An Introduction to Coding Theory
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Module 05
Lecture Number 21
Problem Solving Session-IV

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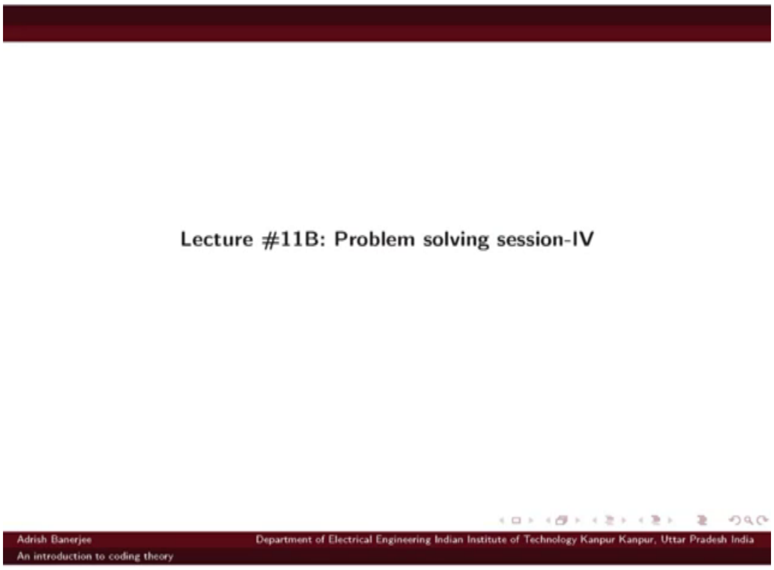

So before we go to concatenated codes, let us spend

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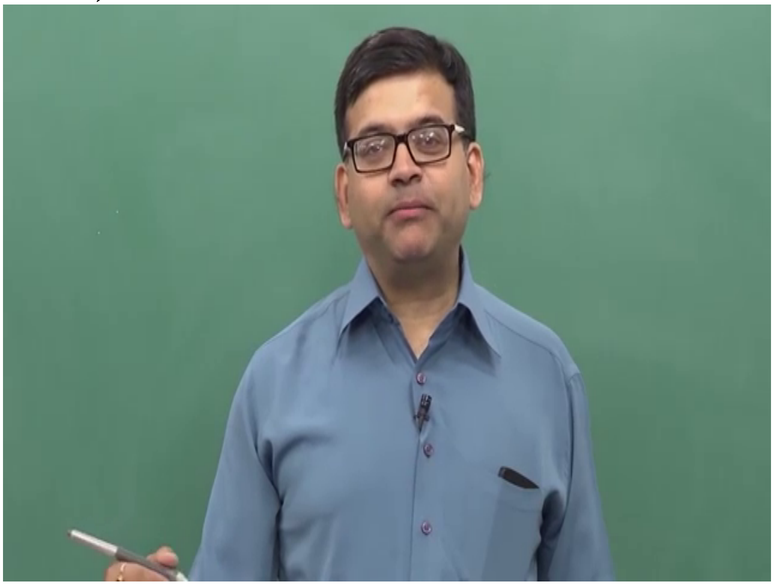

some time solving some problems.

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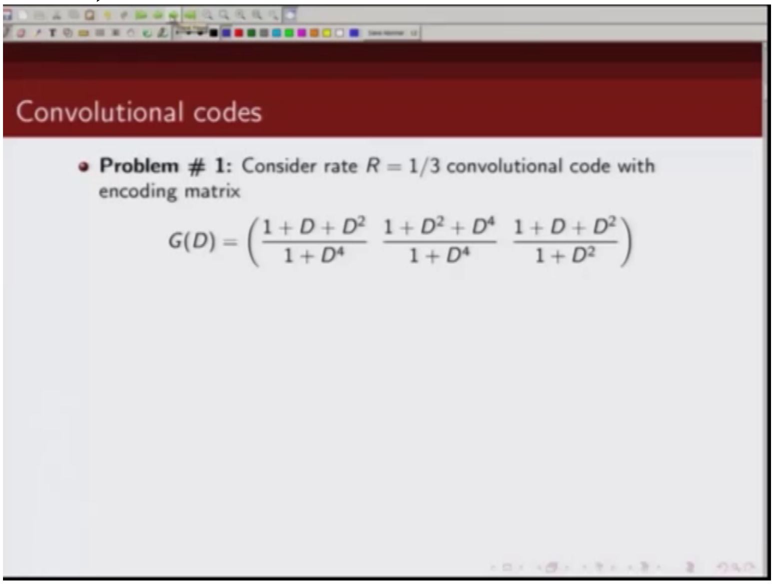

So the first question is you are given a rate one third convolutional code with generator matrix G of D which is given by this.

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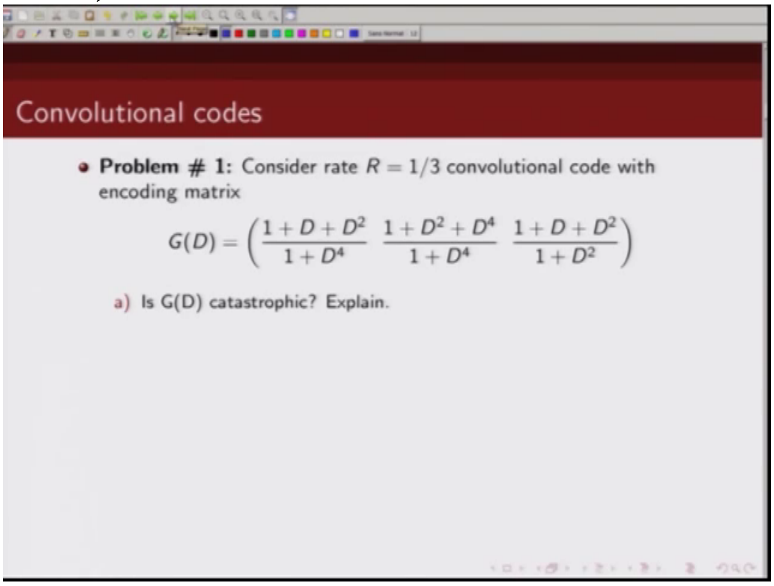

The first question is, is this a catastrophic encoder? Will an encoder which has a generator matrix like this; will this result in a catastrophic encoder? So if

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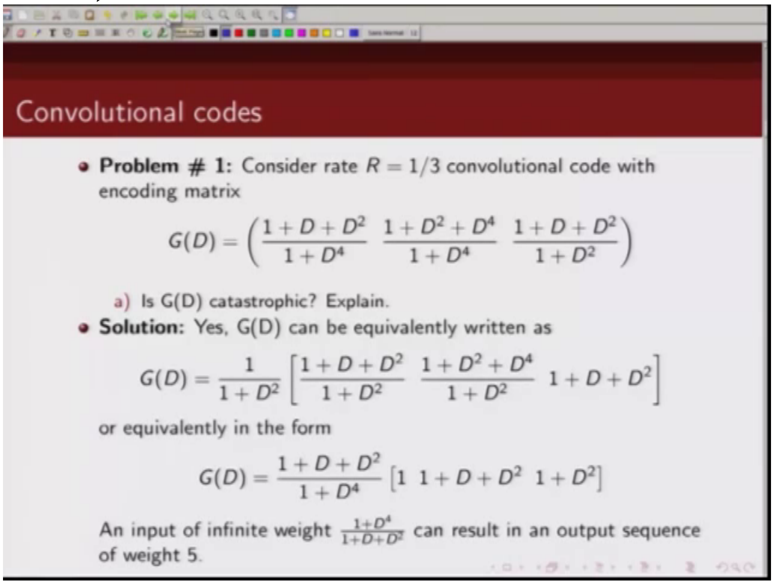

you recall what is a catastrophic encoder, a catastrophic

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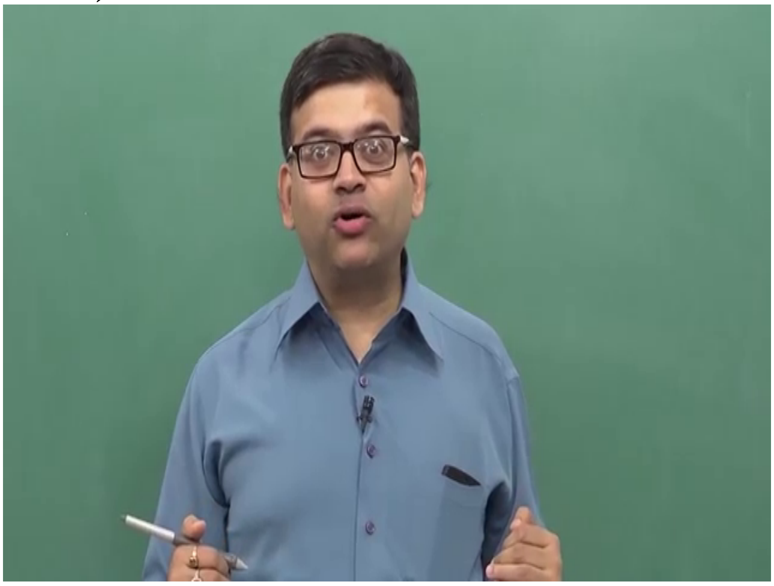

encoder generates a finite weight output corresponding to an infinite weight input sequence. Now if we try to look it in terms of state diagram, in a state away from all zero state, there is a self loop around a state where a non-zero input results in all zero output, right. Now let's look at this generator matrix

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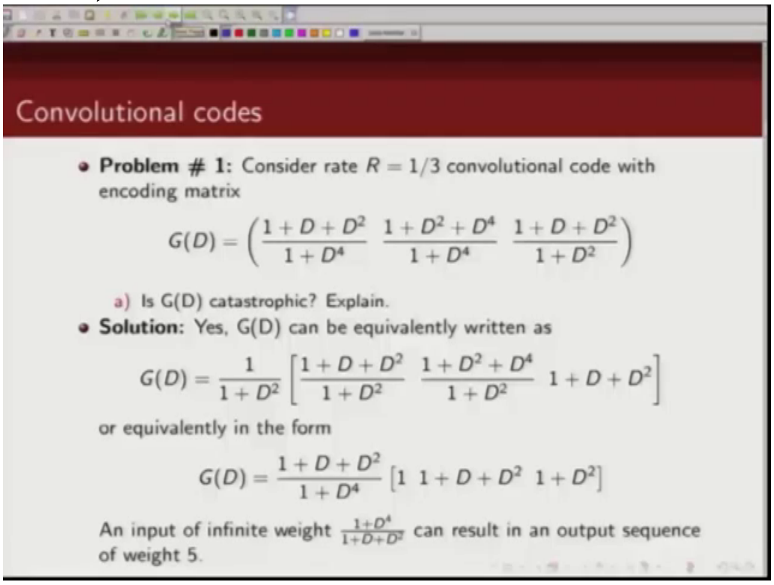

and let us try to simplify, put it in a minimal form. So we can see the denominator, one plus D square is common. So if we take that out, we get here 1 plus D plus D square and this is 1 plus D square, this is 1 plus D square plus D four divided by 1 plus D square and this is 1 plus D plus D square. Similarly we see in the numerator there is a common term 1 plus D plus D square. If we take that out, what we get here is then this is 1, 1 plus D plus D square and 1 plus D square. Now how do we know whether this will result in a catastrophic

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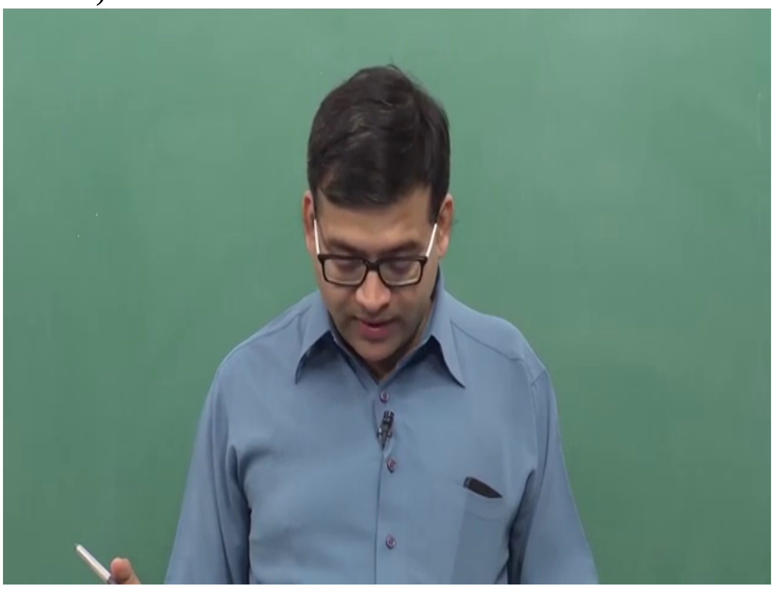

encoder or not. So look at

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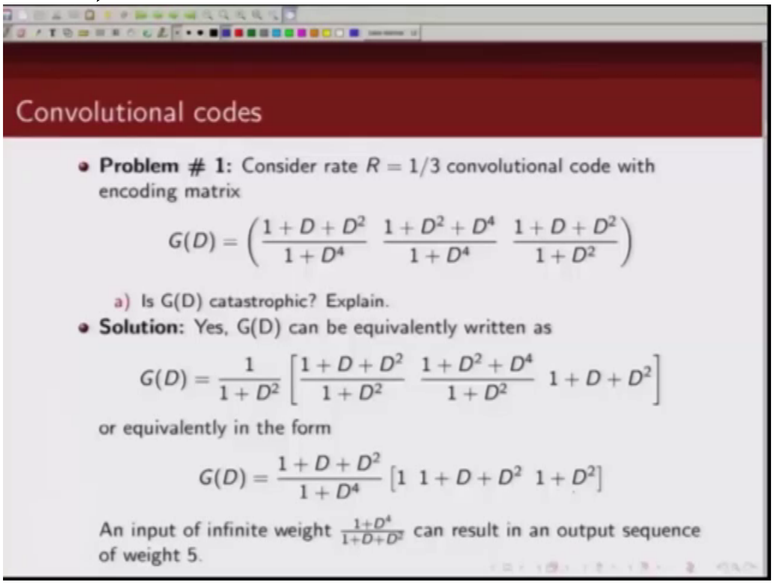

this particular generator matrix. Now what is my output sequence? My output sequence v D is u D times g D.

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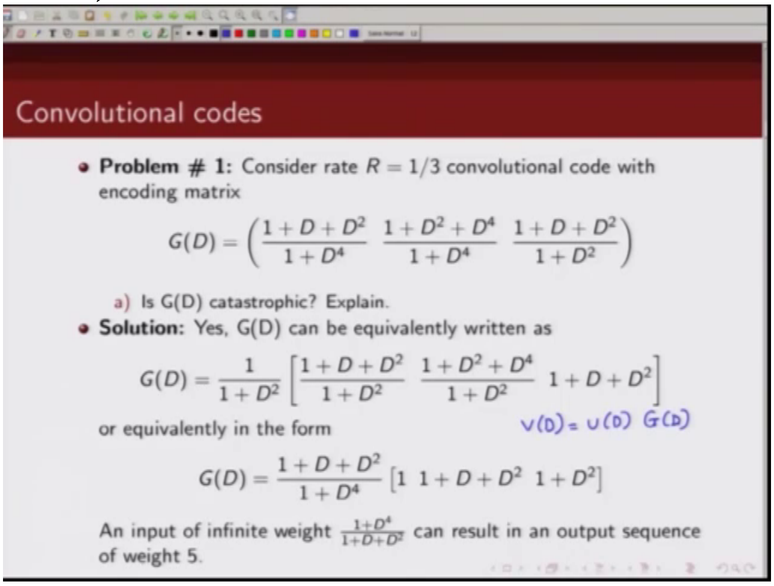

Now is there any input sequence which is of infinite weight but can result in a finite output weight for v D? If you pay close attention to g of D we notice that if our input u of D is chosen as 1 plus D four 1 plus D plus D square,

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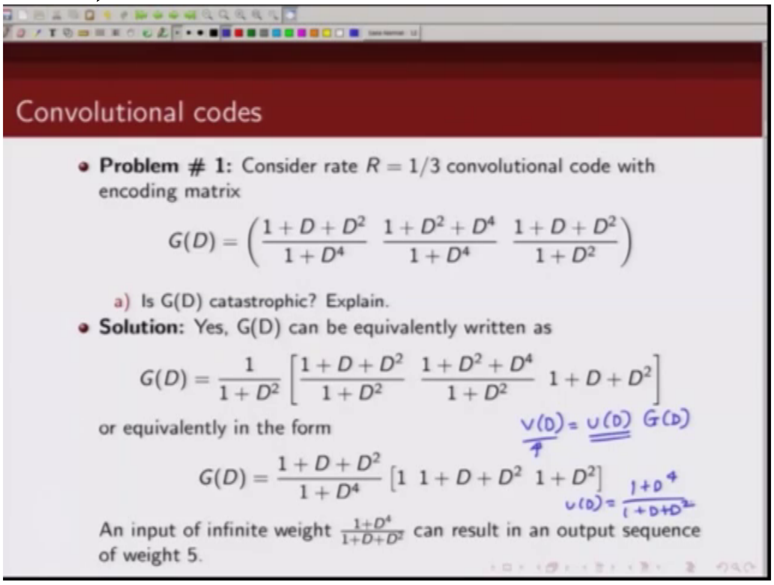

if our input is chosen in this particular fashion, then what would be the corresponding output v D? If the input is chosen this way then output will be u D times g D so this term will cancel this term so what

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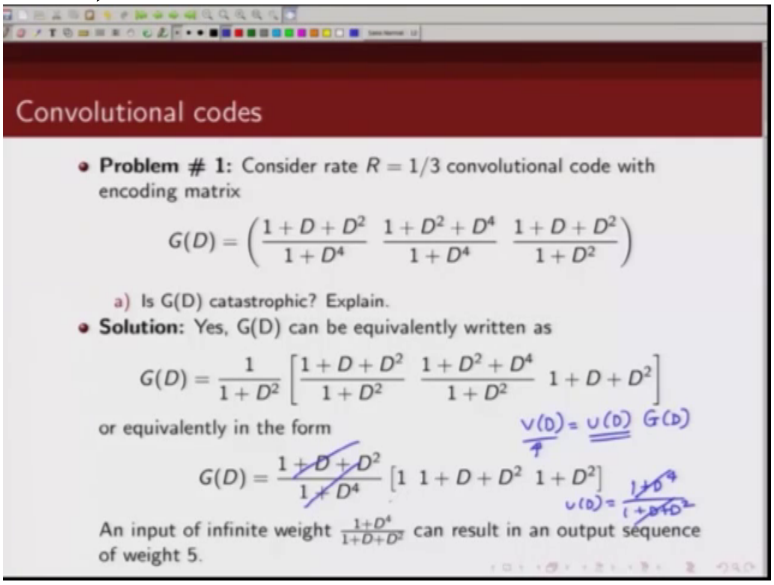

you will be left with is this. So your v of S would be 1, 1 plus D plus D square and 1 plus D square. And what

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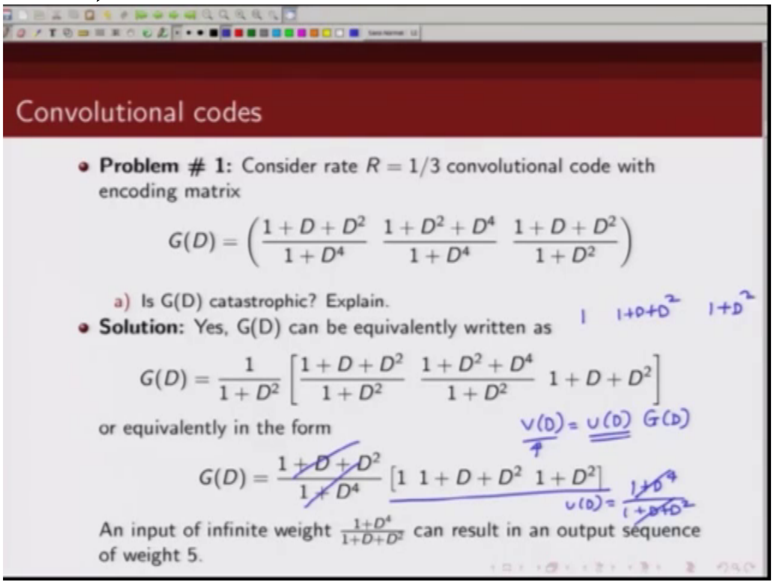

is the weight of this? (Delete kar diya, woh ganda)

