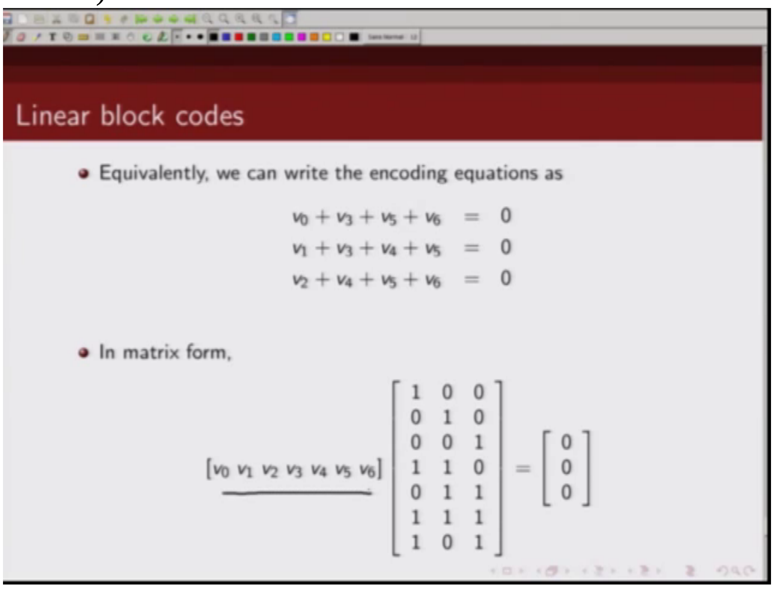

coded bits, v 0 to v 6 I have 3 sets of parity check equations, this, this and this. And the same thing I can write it in a matrix form like this. Now you can see these are equivalent. So look, let's look at first equation. This is v 0 plus v 3 plus v 5 plus v 6 is 0. You can see which are the elements which are so v 0 times 1; this is v 3 times 1 plus v 5 times 1 plus v 6 times 1. So that's what is defined in this equation. Similarly we can see this equation. This v 1 plus v 3 plus v 4 plus v 5 is equal to 0 and this last equation, this is v 2 plus v 4 plus v 5 plus v 6 is 0. And what we did we say about parity check matrix?

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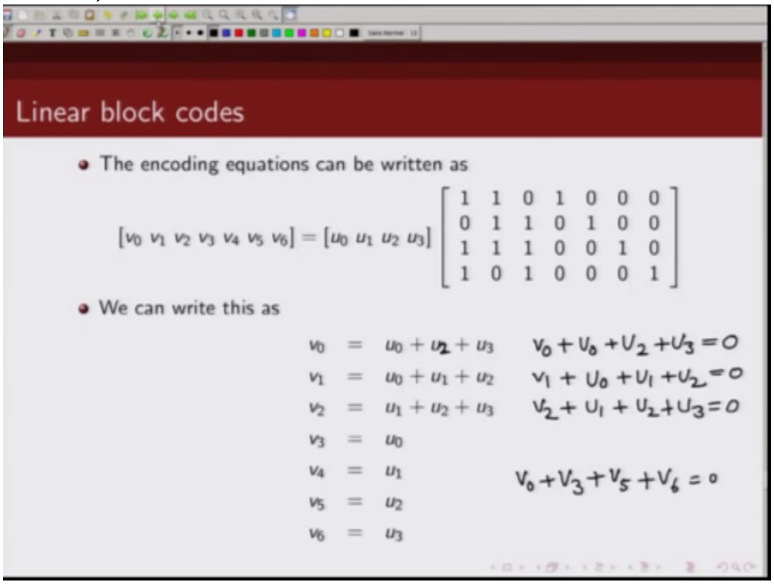

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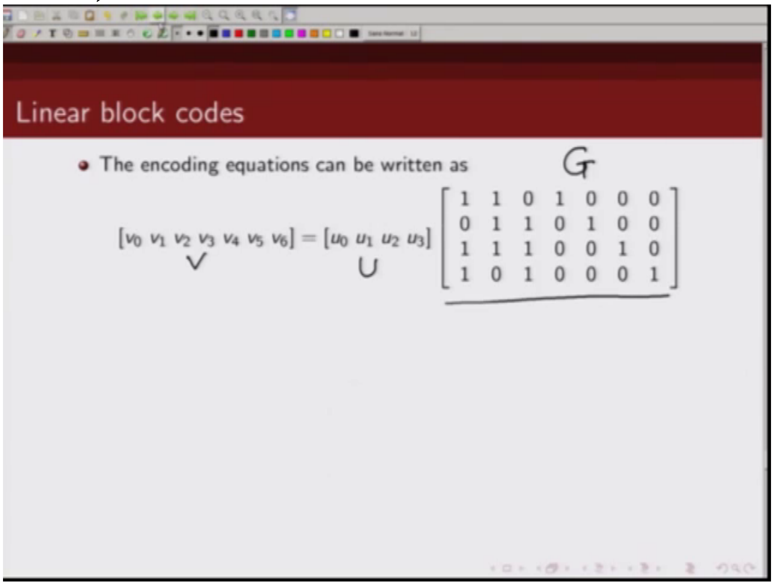