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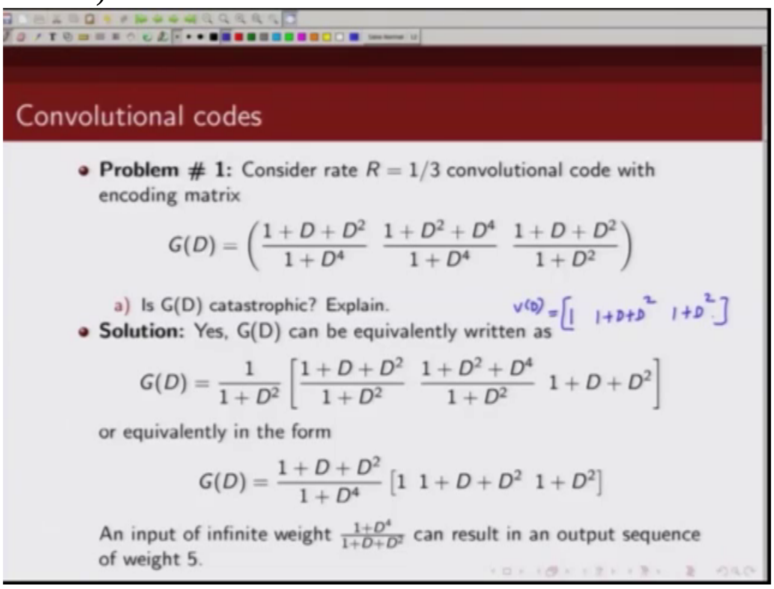

So note here, so the input that will cause this output is given by 1 plus D four by 1 plus D plus D square,

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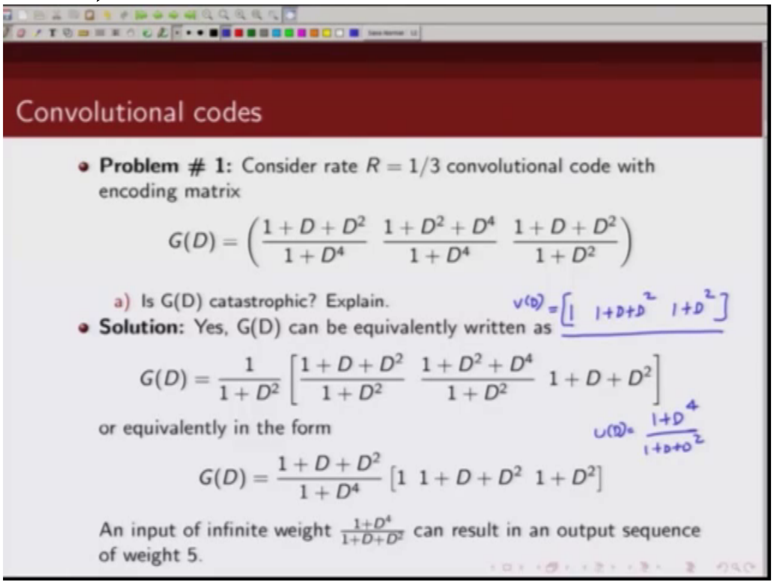

right? Now we can expand this. So let's say 1 plus D 4, this is 1 plus D plus D square. So let's just take

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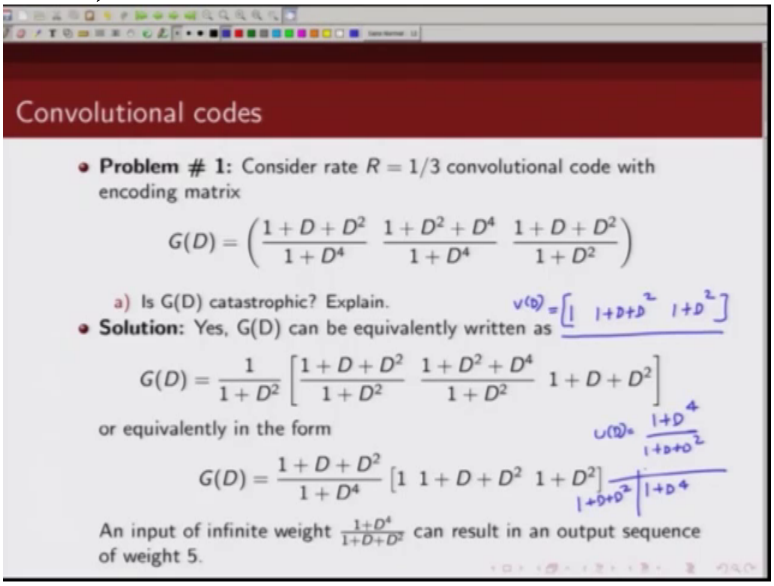

1. 1 plus D plus D square, this will be D plus D square plus D 4, now

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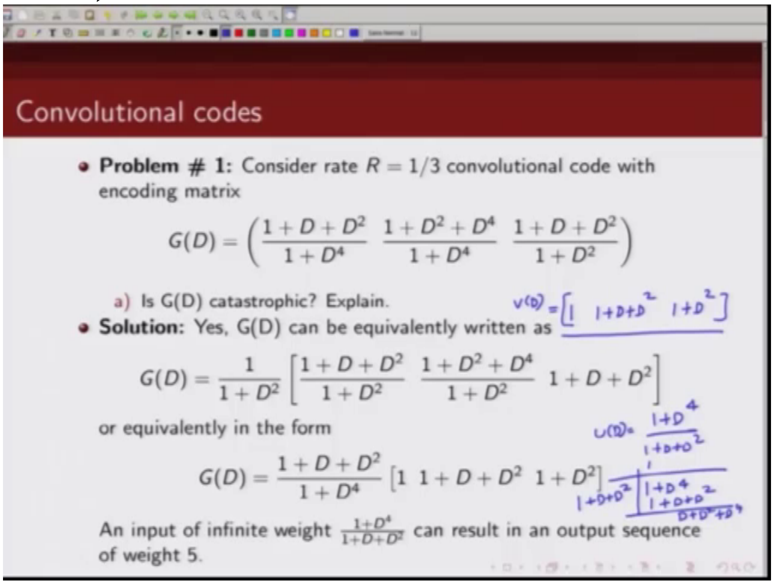

this will be plus D, this will be D plus D square plus D cube, then this will be D cube plus D 4, you can write D square, so like that basically we, we can see that this is a infinite series. The input is an infinite series, 1 plus D plus D square is essentially an infinite series. We can

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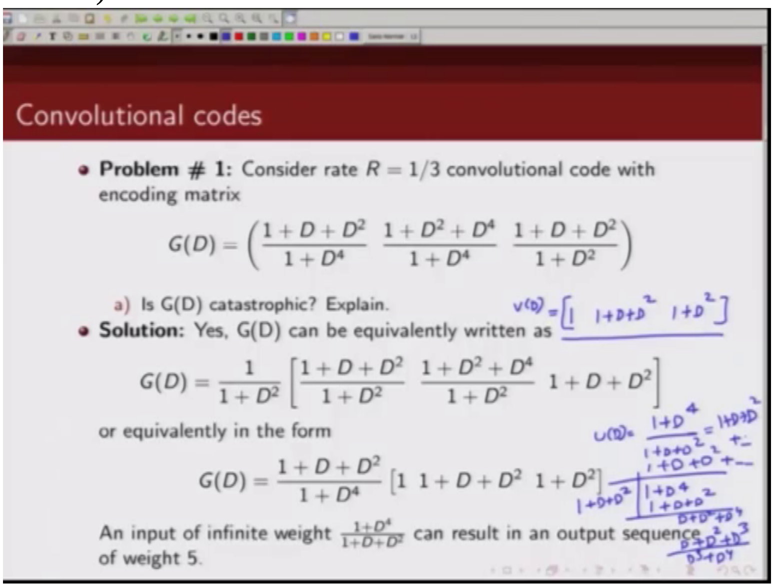

expand it like that where as output is a finite series, it is just 1, 1 plus D plus D square, and the third bit is 1 plus D square. So you can see, input has lots of 1's in it but the output has finite 1s. So this is a case of catastrophic

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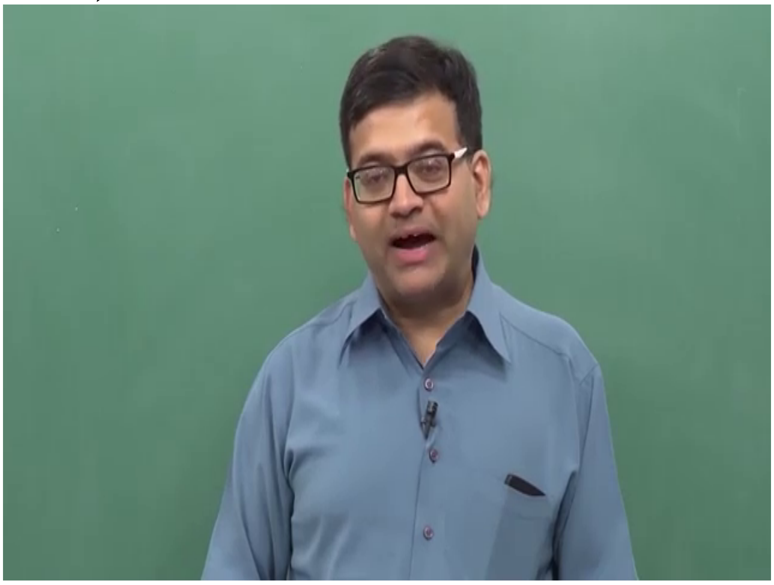

encoder.

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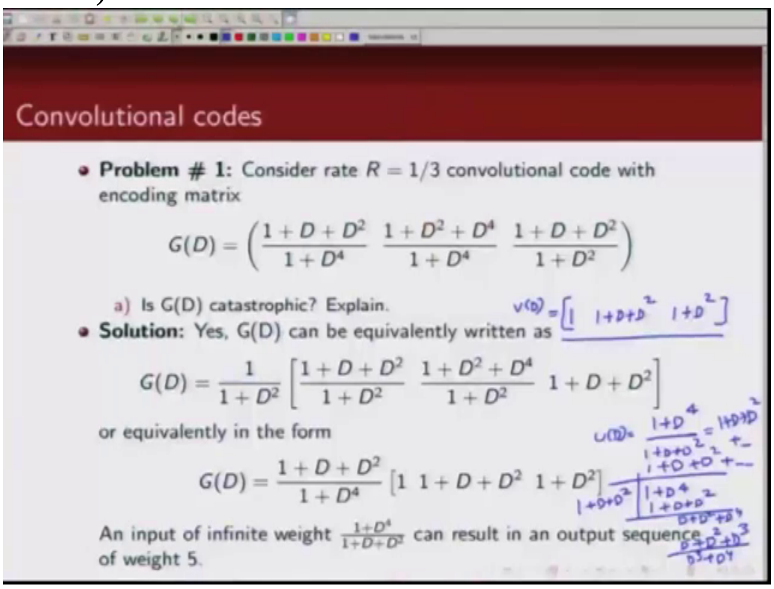

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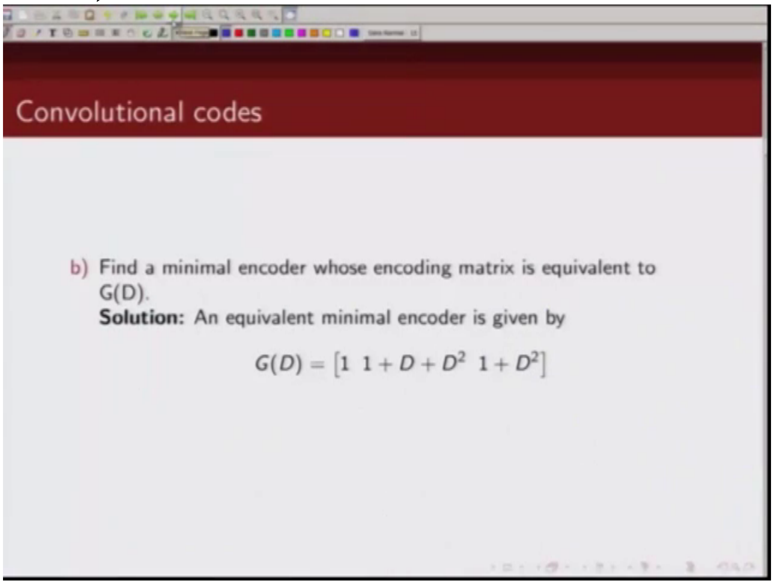

Now the second question is what would be the minimal encoding matrix for the generator matrix given in the previous example.

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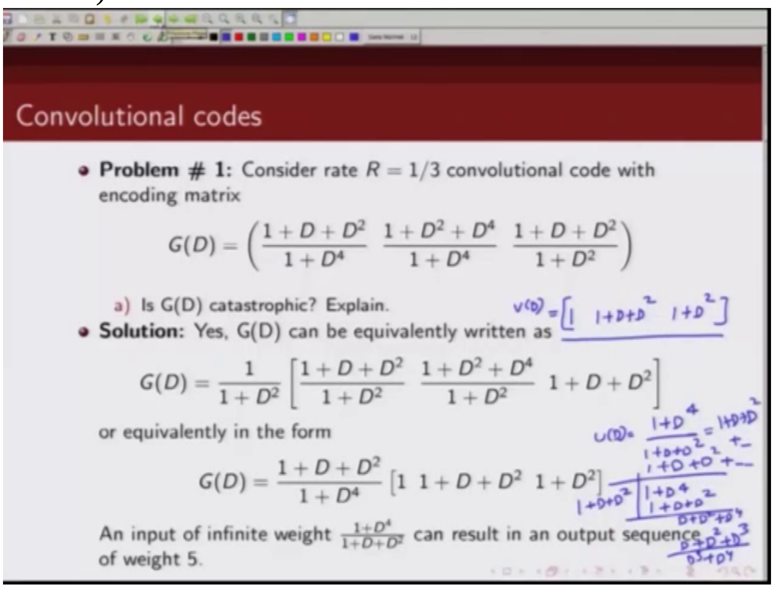

If I ask you, find out the minimal encoding matrix for this encoder. So what do we do? We take out all the common factors. If we take out common factors, then we basically what we get is like this is our minimal encoding

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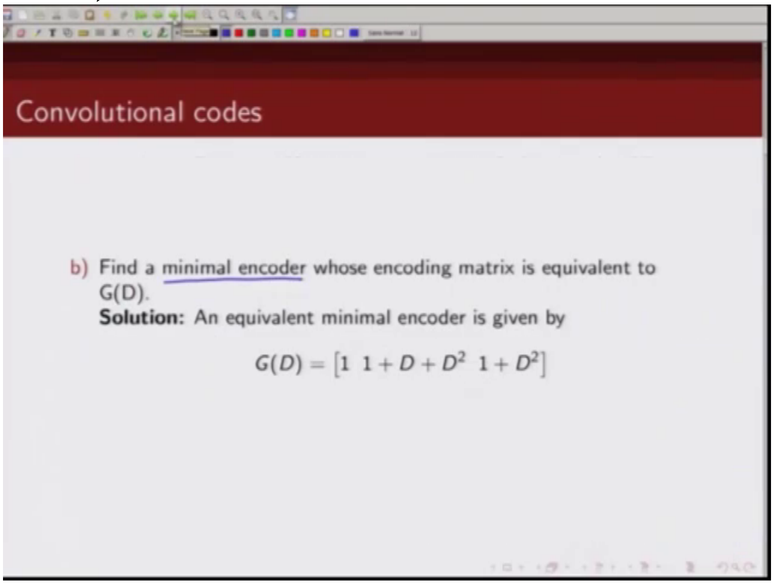

matrix and if we can write, if I ask you to draw this encoder, we can, this is, k is 1, n is 3, the maximum memory is 2 so I am drawing 2 memory elements here. The first coded bit is just 1, so this is the information sequence that goes in. Second one is 1 plus D plus D square. So that's your, let's call it v 0, this is v 1, this is u 0, and the third bit, coded bit is 1, and D square this is your v 2,

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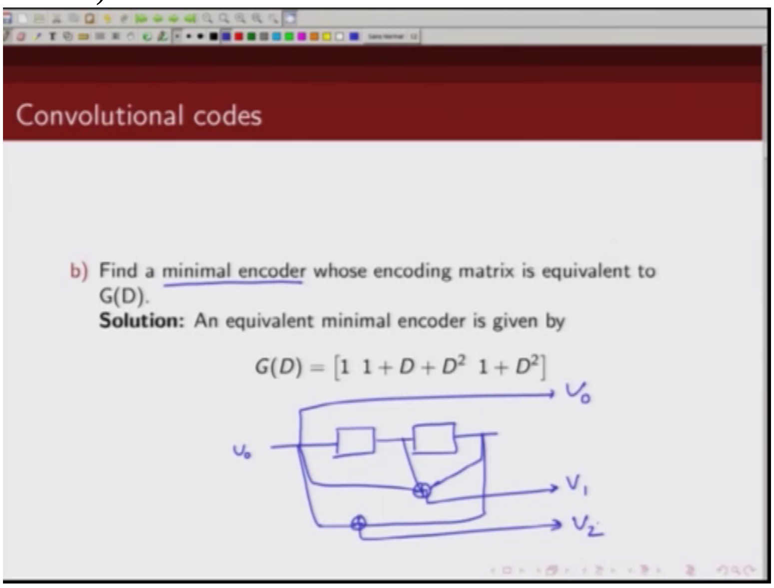

Ok. So this is the minimal encoder for the same generator matrix given in the previous example.

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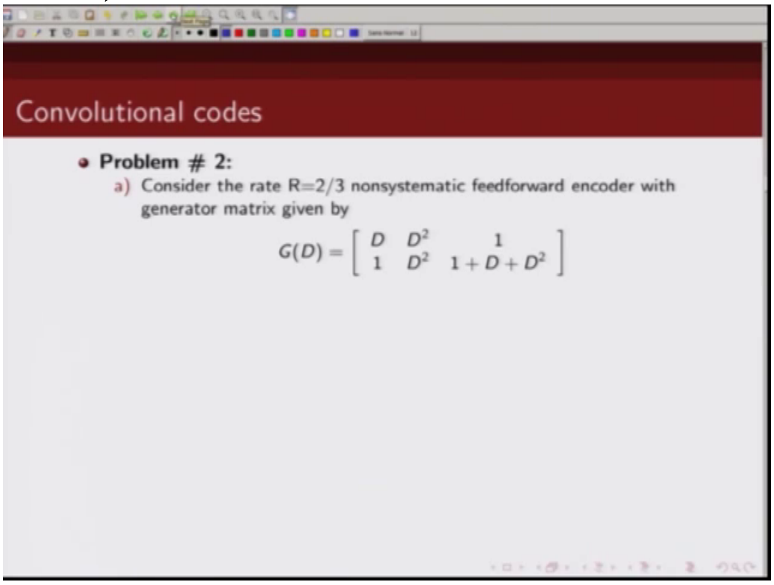

So consider a rate two third non systematic feed forward encoder. So this is a generator matrix for a non systematic code, rate two third and it is a feed forward encoder. There are no feedback polynomials here.