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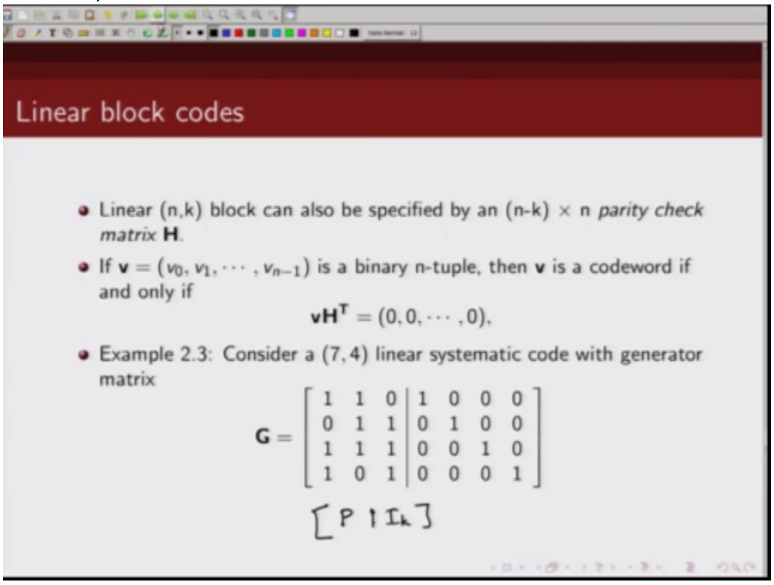

We said that if H is a parity check matrix it is

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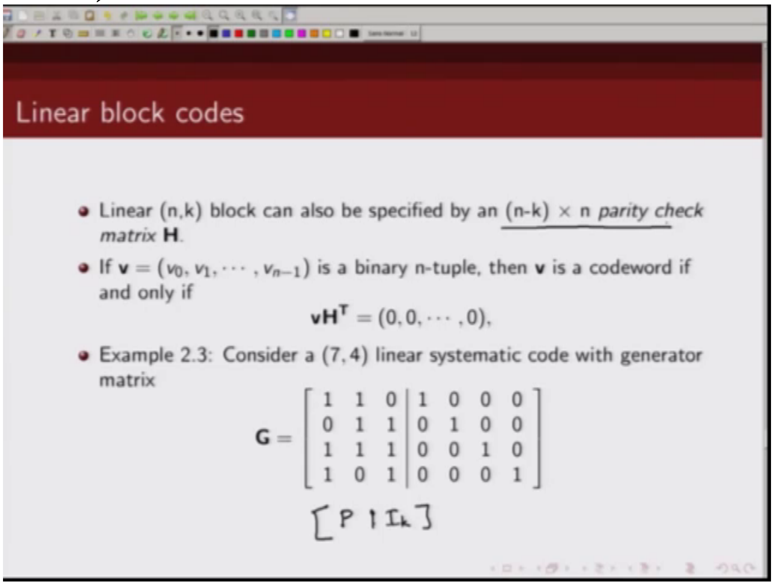

n minus k cross matrix and it has this property that v H transpose is

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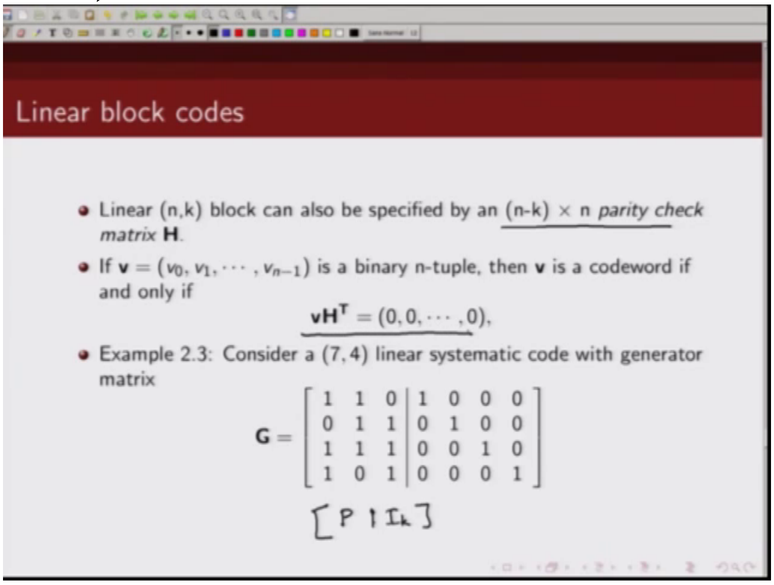

0. So we have,

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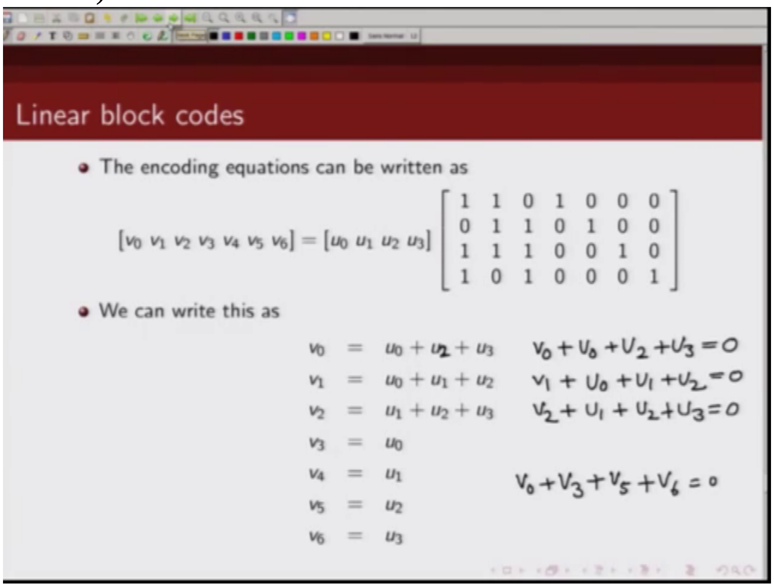

we can write this as,

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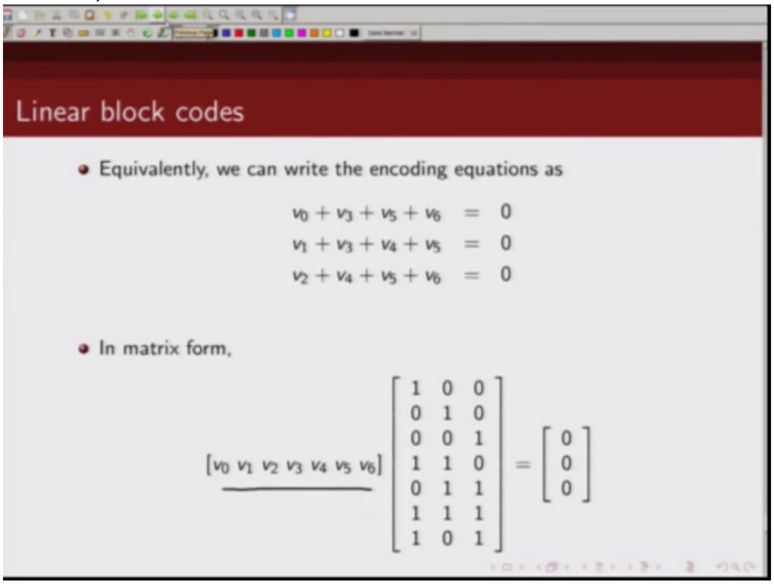

this is my v,

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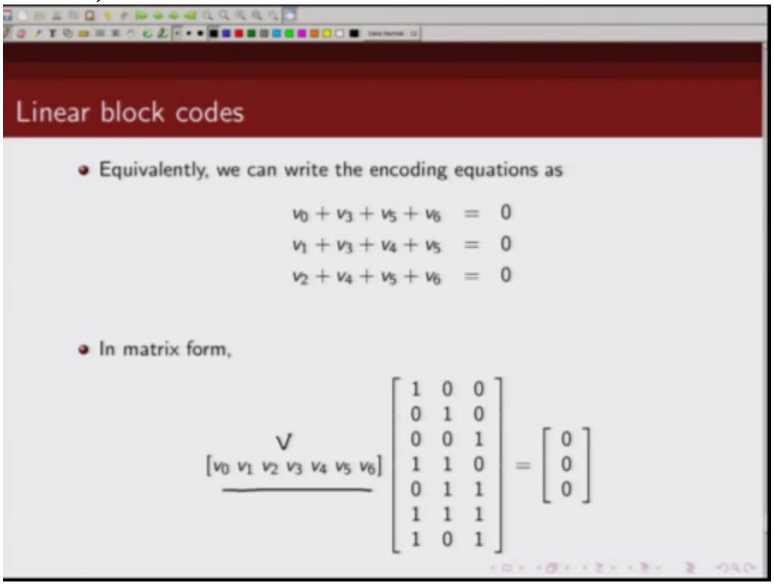

this is my H transpose. v H transpose is

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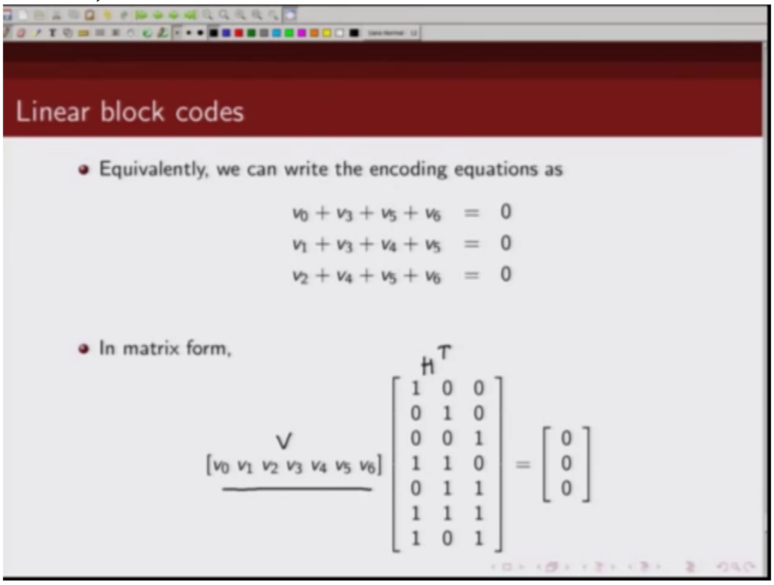

0, so then what is my H matrix? H matrix is a transpose of this matrix, so this will be 1 0 0, 0 1 0, 0 0 1, 1 1 0, 0 1 1, 1 1 1 and 1 0 1. This is my, so for the 7 4 code, 7 4 code this is basically 3 cross 7. As I said, n minus k cross n matrix,