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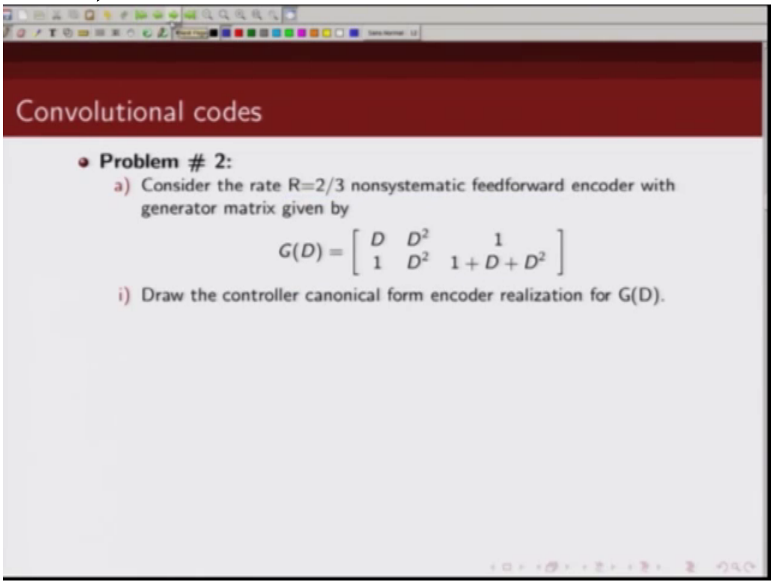

The first question is draw the controller canonical form realization for this generator matrix. Now in controller canonical form realization we have 1 set

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of shift register for input. Now how many inputs do we have here? k is 2

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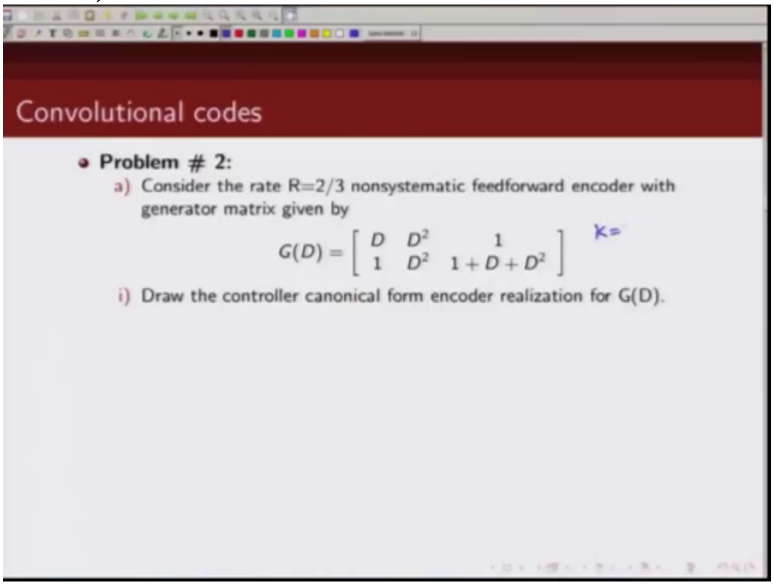

so we will have 2 sets of shift registers for this. One for this and second

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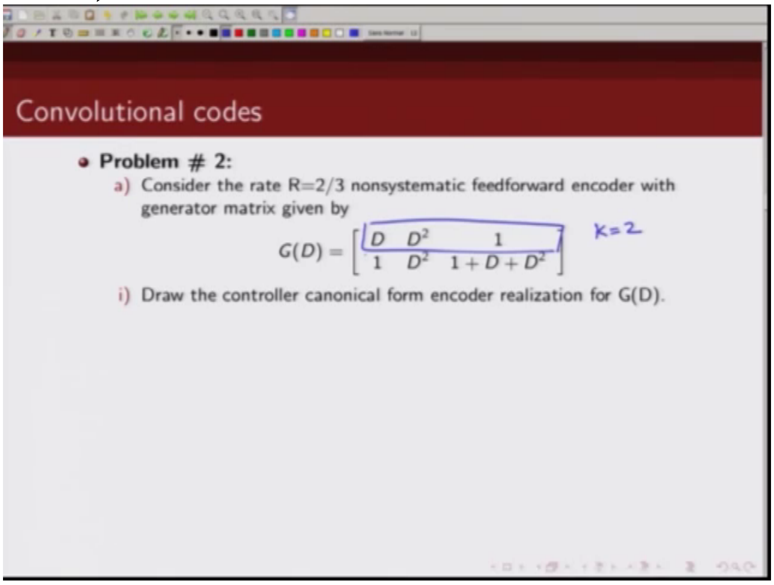

set of shift registers for this. Now what

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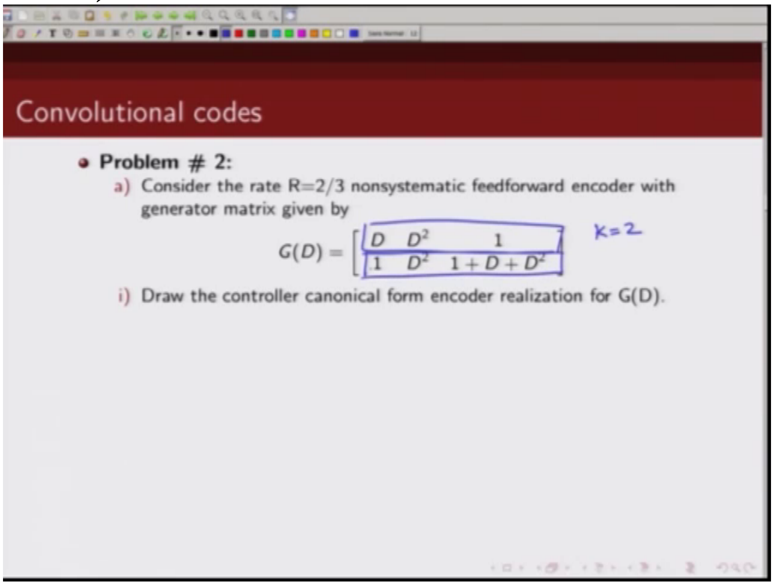

should be the maximum memory for each of these shift registers? You can see here the maximum power of d is 2. So we should have 2 memory elements for the first input. Similarly for the second input also we should have 2 memory elements. Let's call it u 0 and u 1.

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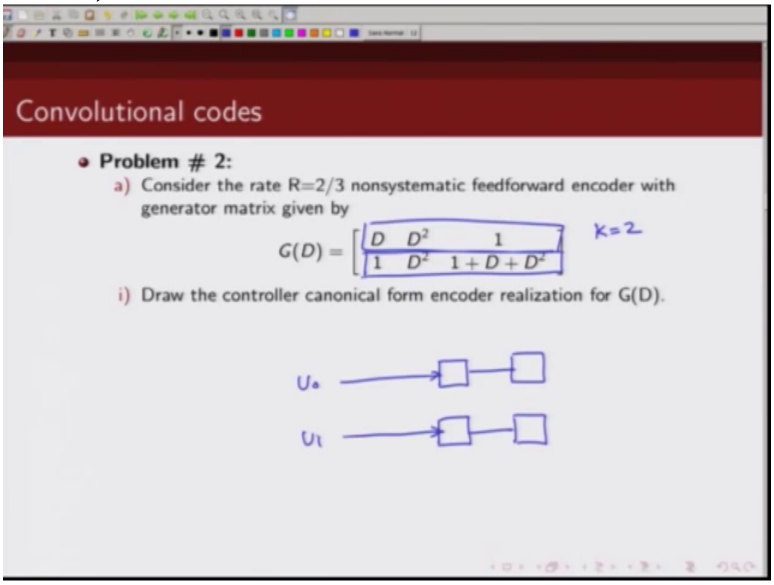

Now there are 3 outputs. So the outputs are, this is 1 output, D times

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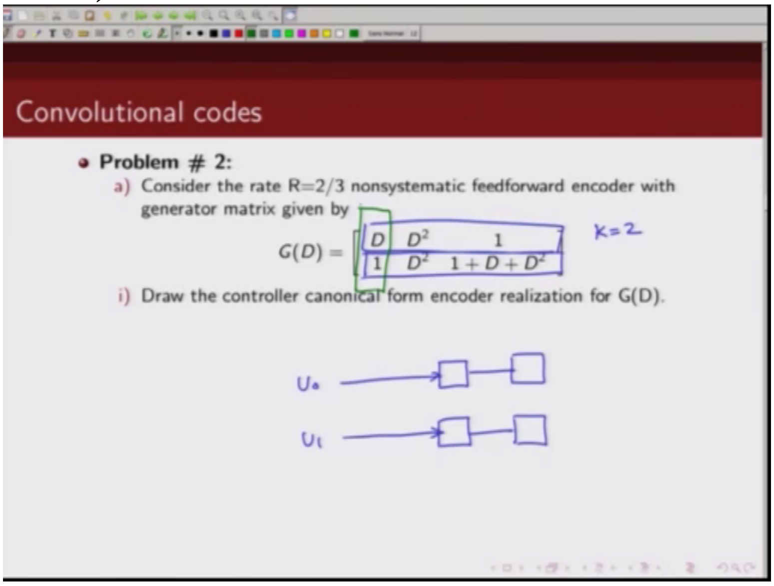

the first and one times u 1, so D times the first input is this, and one times second input so that is this. So this is your first coded bit. Let's call it v 0.

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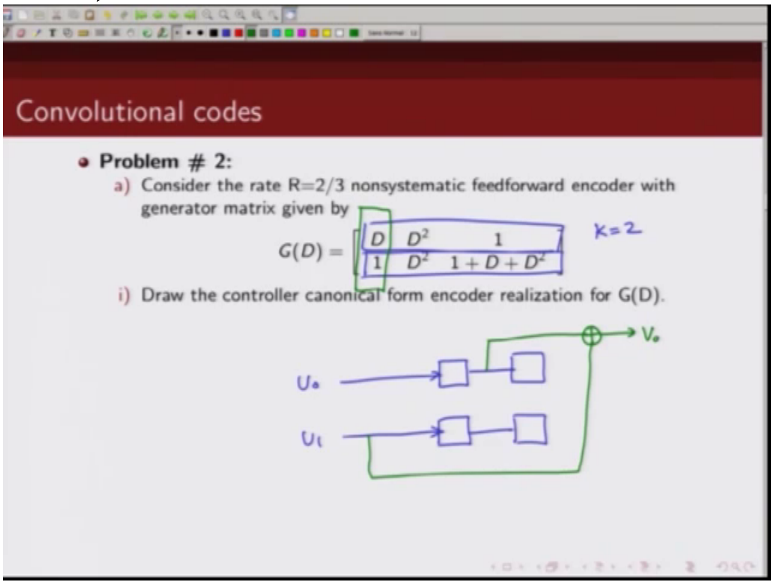

Now what's the second coded bit? This is this term,

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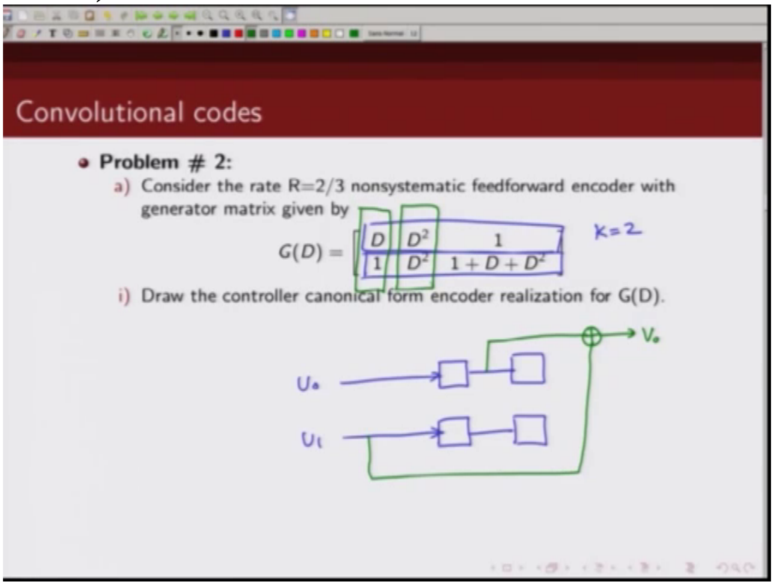

D square u 0 is this term and D square u 1 is this term, so this is your v 1

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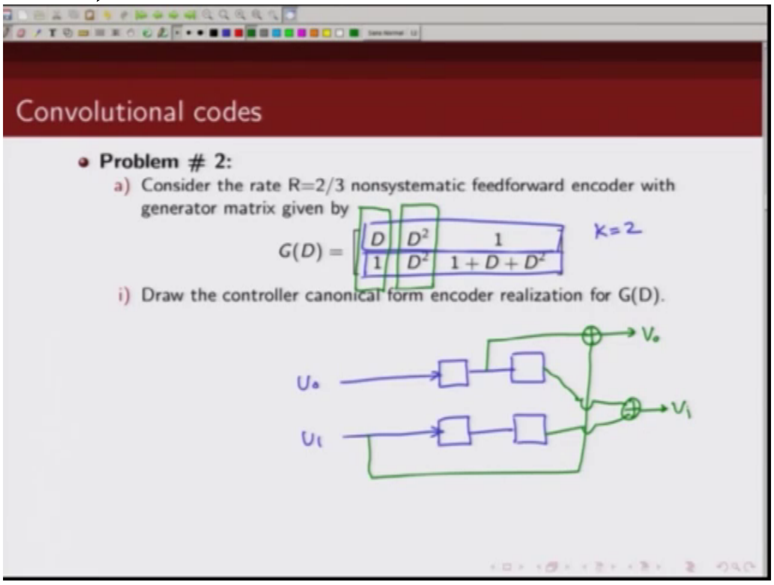

and the third output is this.

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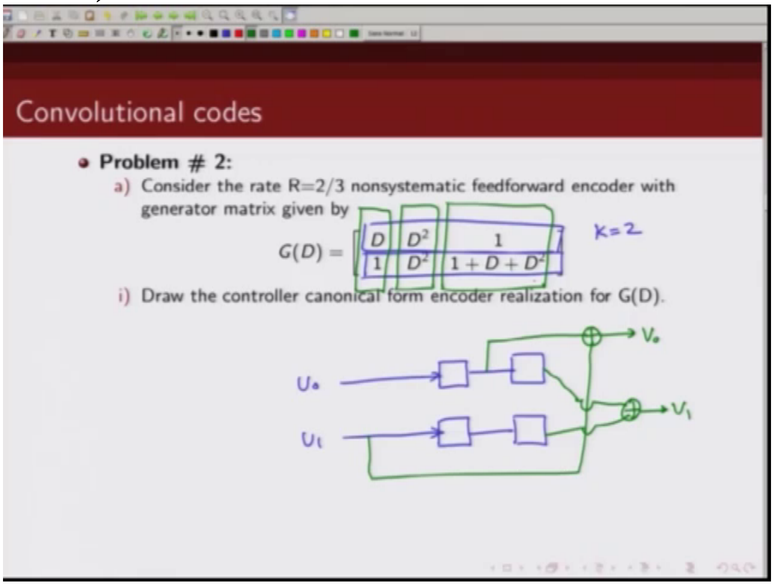

So this is, just a minute, u 1 and one D term and D square term.

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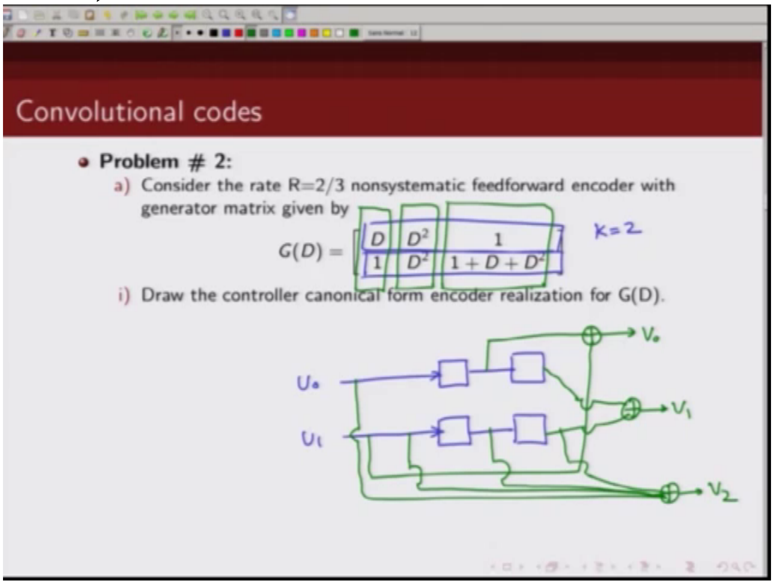

So this is your controller canonical form realization for this generator matrix. So this is

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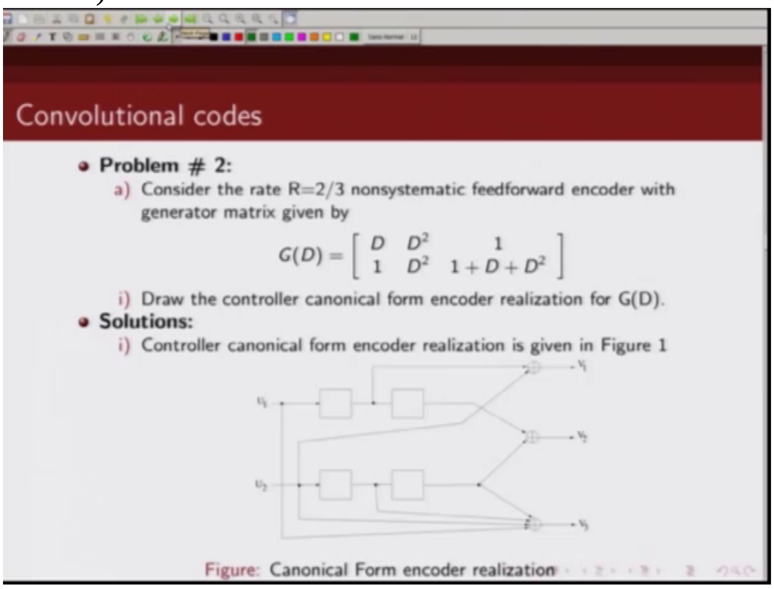

precisely what we have here. You can see, so this shift register is for

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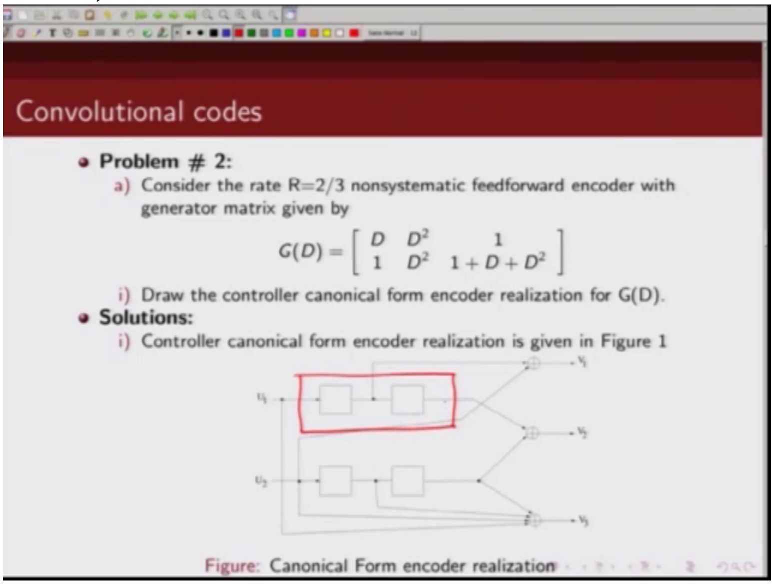

this input and this

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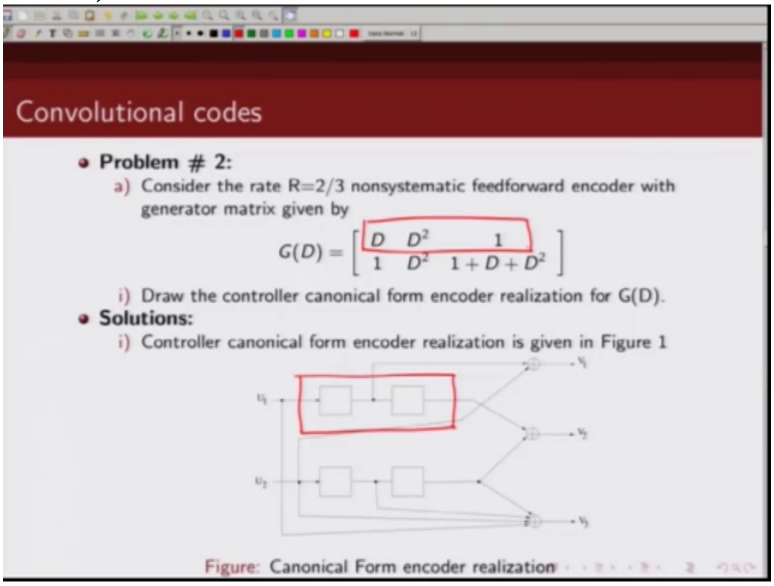

this shift register is for

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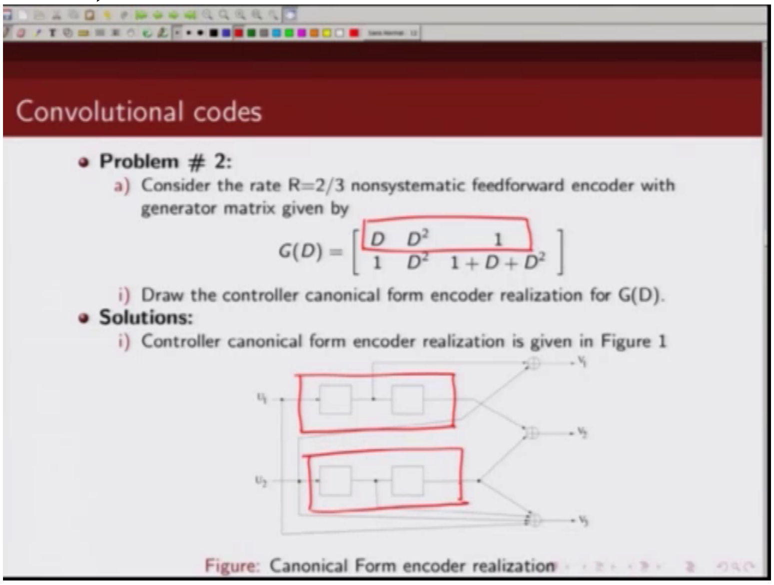

this input.