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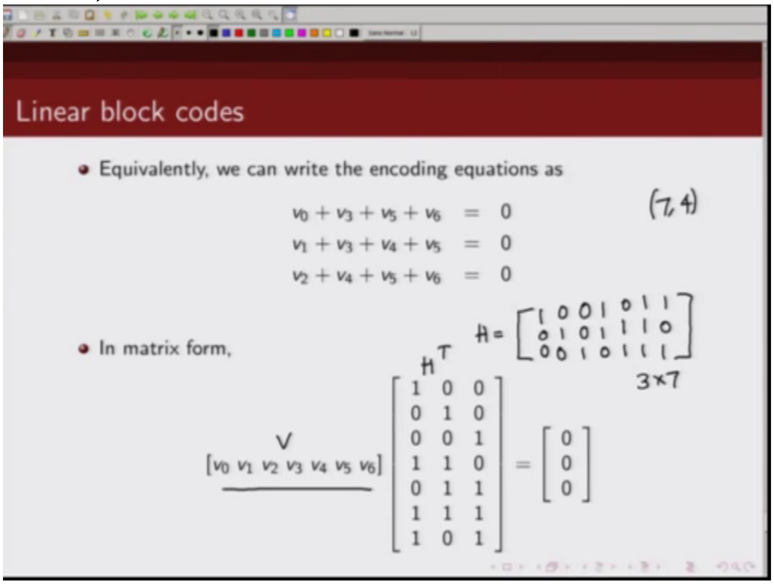

this is my parity check matrix corresponding to this same code which is generated by

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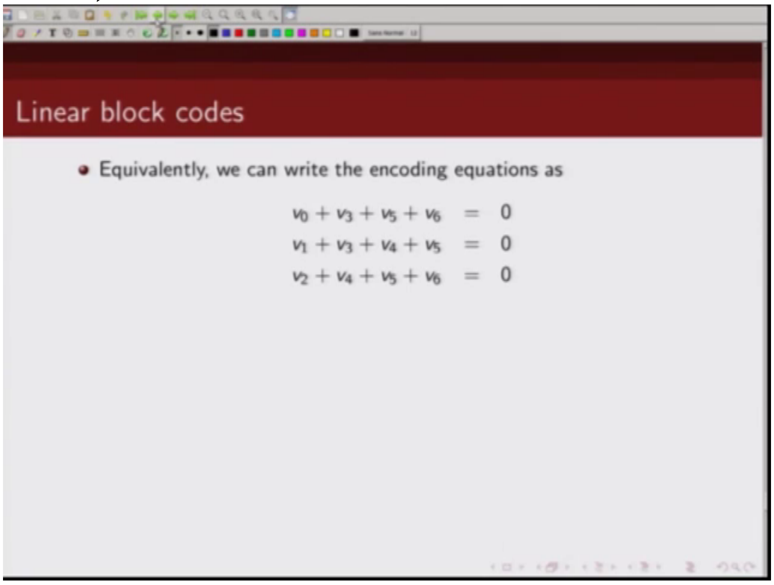

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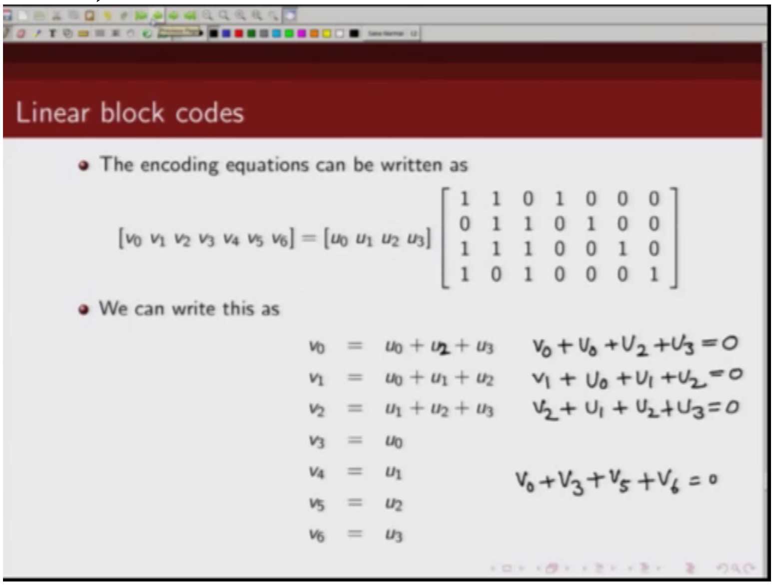

this

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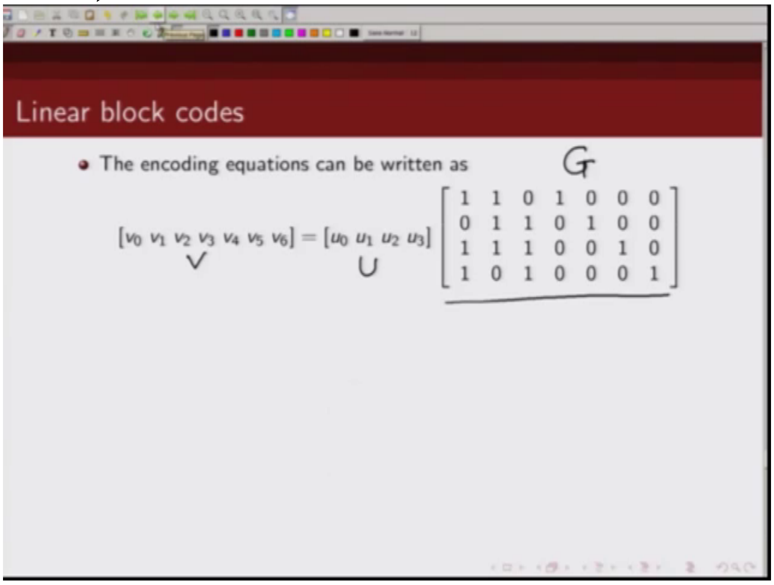

Another interesting

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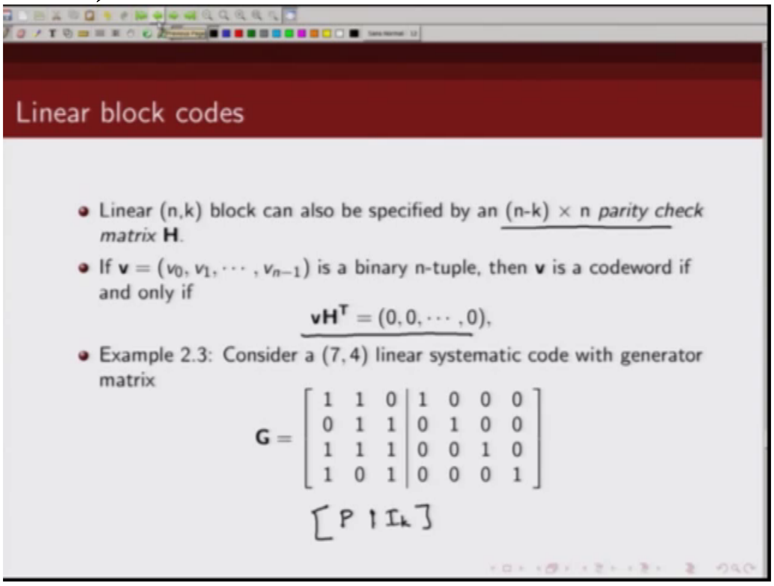

property which you can generally see is, so v H transpose is 0; I can write this u times v H transpose is equal to 0. In other words,

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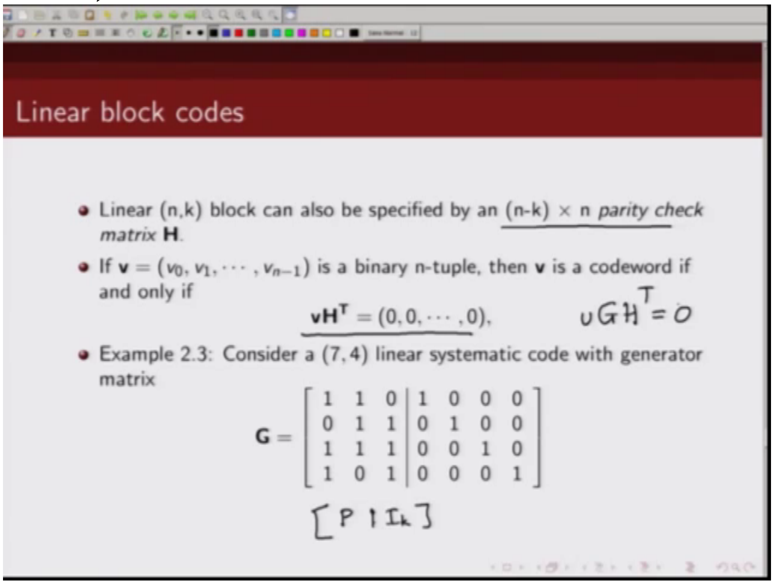

v H transpose is

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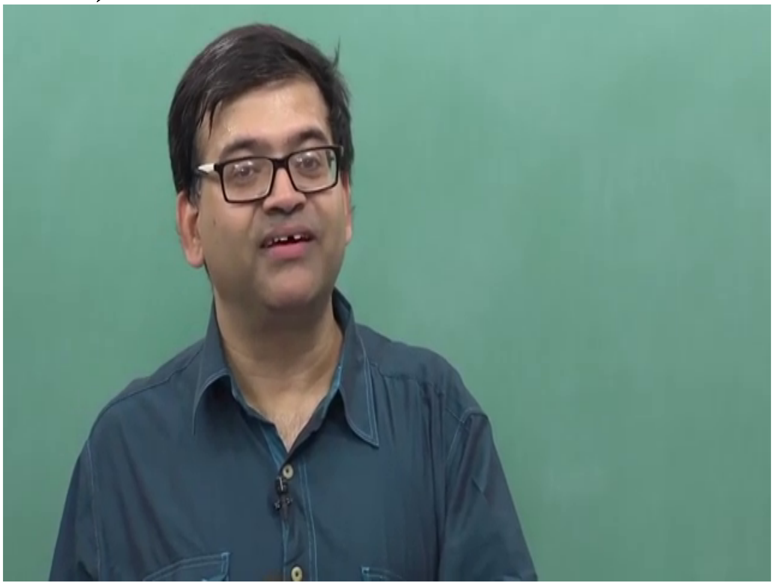

0, so what does that mean? The rows of G matrix and rows of H matrix are orthogonal to each other.