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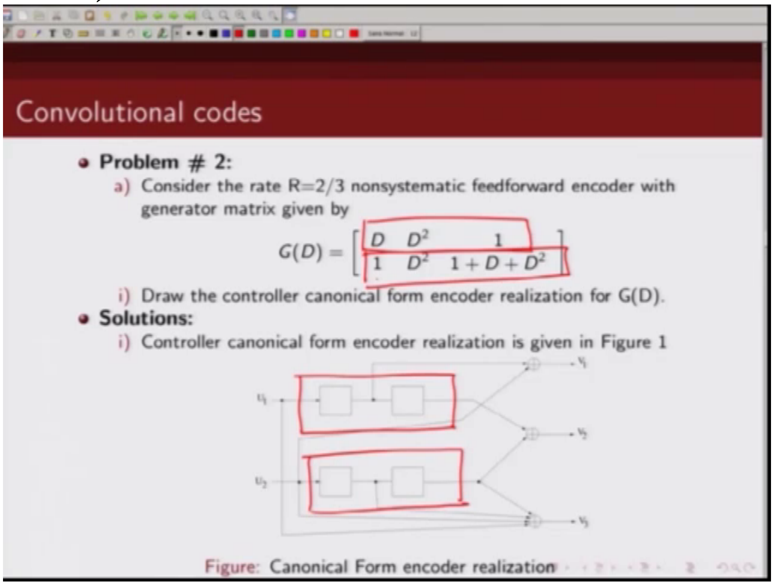

Maximum memory element for the first one is 2, second one also 2 and we can see now, the first output is D times u 1 which is this plus u 2 times this.

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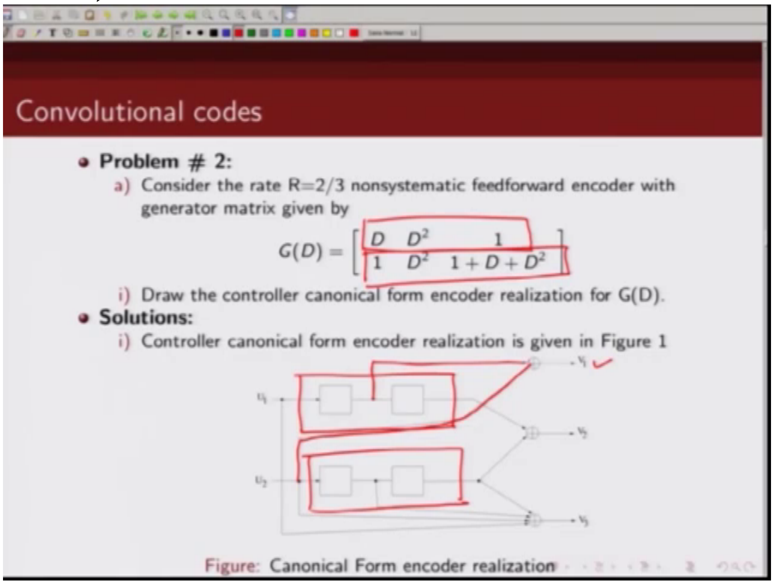

The second output is D square times u 1 and D square times u 2, so that is this.

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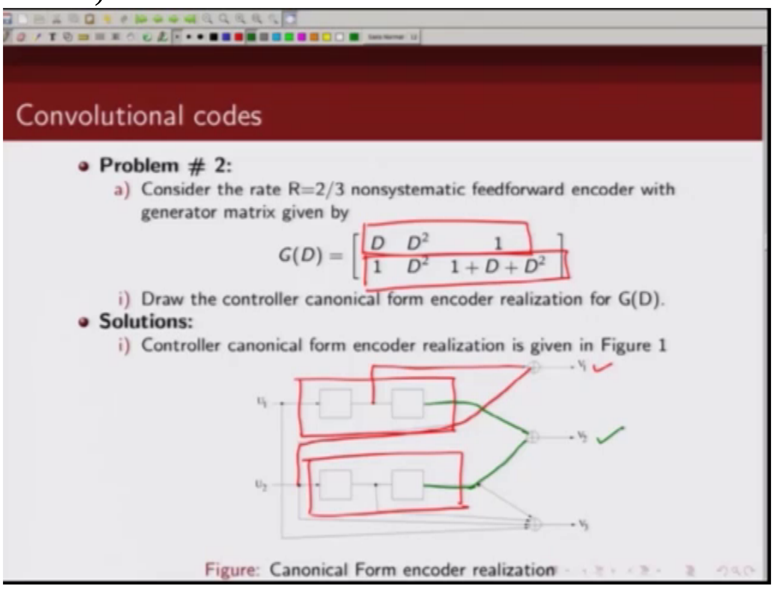

And the third output is u 1 which is this and this is u 1 D times u 1, D square times u 1, so that's your third output,

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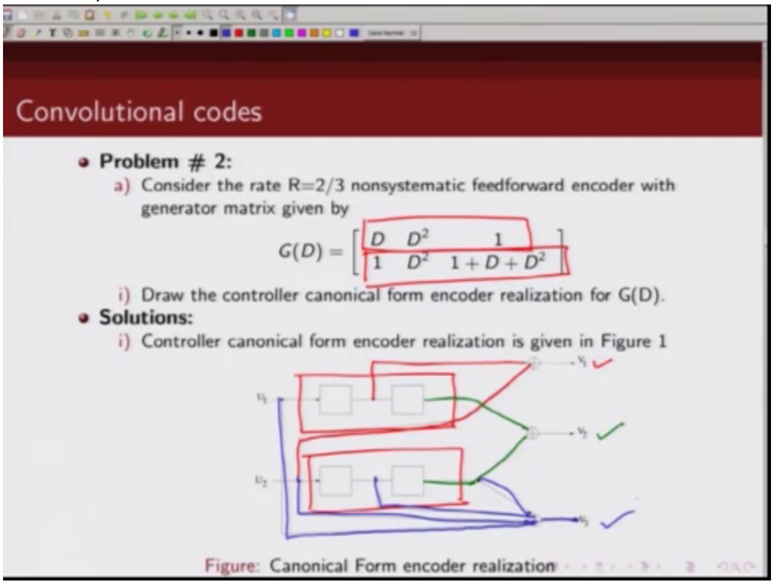

Ok.

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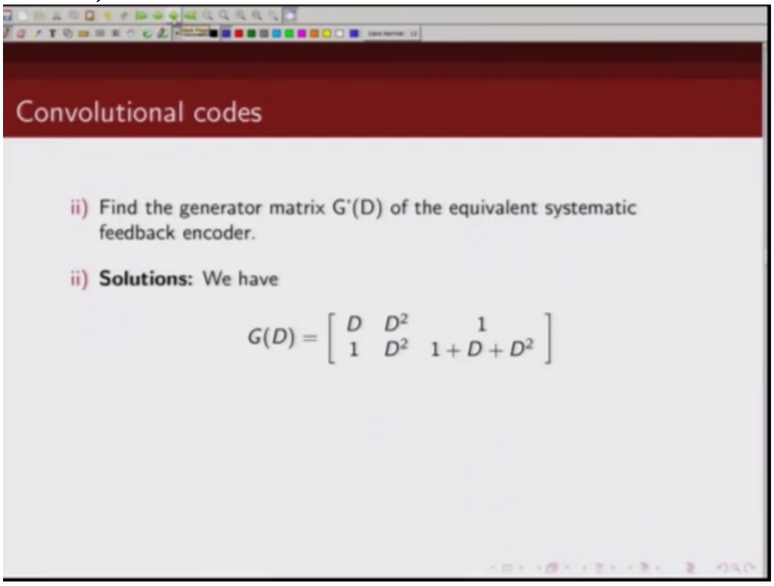

Now this was a non systematic encoder. Can we find an equivalent systematic encoder or systematic encoding matrix for this generator matrix, the answer is yes. So how do we find a systematic encoding matrix? So this has to be put in the form like this; 1 0 0 1 and some matrix here let's call it a 1 D times a 2 D and b 1 D times b 2 D. So we will have to bring

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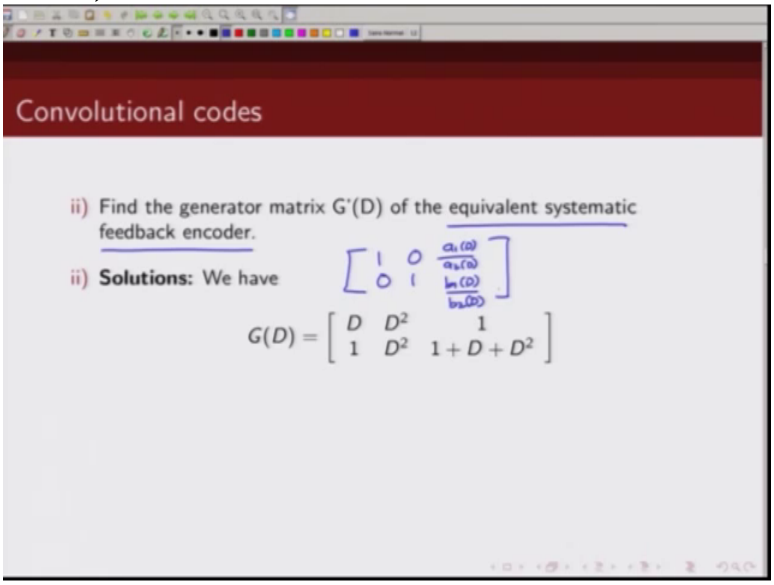

this matrix in this particular form. So we have to get this to 1, this to, this has to be changed to 1, this has to be brought to 0,

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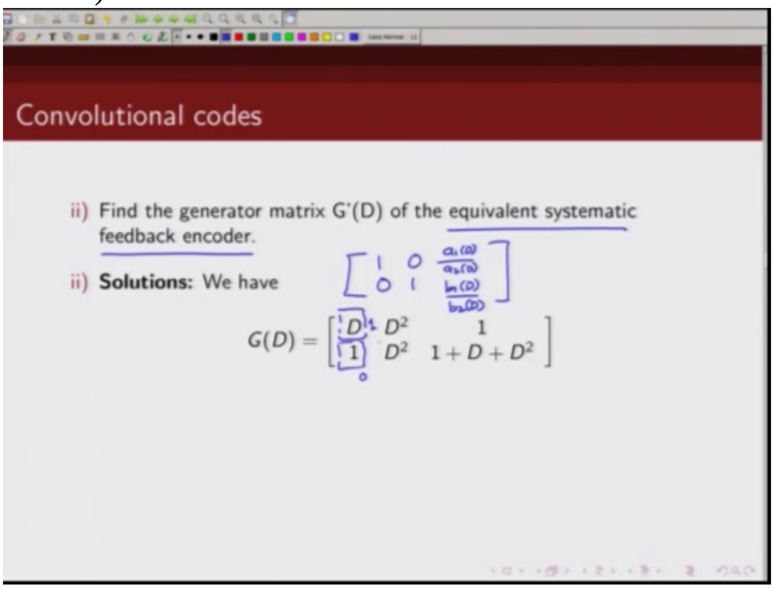

, this has, this we have to brought to 1 and this we have to bring to

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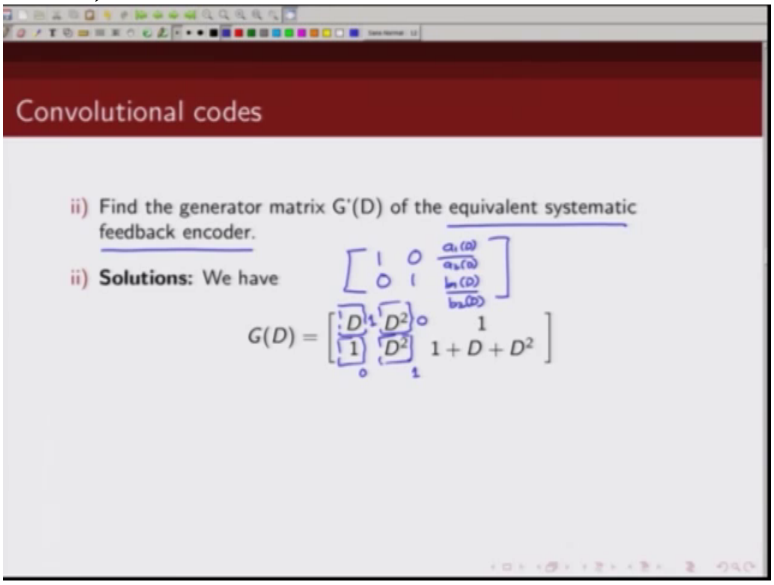

0. Now we will do elementary row operation to get an identity matrix here. So let's do that.

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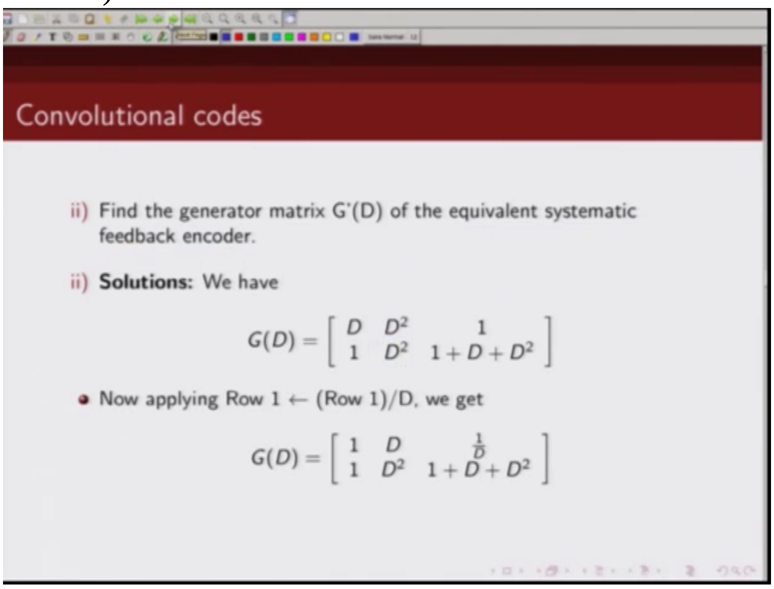

So first thing that we do is, we make this a 1. How do we make this a 1? We do this transformation that row 1 is, row 1 by D. So we divide this whole thing by D, what we get is 1 D 1 by D. Next we would like to get a zero here.

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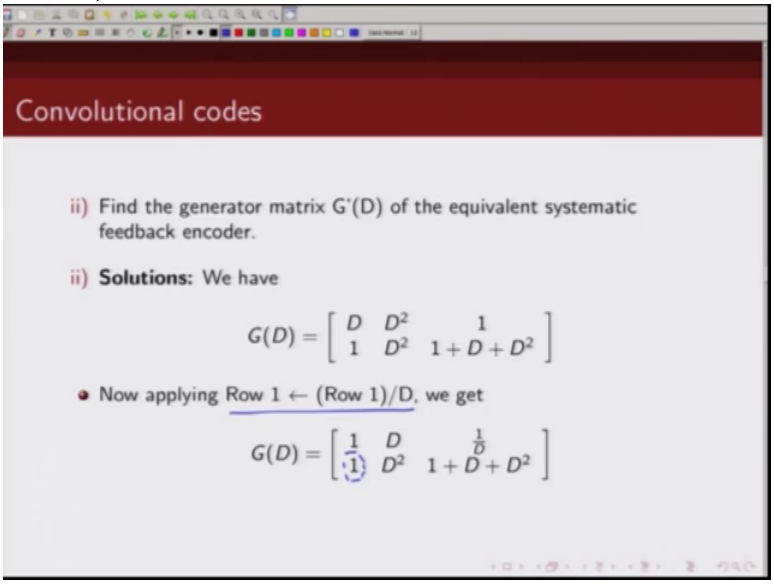

Here we would like to get a zero. How can we get a zero here?

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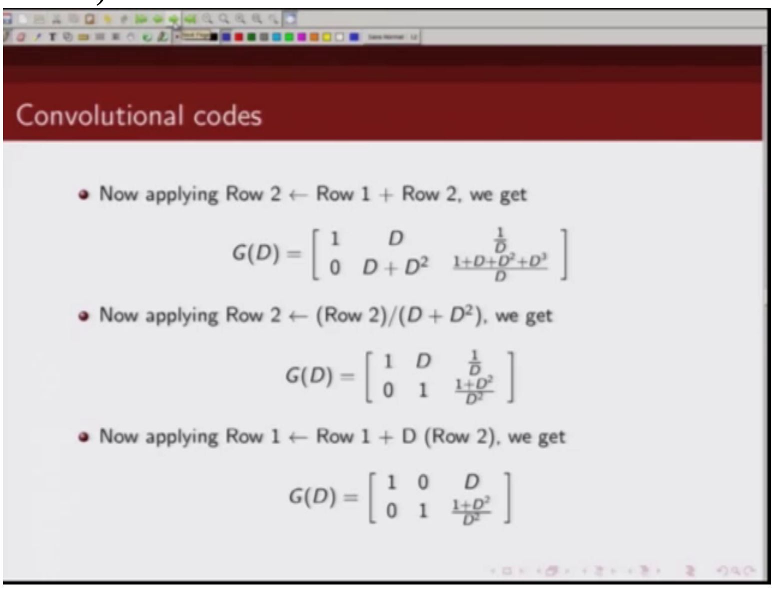

So we will do this transformation, row 2 is row 1 plus row 2. If we do that, so we

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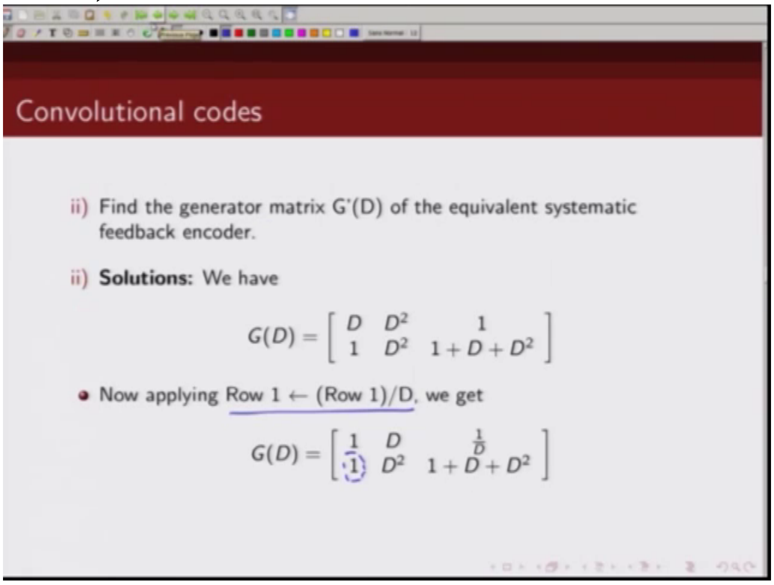

add these two, this will become 0, this will become D plus D square and what we will get is

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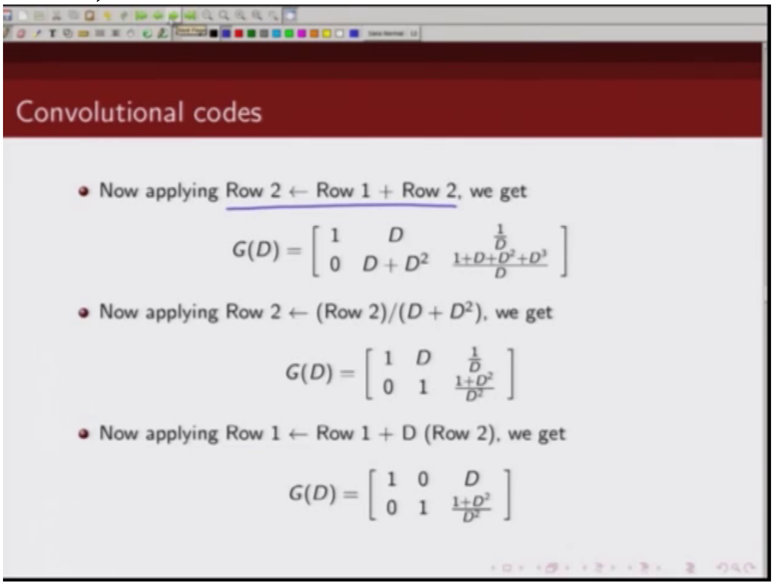

this. Next we would like to get a 1 here,

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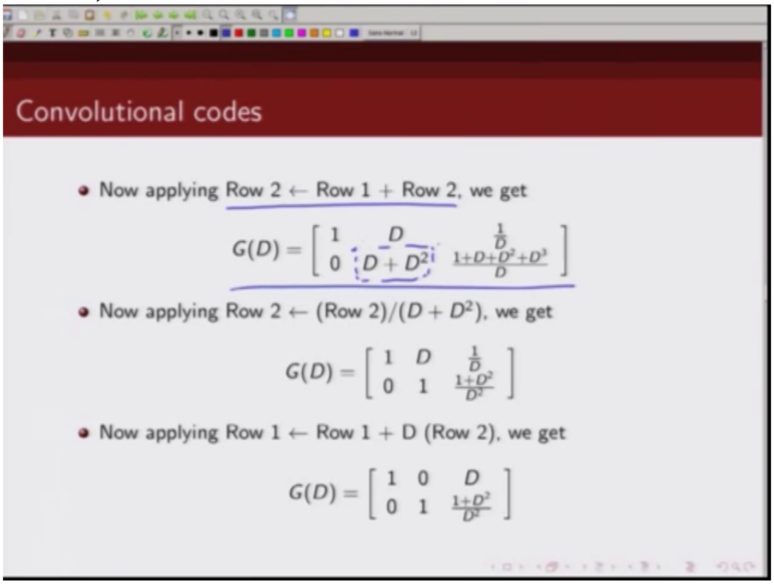

like to get a 1 here. How can we get a 1 here? We divide row 2 by this. So we do this transformation that row 2 is row 2 divided by D plus D square and once we do that, we get this. Next we would like to get a zero here.

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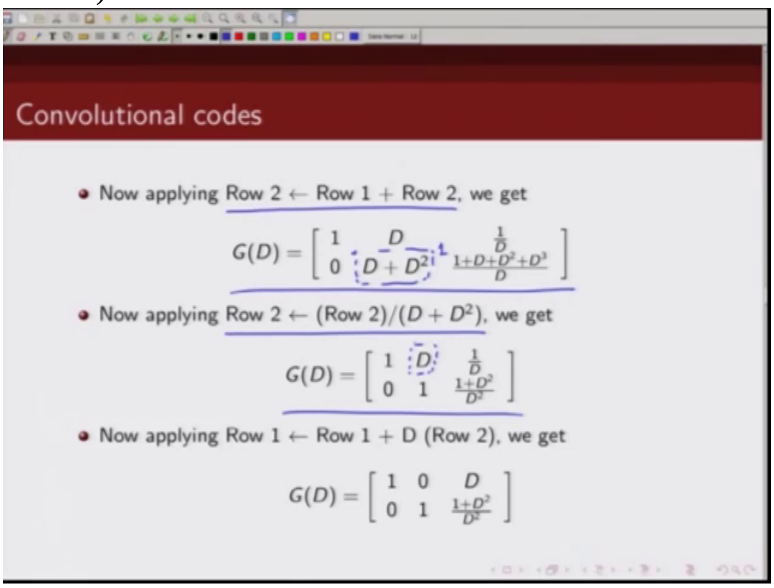

How do we get a zero here? We multiply row 2 by D and add it to row 1. So we do this transformation that row 1 is row 1 plus D times row 2. And when we do that, we get this. So this is our equivalent systematic encoder for the

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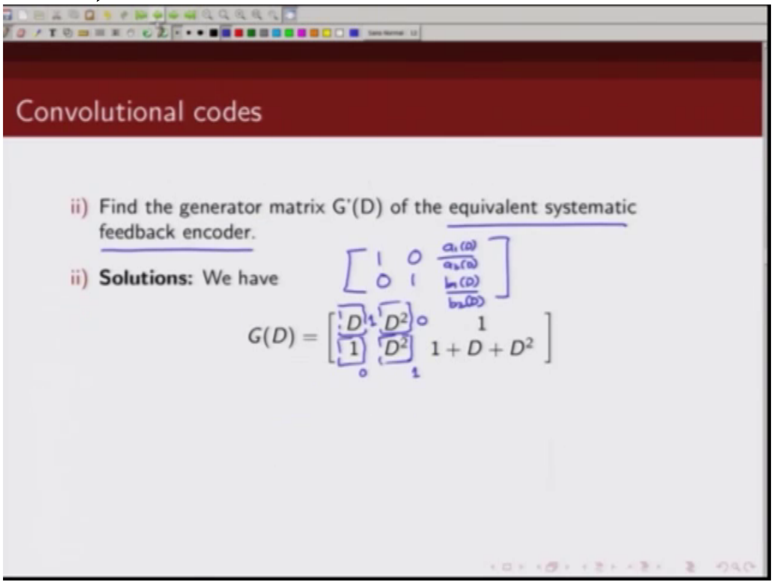

generator matrix this, Ok.

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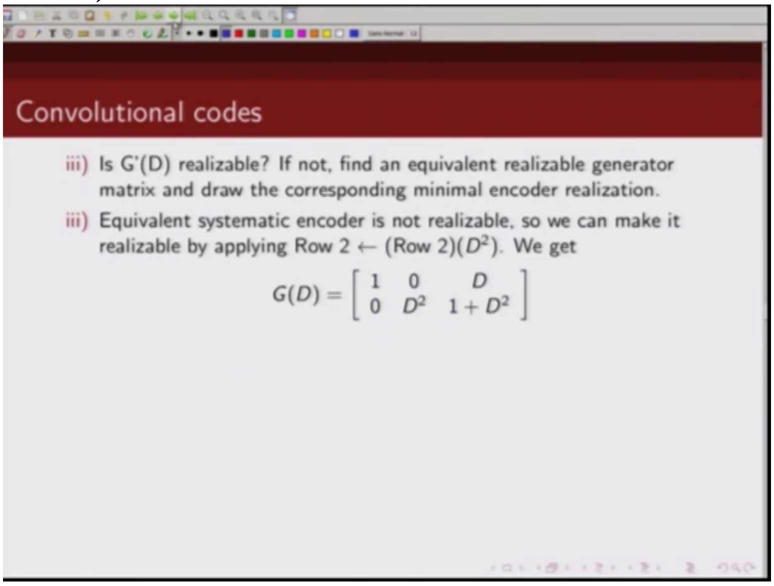

Next. Is this equivalent systematic generator matrix, is it realizable? If it is not, find out an equivalent realizable generator matrix and draw its corresponding minimal encoder realization. Now note here

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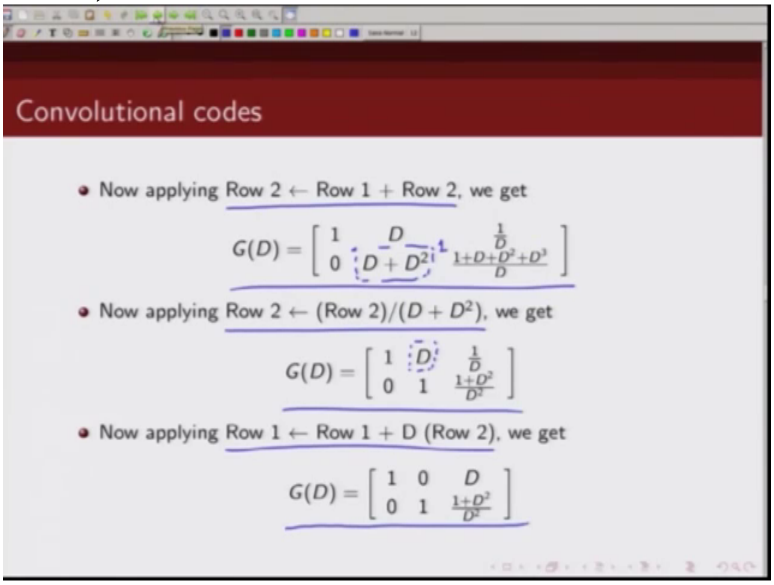

this generator matrix has a term 1 plus

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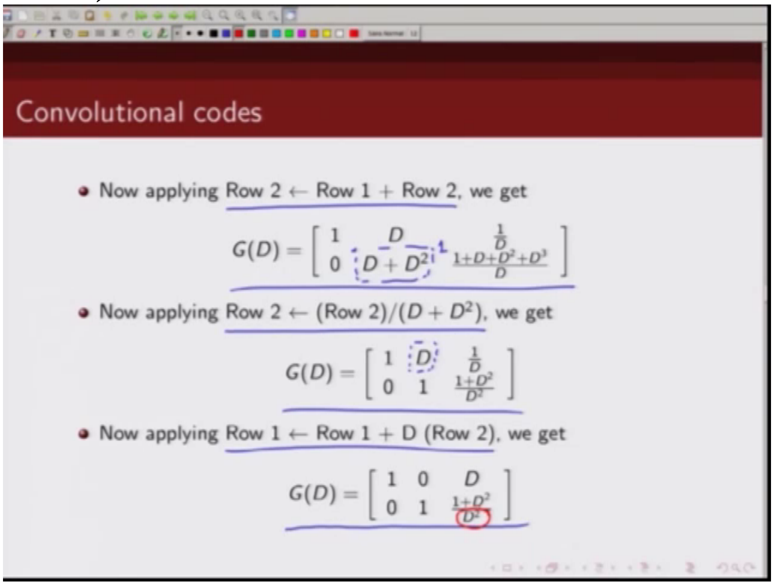

D square in the denominator. Now this cannot be realizable. So any denominator term that we have, it has to be of the form 1 plus some polynomial here

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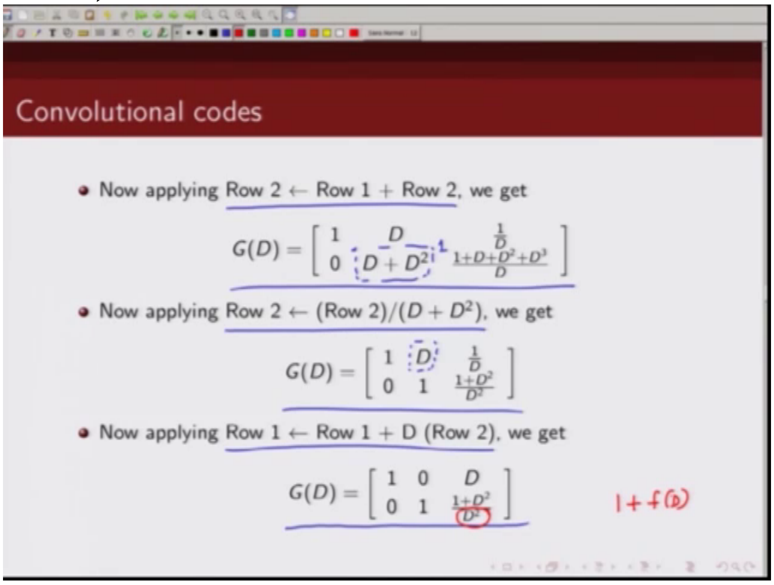

but here this 1 is not here. So we cannot realize a rational function of this form using our shift register. So this particular equivalent systematic encoder is not realizable. However if we multiply this by D square, what we

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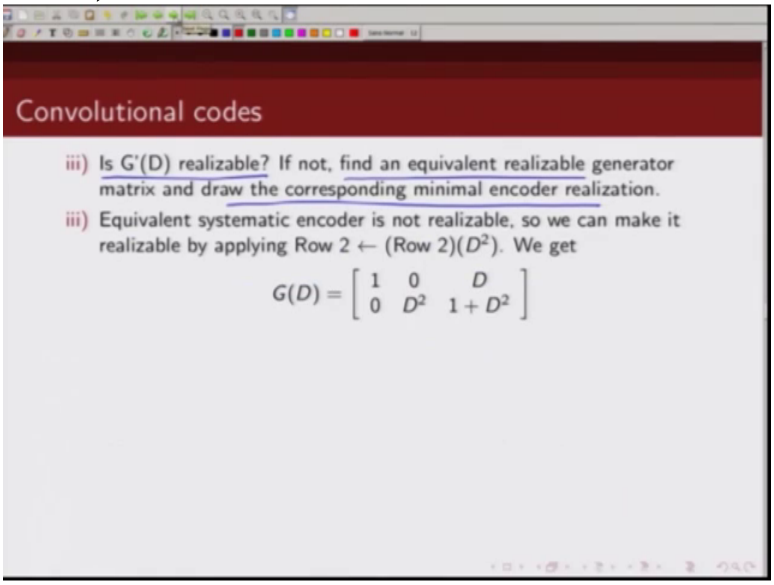

get if we do this transformation, what we get is this. This is no longer, so what we are getting now is basically a new equivalent encoder which is in the feed forward form and it is realizable.

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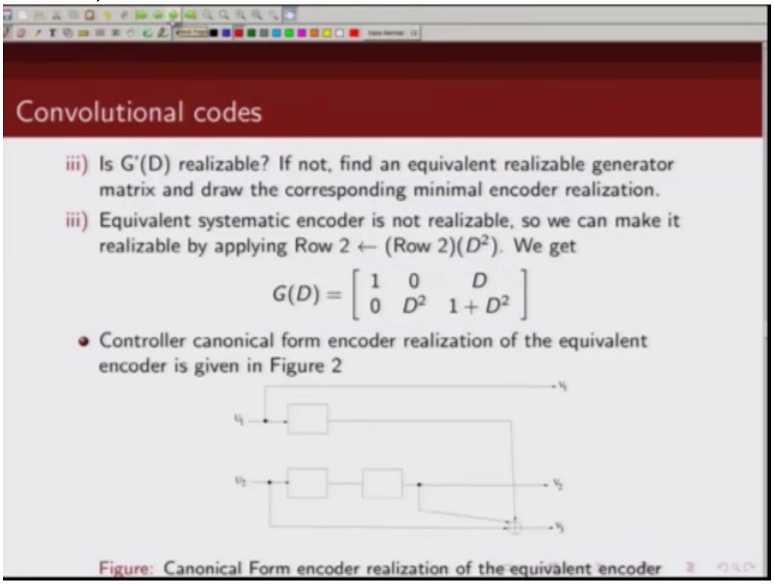

So how do we realize it? Again if we using controller canonical form realization we will have one set of shift registers for this

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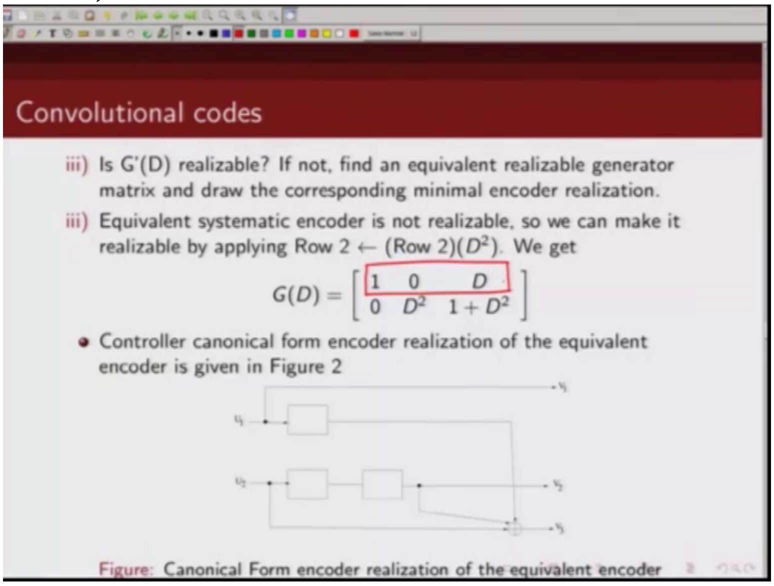

input, another set of shift registers for this input.

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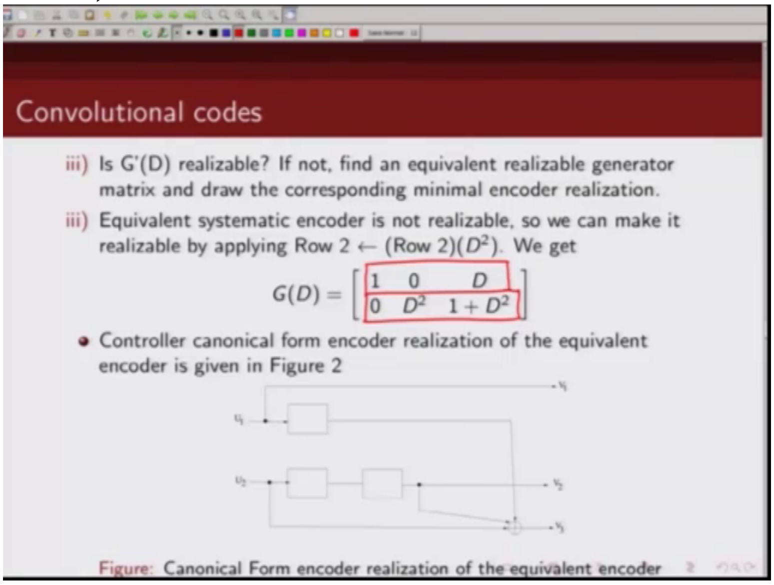

What's the maximum memory for the first row? The maximum power of D is 1. So we will have 1 memory element for the first input and what's the maximum power of D for the second? That's 2 here, 2 here, 2 here

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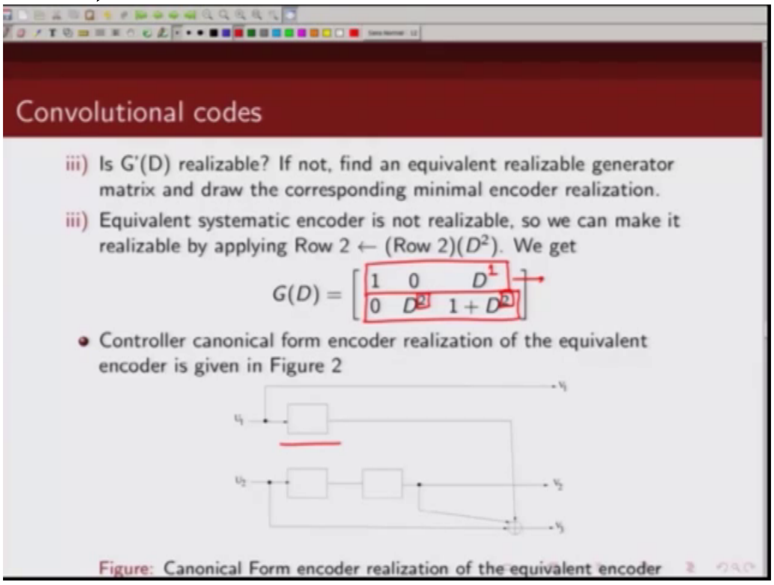

so we will use 2 memory element

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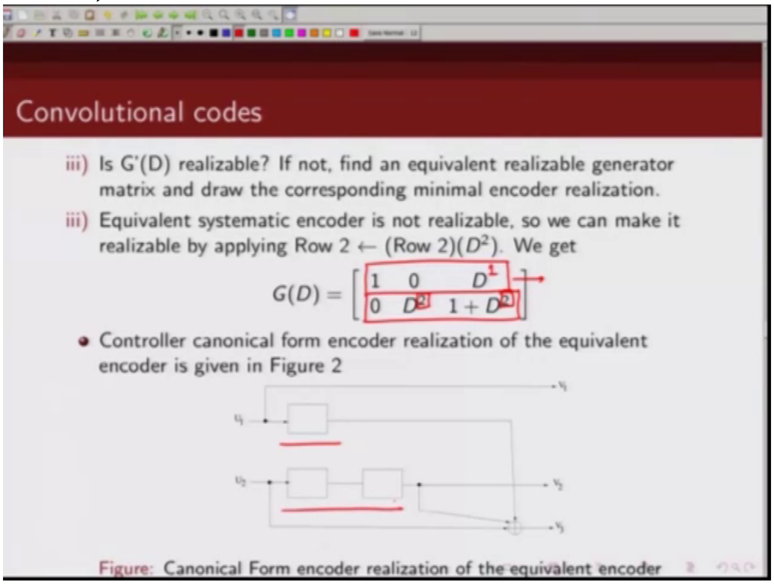

for the second input. And again what are our outputs? There are 3 outputs. The first output is this,

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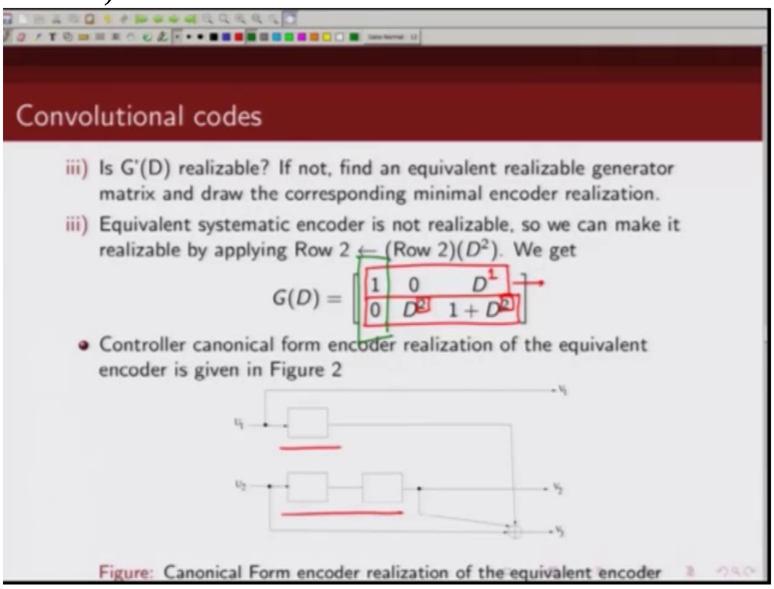

this is u 1 times, this is just u 1 so this is this, second one is D square

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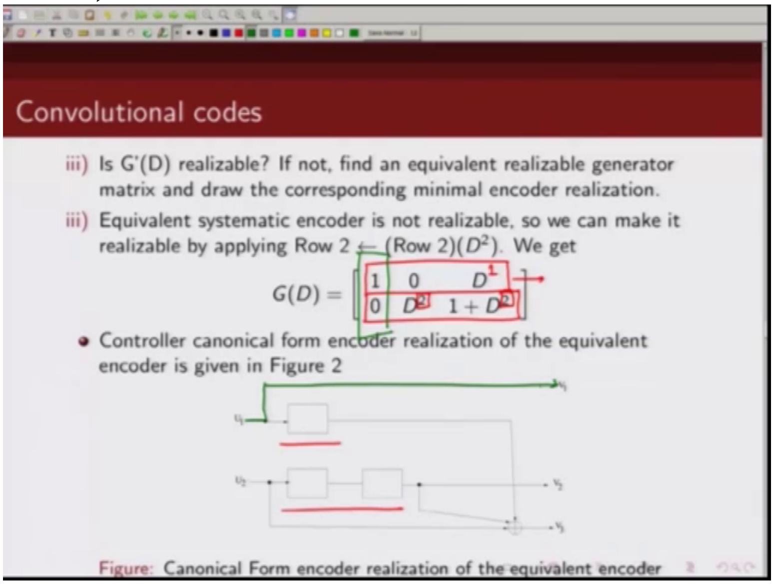

of u 2. So D square of u 2 is just this term.