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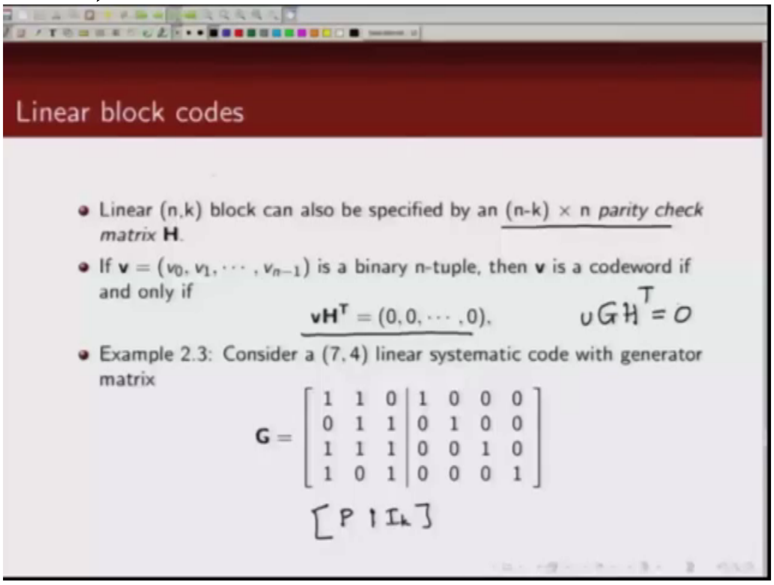

So the H lies in

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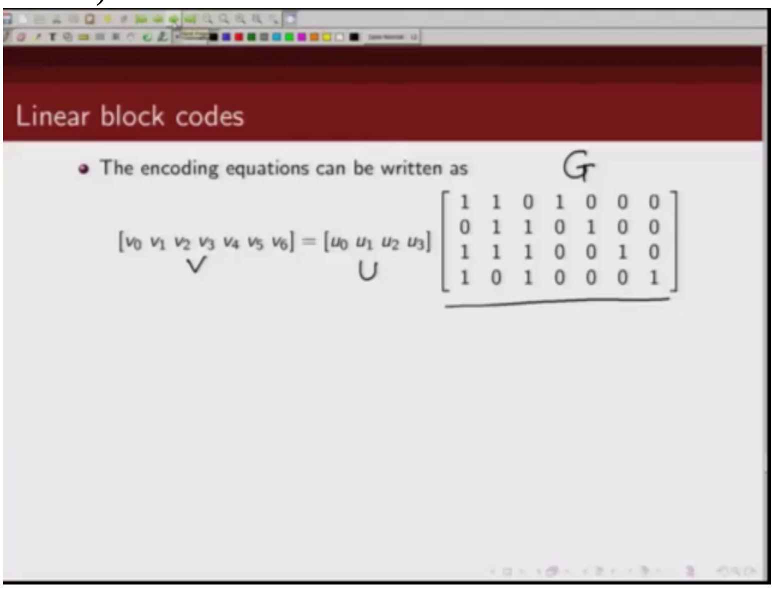

the null space of G, so as we can see from this that generator matrix and parity check matrix are related to each other.

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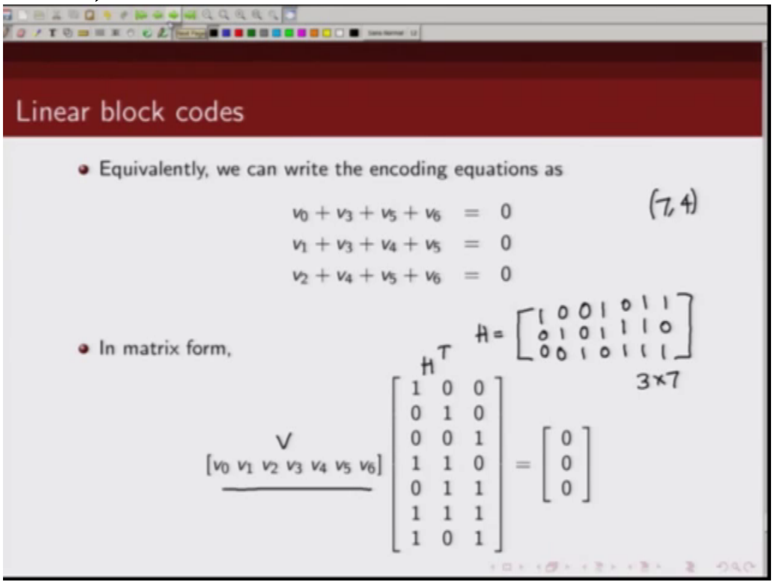

And they have this property that rows of G matrix and H matrix are basically orthogonal to each other. So if you have