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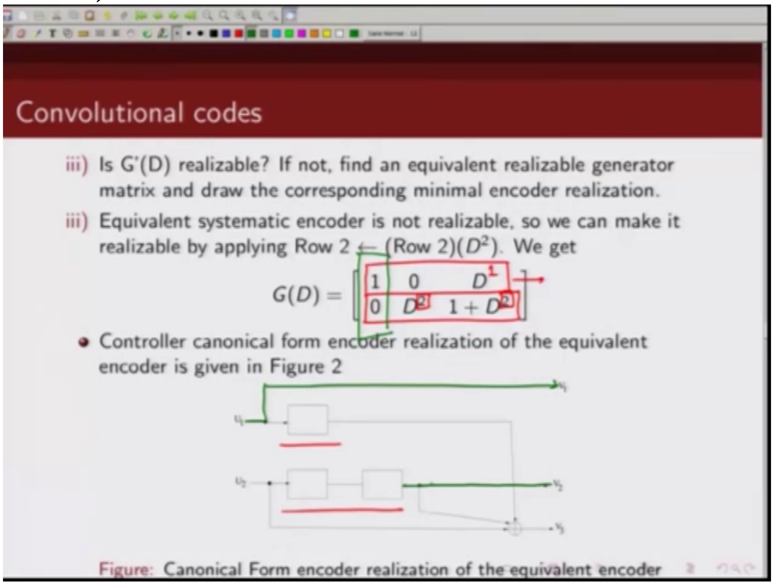

So this is my second output and the third

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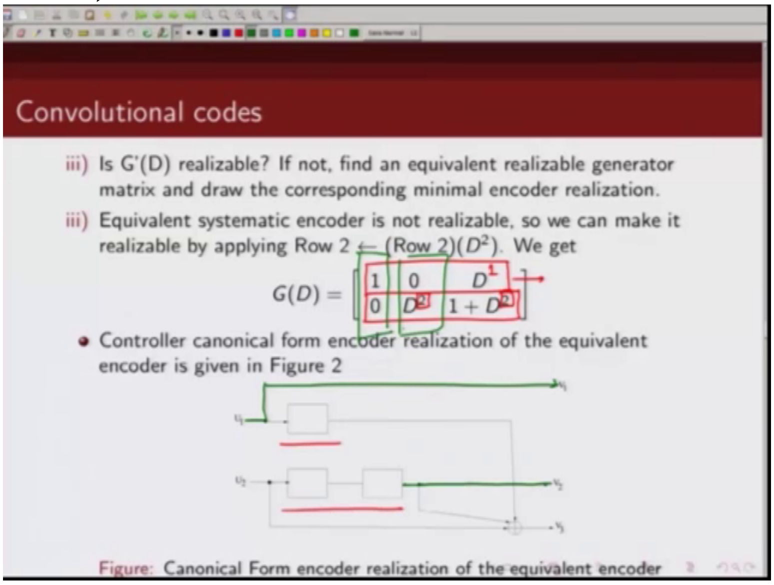

output is this,

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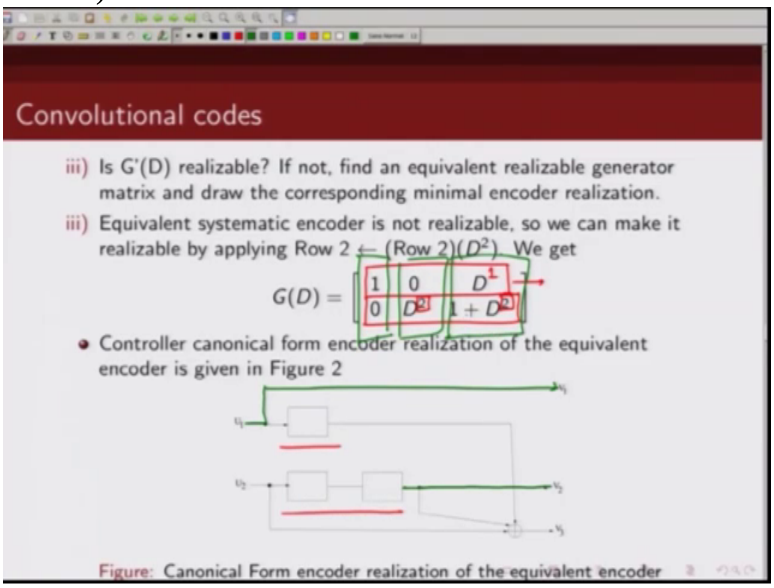

D times u 1 D which is this one and one times u 2 D and D square times u 2 D, so that's this. This is our third

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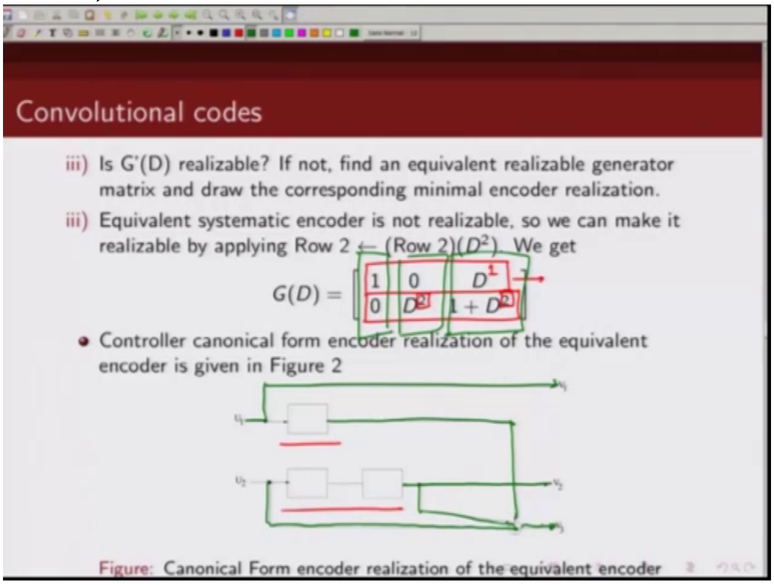

output. Now given

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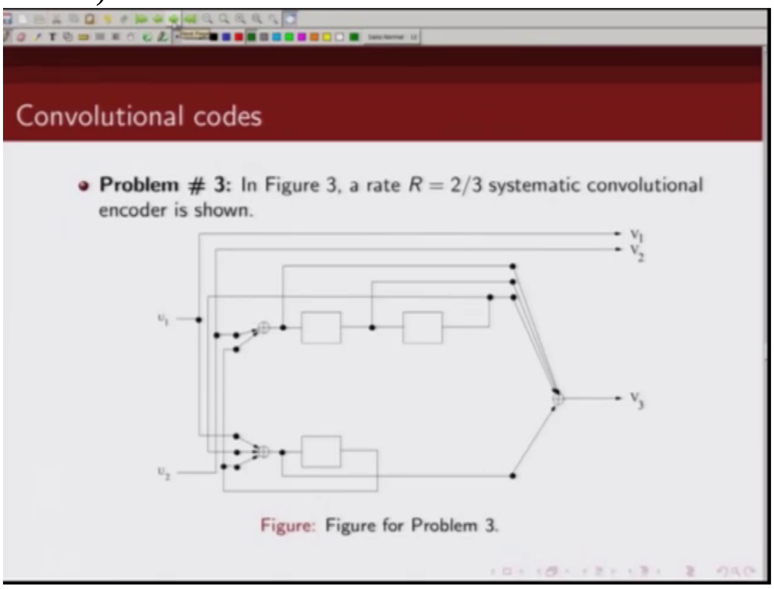

below is a rate two third systematic convolutional encoder. Please note this is neither in the controller canonical form realization or in the observer canonical form realization. Note here the feedback terms that are coming here are not only coming from the same encoders like this, feedback is not only, so if you look at the feedback, feedback from this is going to this encoder and feedback from here is going to this encoder. So not only feedback is coming to the same encoder but it is also going to the other encoder.

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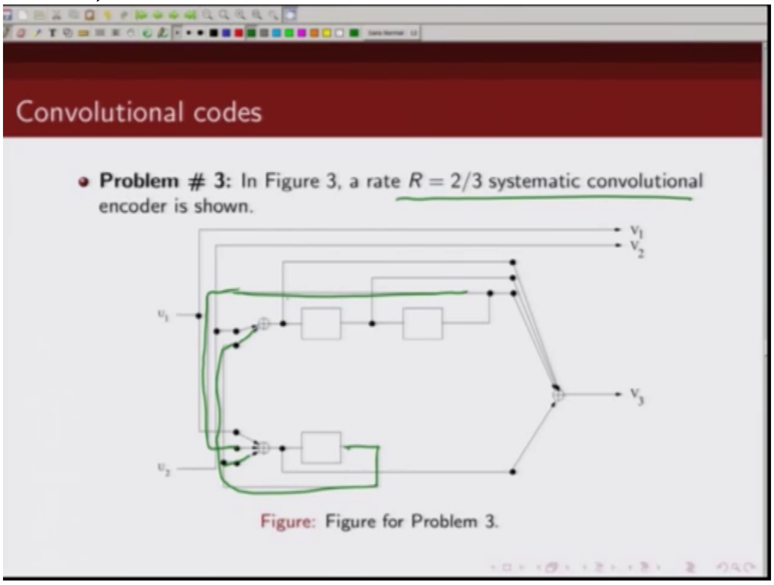

So this realization is

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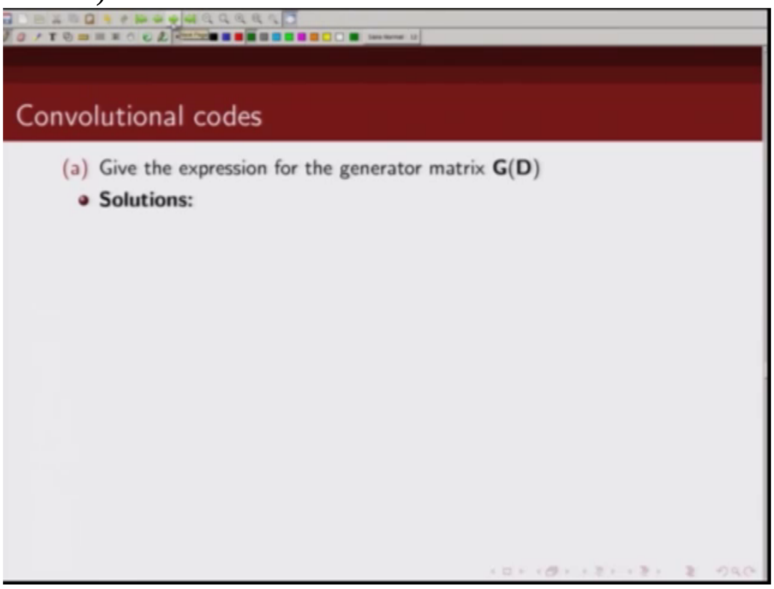

a very compact realization. The question that has been asked is can you find out the generator matrix corresponding to this encoder? So

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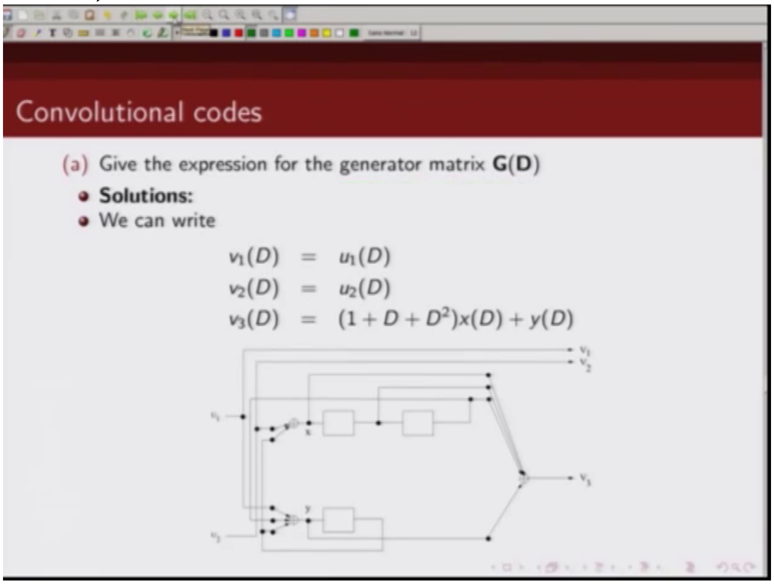

how do we find the generator matrix? We know this is a relation between the input and the output. So how these

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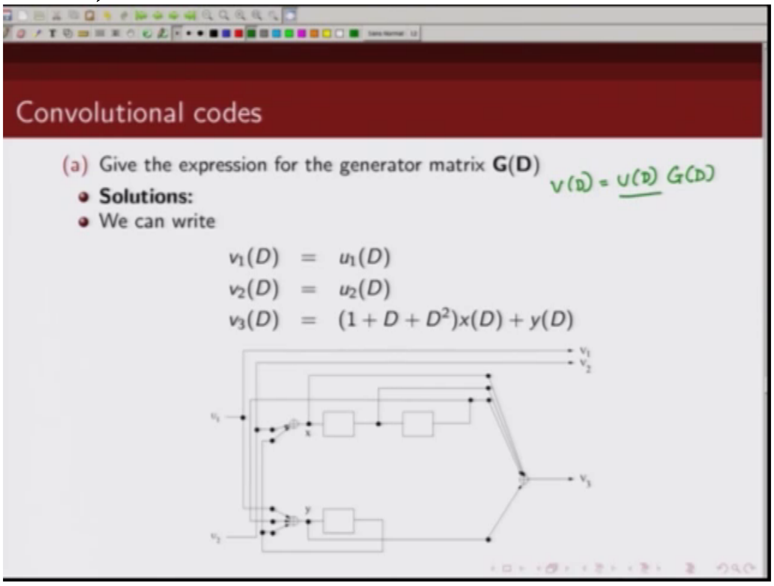

inputs are getting mapped to the output, that is governed by this generator matrix.

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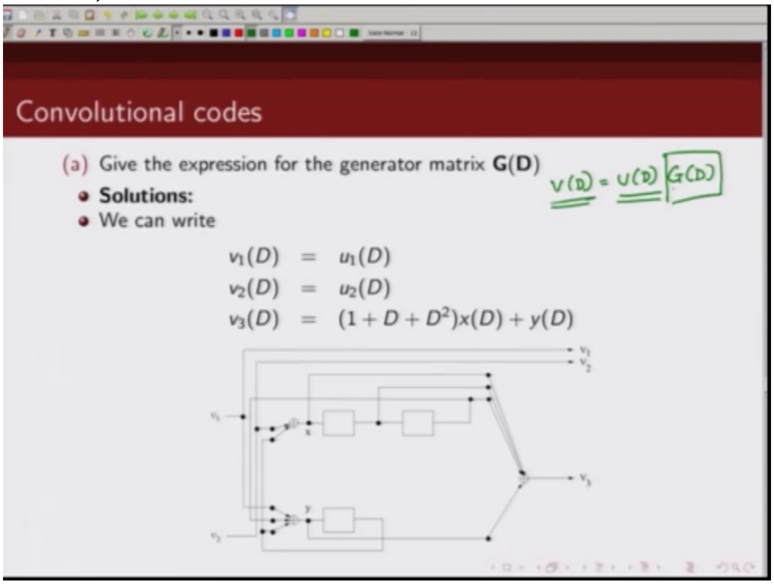

So what we are going to do is we are going to write the output v D in terms of input u D. And then that would give us our generator matrix. So our objective is to write v 1, v 2, v 3 in terms of u 1 and u 2,

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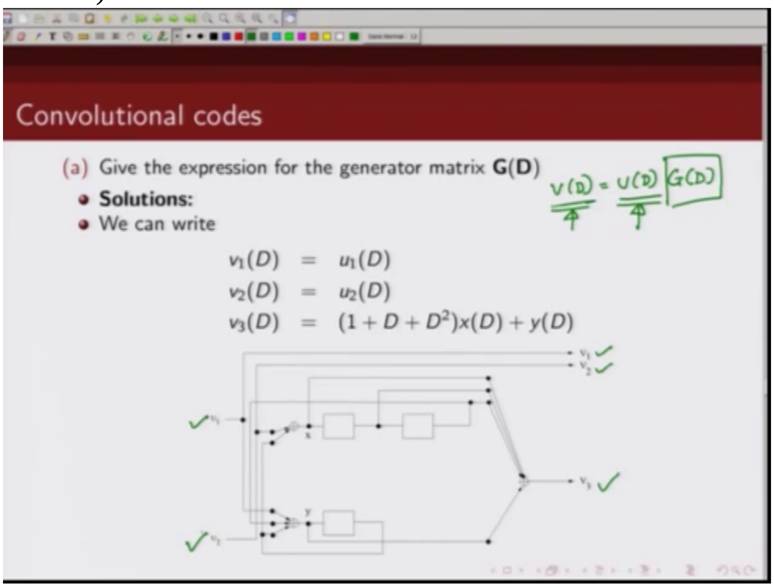

fine. We use some auxiliary variables; x and y which basically will help us find the contents here. So if this is x of D, this term will be D times x of D and this will be D square times x times D.

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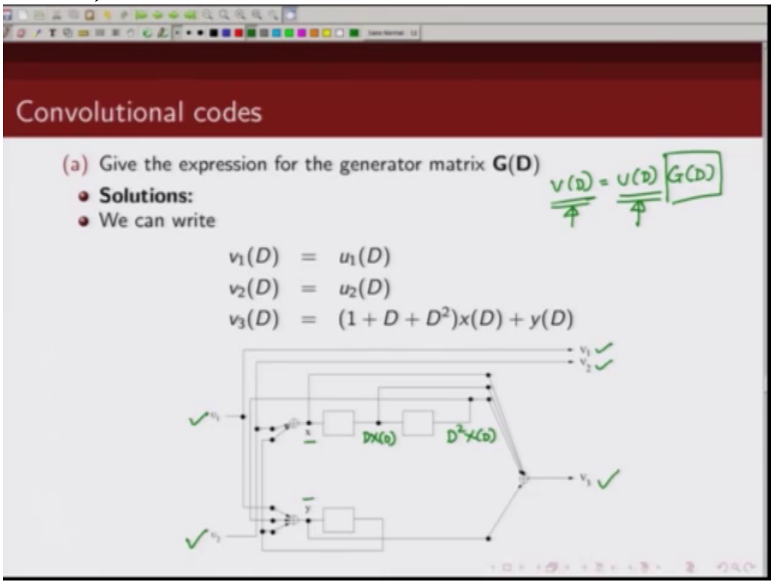

Similarly if this is y, this term will be D times y of D.

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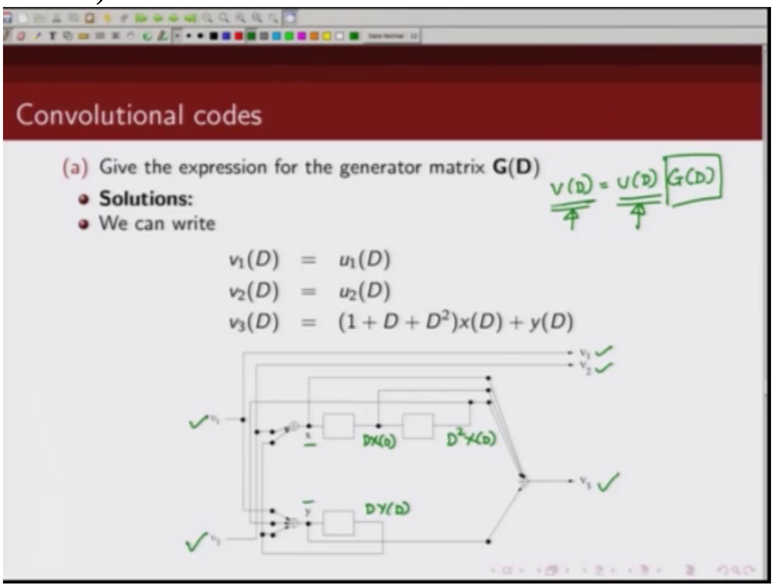

So what is v 1 of D? v 1 of D is u 1 of D, you can see u directly goes, this input directly goes here. So v 1

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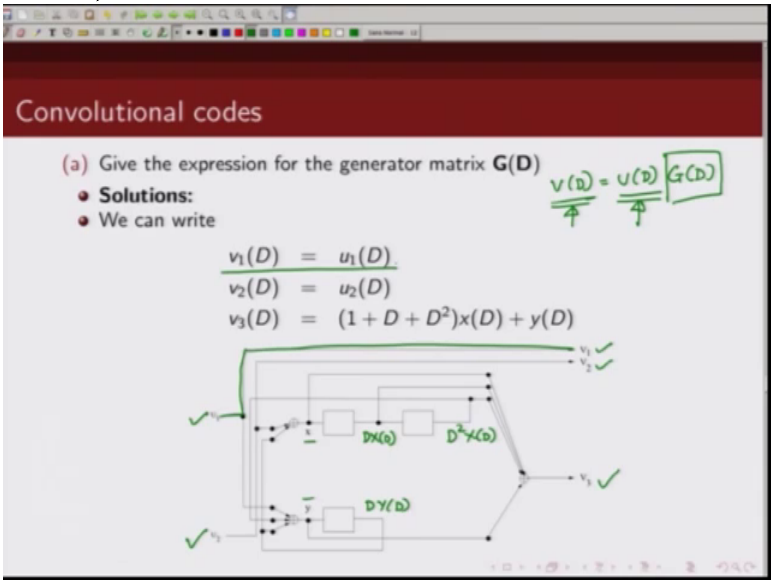

of D is u 1 of D. Similarly this input u D directly goes to the output here.

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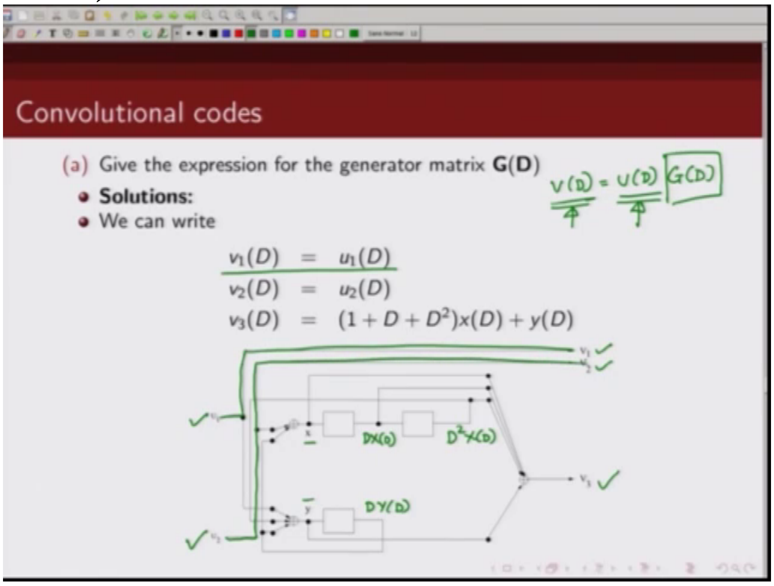

So v 2 of D is u 2 of D. Now what is v 3 of D? v 3 of D is this term which is x of D, this term D times x of D and this term which is D square x of D. So this is this term plus this term. So it is these 3 terms. Now what is this term? This is y of D. So we have

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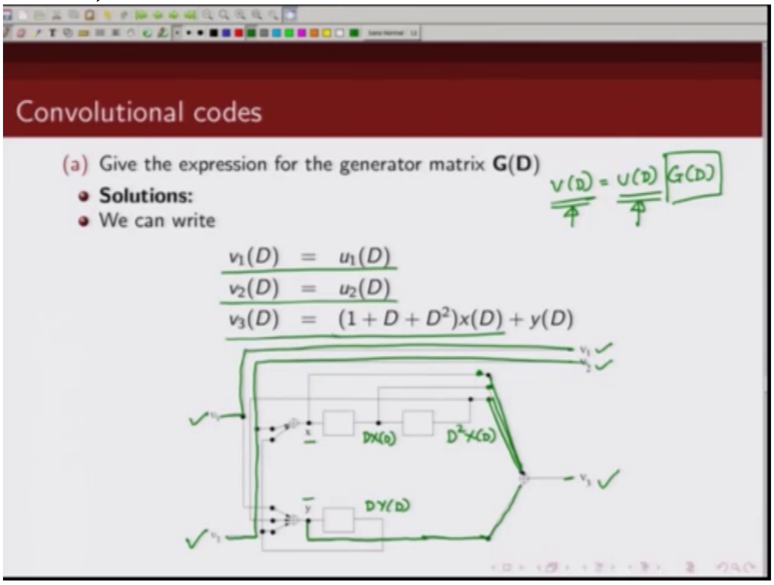

written v 1 of D, v 2 of D, v 3 of D in terms of u 1, u 2, x of D and y of D. Now note we need to get rid of x of D and

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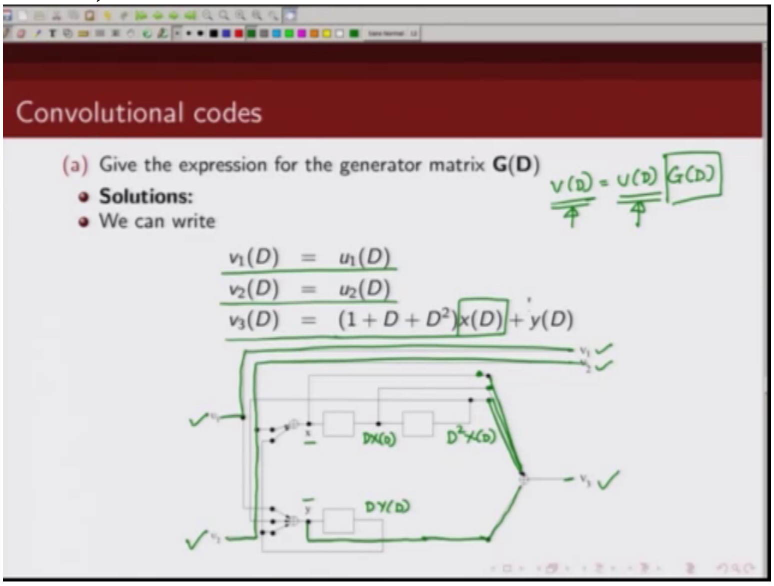

y of D, and we have to

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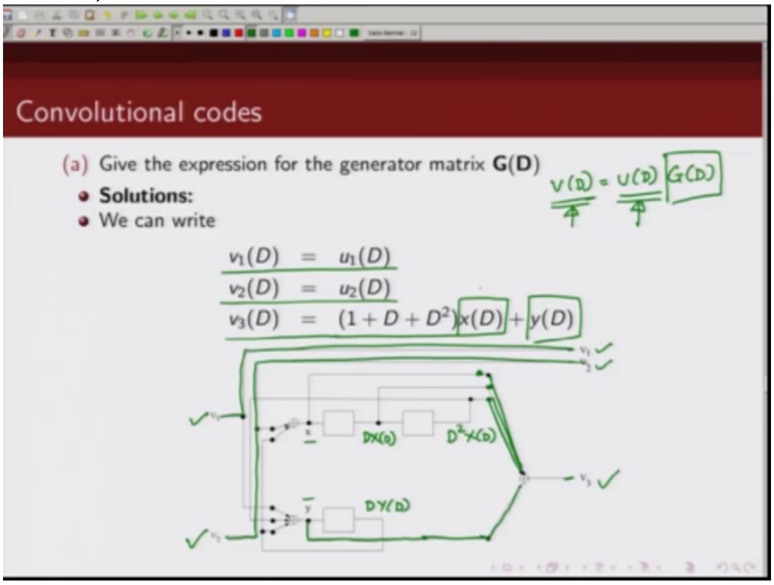

write these in terms of u 1 and u 2. Now what is x of D? x of D is this and this. Similarly what is y of D?

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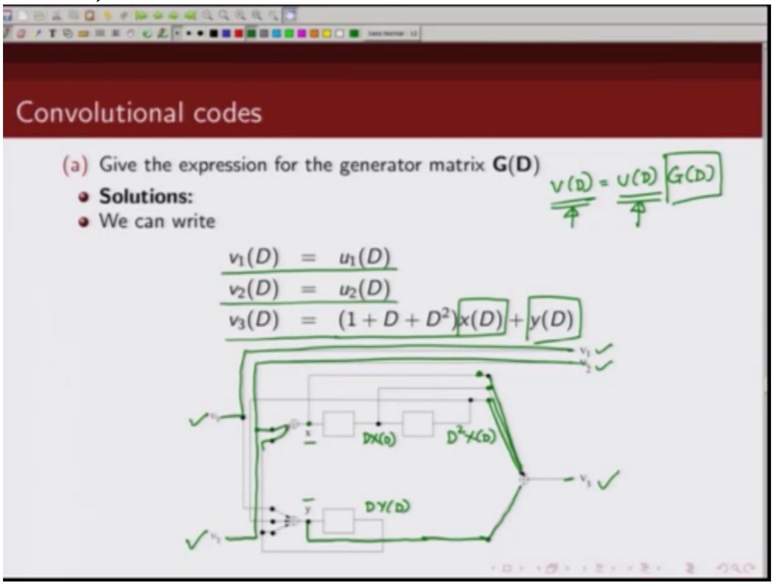

y of D is this term, this, this term sorry this term and this term, Ok. So we can write

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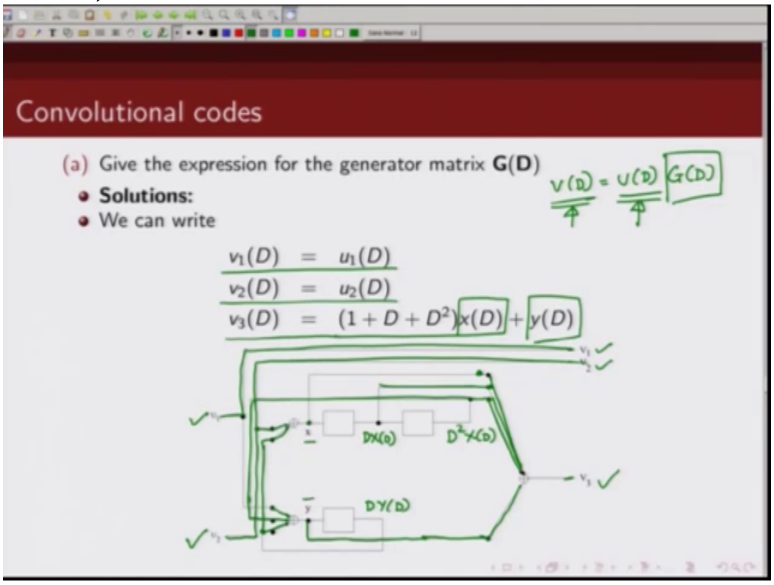

2 more equations

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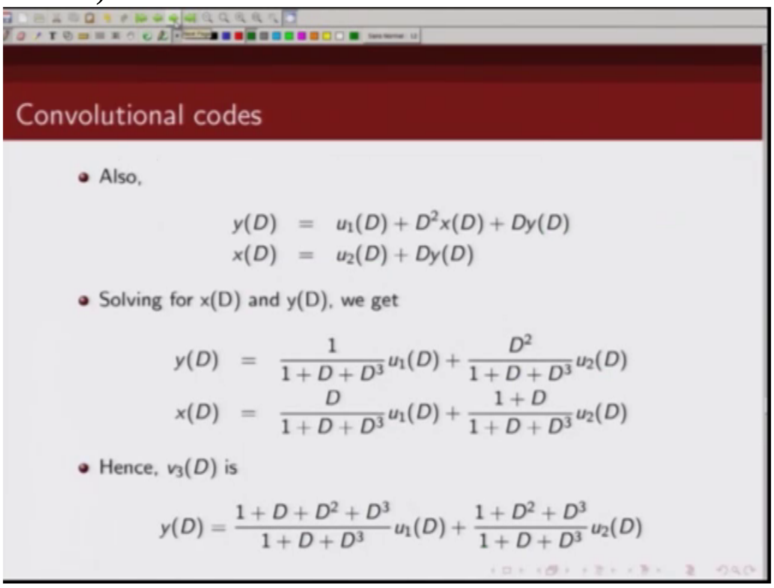

for x of D and y of D. So again y of D as I said is u 1 of D,

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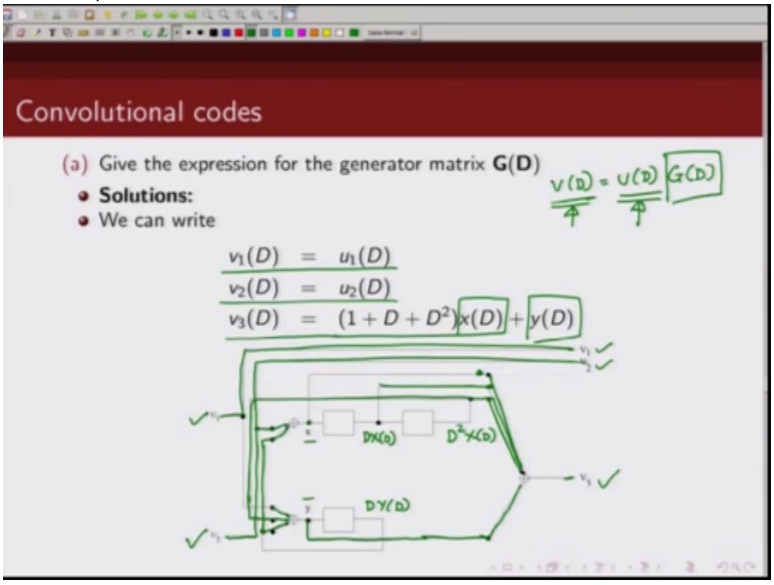

y of D is u 1 of D which is this one, this is u 1 of D

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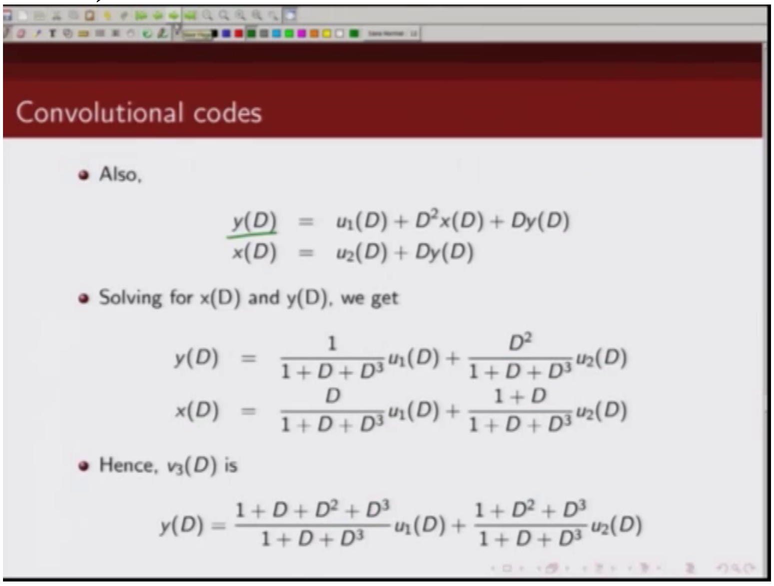

plus D square x of D.

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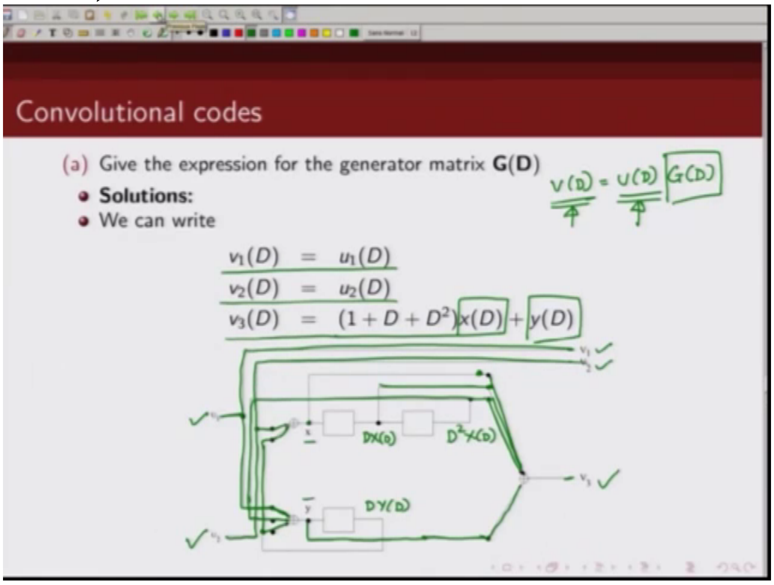

D square x of D is this term, D square of x of D is this term which is coming here, this term and there is another term