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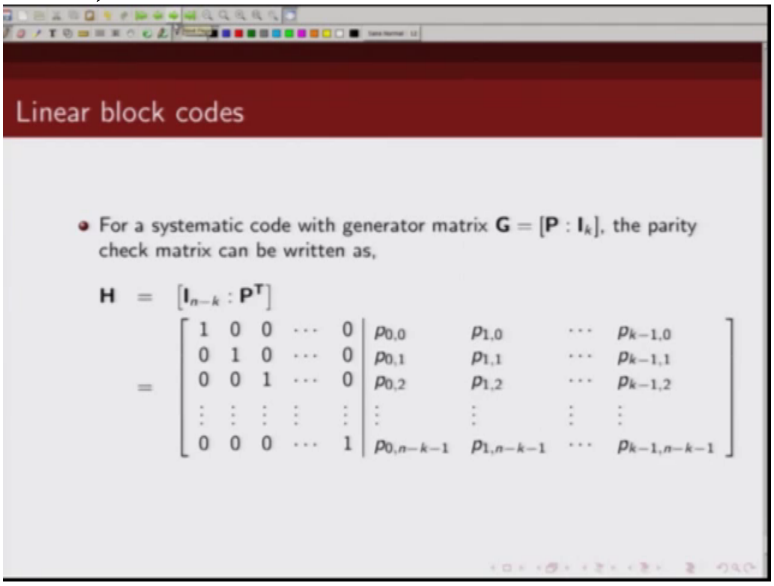

a systematic code whose generator matrix can be written

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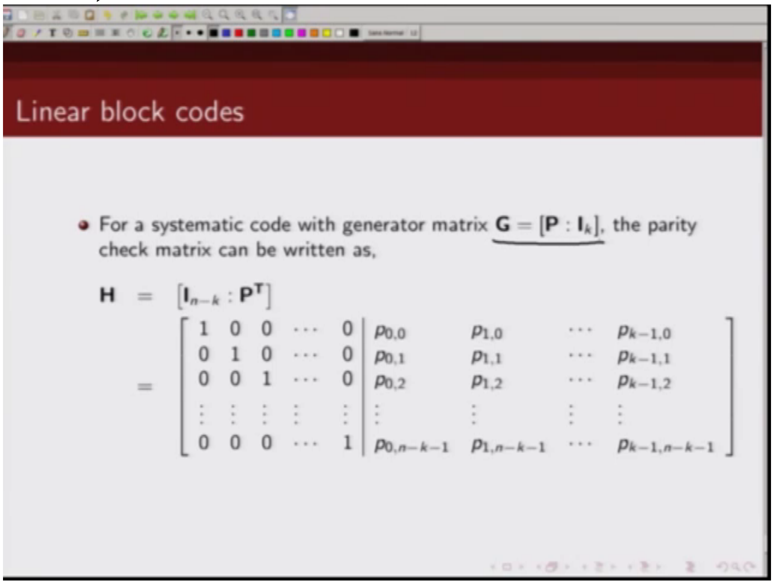

in this form, because H lies in the null space of G, we can write down its corresponding H matrix very easily. And this is basically given by, so if a generator matrix can be written of the form P and identity matrix, we can write its parity check matrix as identity matrix and P transpose.

So let's take an example

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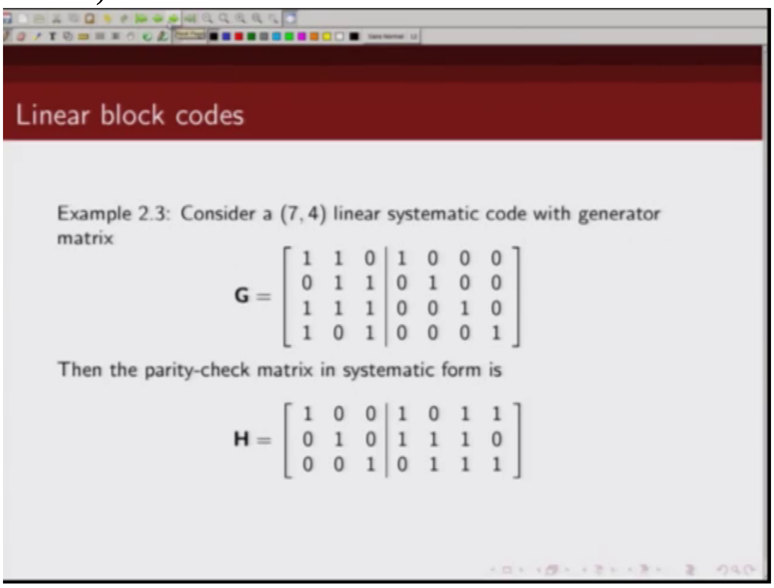

of a generator matrix of a systematic code This is systematic code we can see, we can separate out this generator matrix as some matrix P and some identity matrix. So this we can write as, H matrix we can write as identity matrix and P transpose. So then this can be written as 1 1 0 is 1 1 0; 0 1 1, 0 1 1; 1 1 1, 1 1 1; 0 1 0 so this is my H matrix

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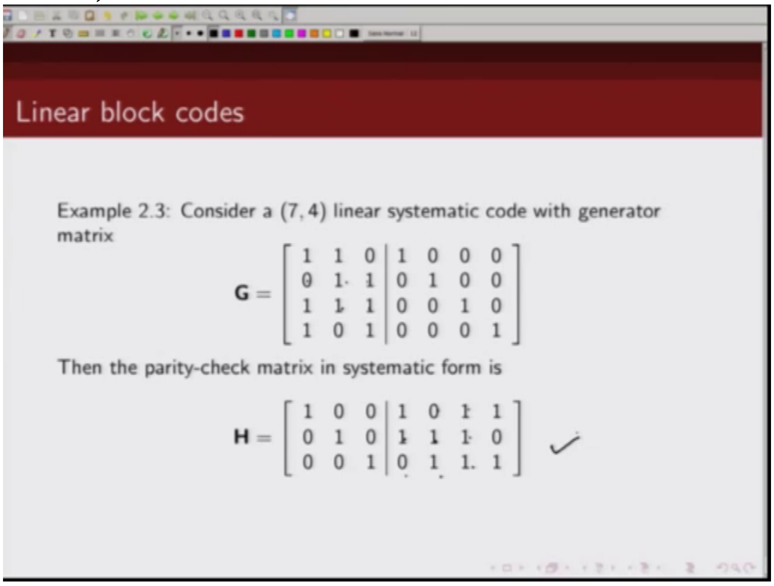

corresponding to this So whether you are given a generator matrix or a parity check matrix, your linear block code is completely specified by either of them. And as I said, we use the generator matrix to generate our code,

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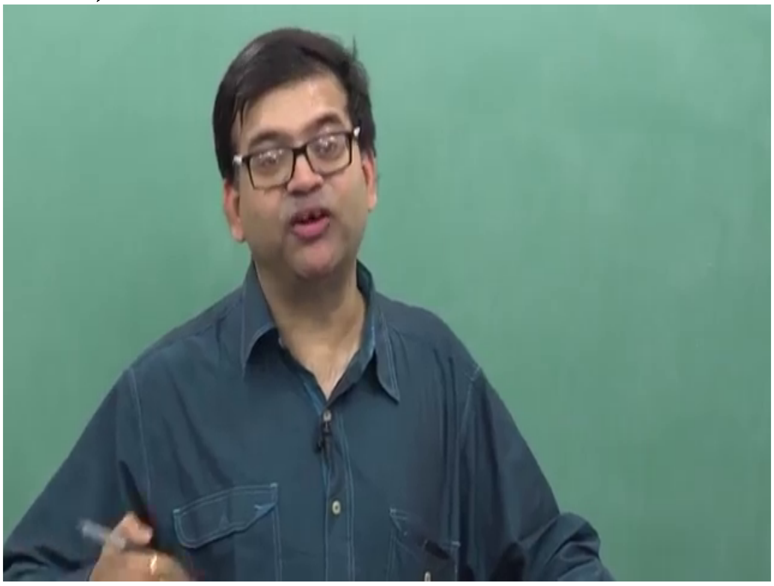

set of codewords where as parity check matrix as the name suggests is used to check whether the parity check constraints are satisfied. As we said basically parity check matrix has this property that, if v is

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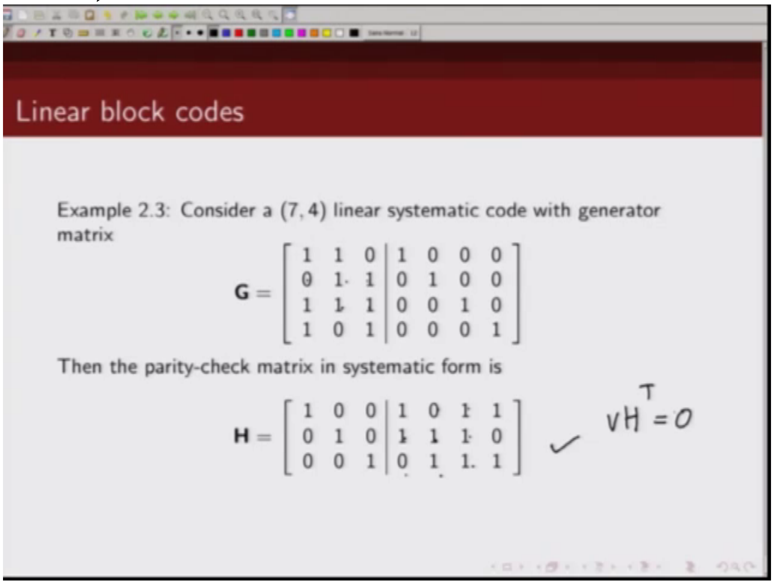

a valid codeword, if and only if v H transpose is zero and we use this property in decoding, so that's why you see the name parity check

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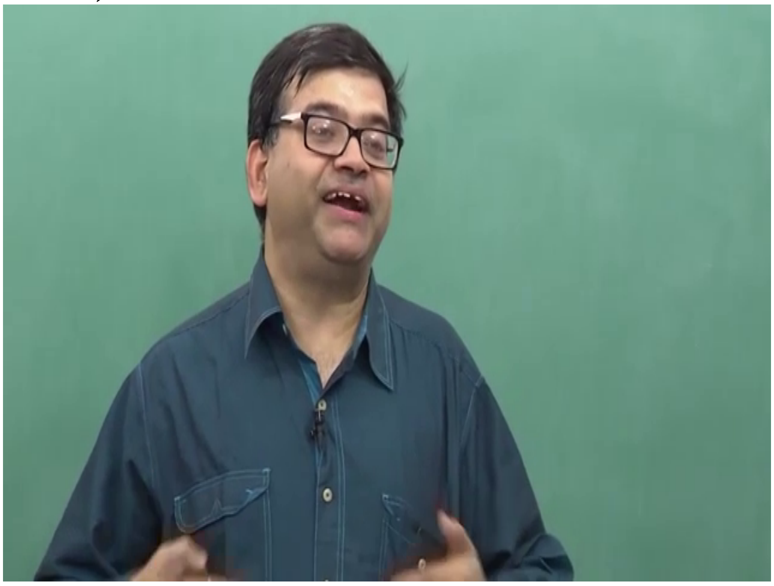

matrix because this matrix H is essentially used to, in some sense check whether the parity check constraints of the code are satisfied or not Thank you.