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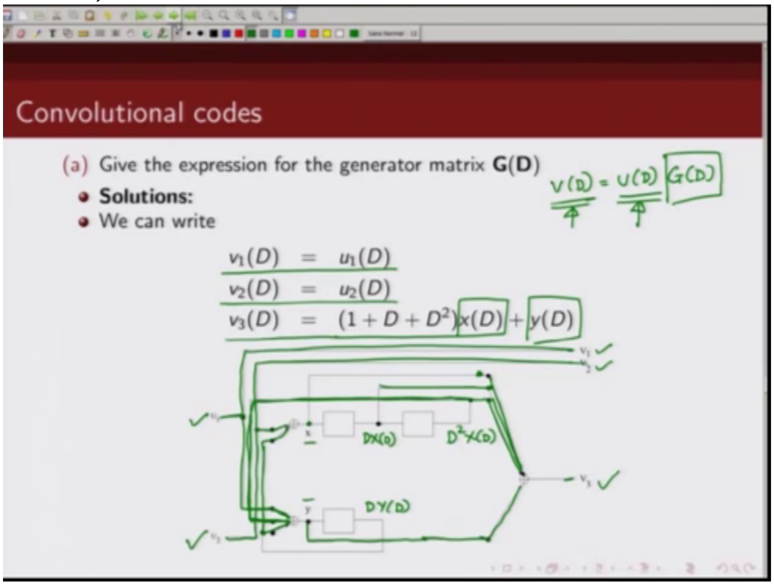

here

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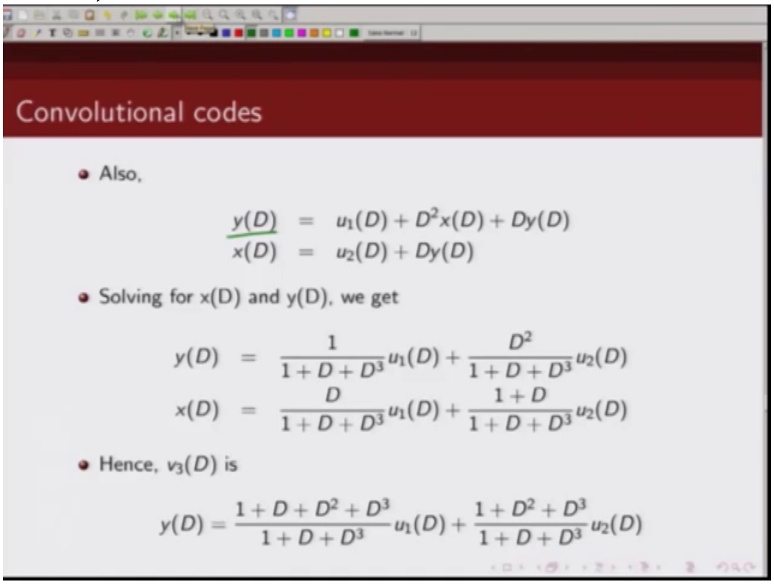

which is D times y of D. So D

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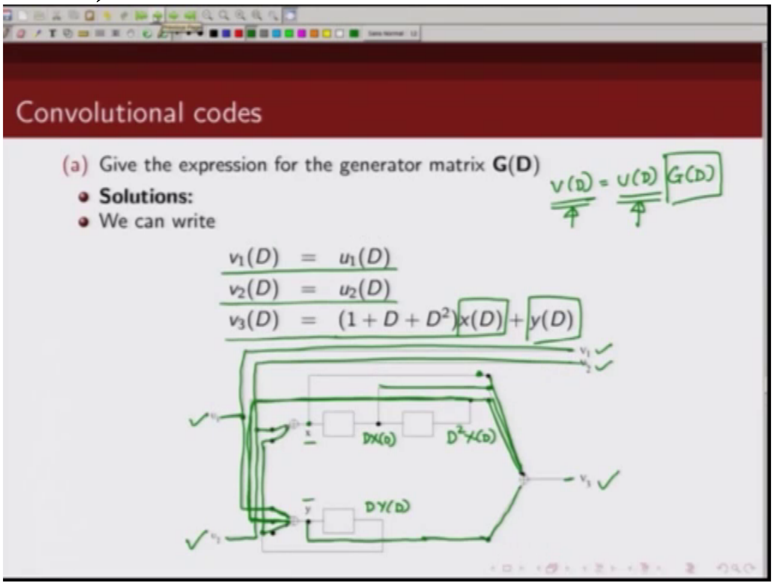

times y of D, note here, the third input here is this one which is D times y

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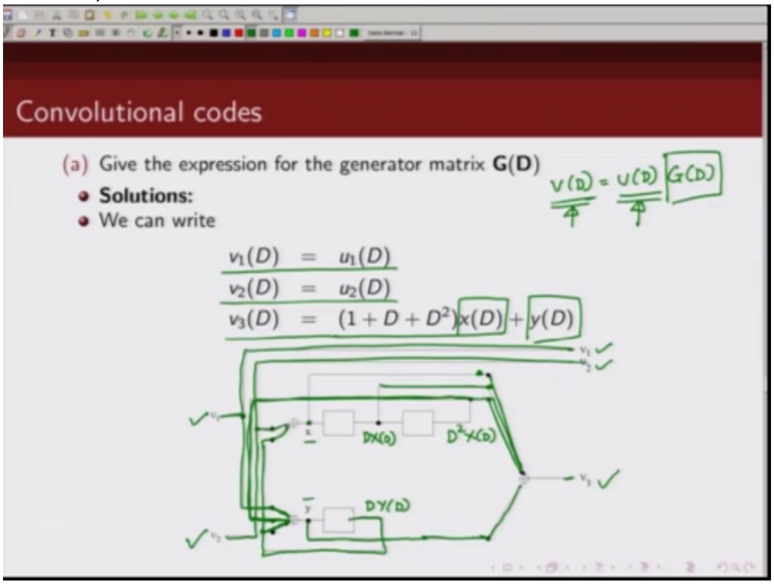

of D. Similarly x of D is, first one is this term which is u of D, so this is u

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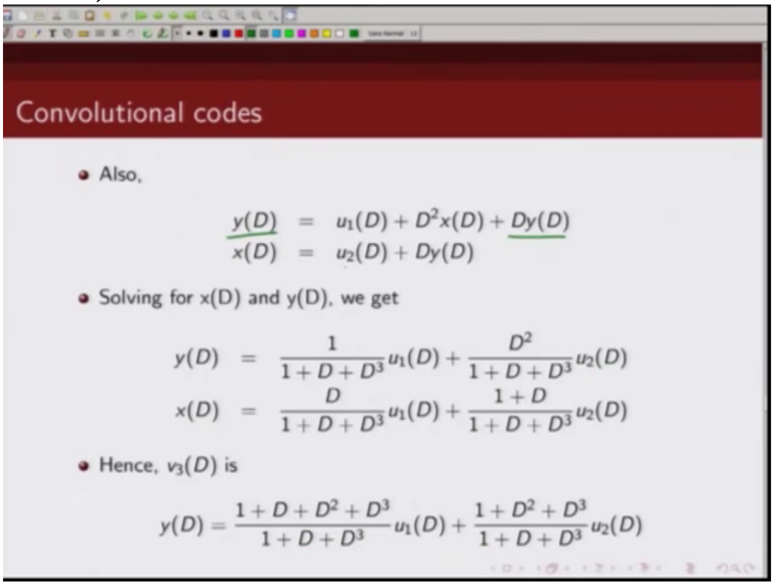

of D and the second term is

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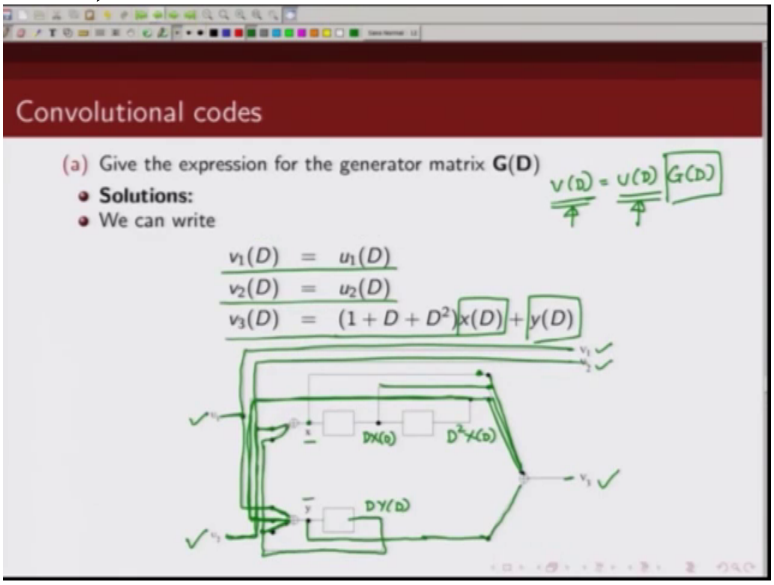

this term which is D times y of D,

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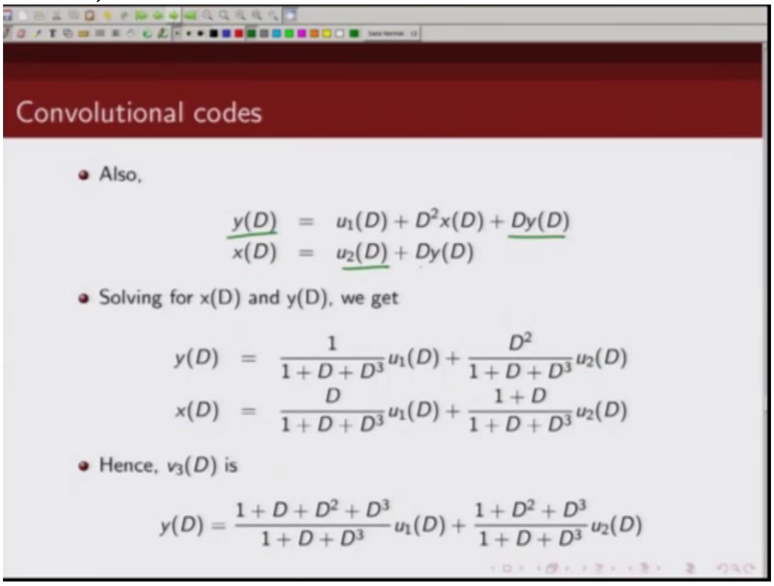

this one, Ok. So now we have got equations of y of D, x of D in terms of u 1 D and u 2 D. So let's write, bring y of D at one side and x of D at one side and write them in terms of, y of D and x of D in terms of u 1 D and u 2 D. So if we solve this, what we get is y of D is given by this and x of D is given by this. Now we plug these values of y of D and x of D given by this into

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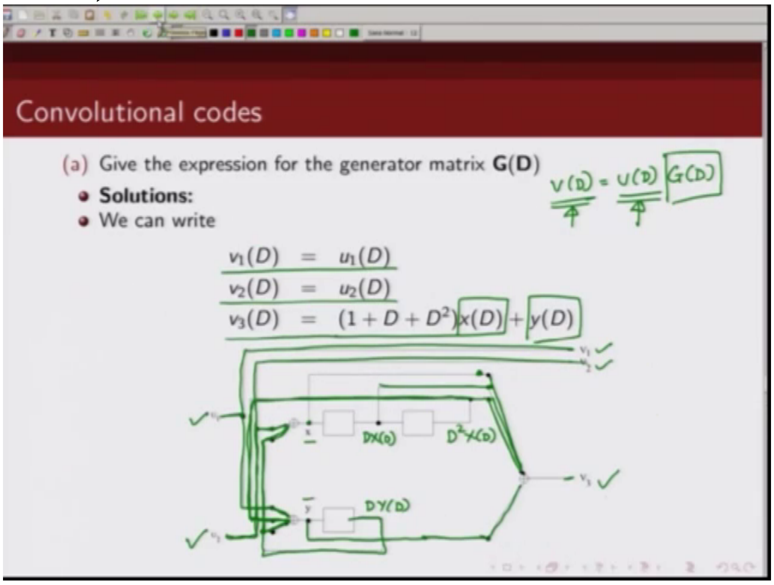

here, into this expression of

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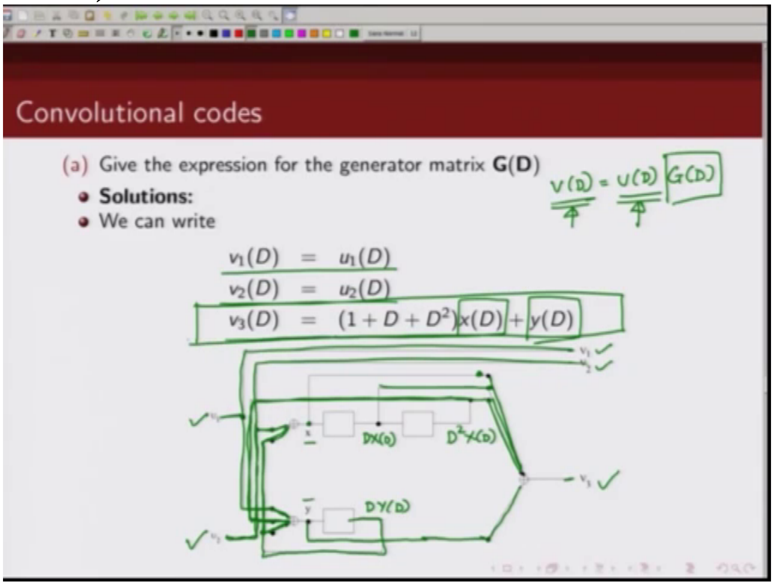

y 3 of D. So we plug this value of x D and y D which we just computed,

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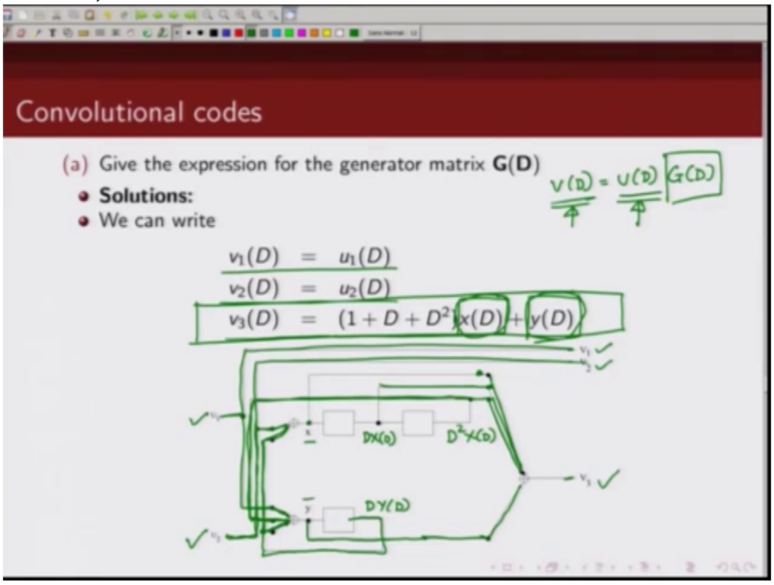

we plug those values in here. If we do that, we

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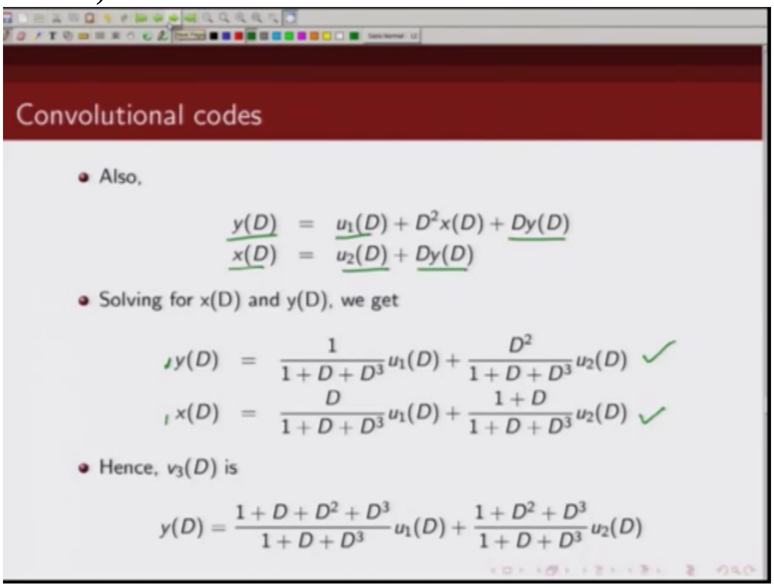

will get the expression of y 3 of D, Ok. Now,

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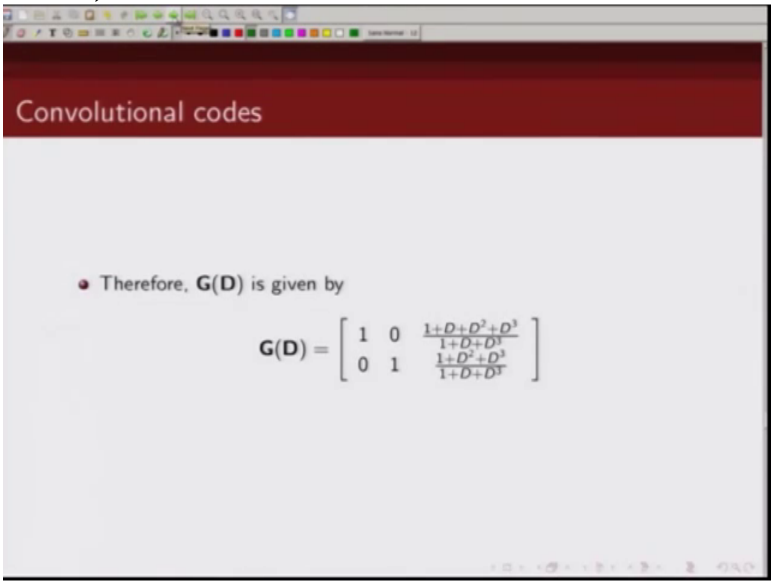

so if we do that finally

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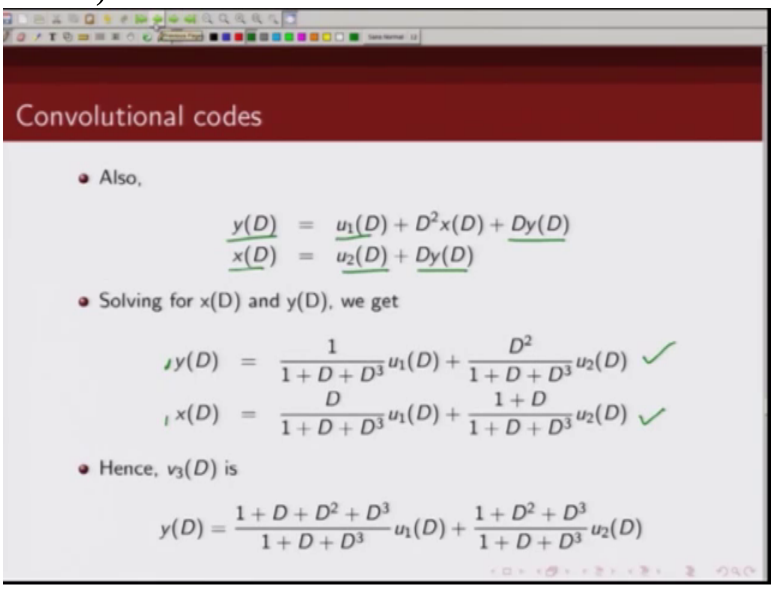

this is v 3 of D, so if we do that what we get is then v 3

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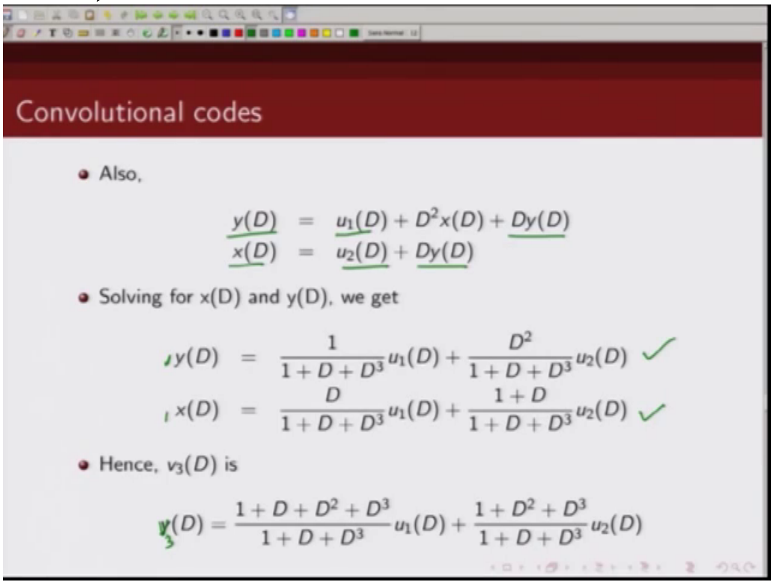

of D is this times u 1 D plus this times u 2 of D. So now we are in a position to

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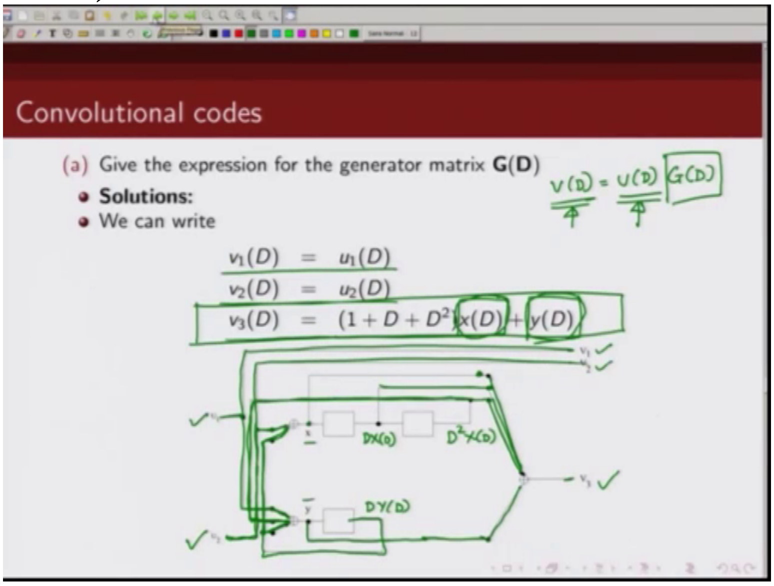

write the generator matrix. The first equation that we will require is this one. Second equation we will require is this one.

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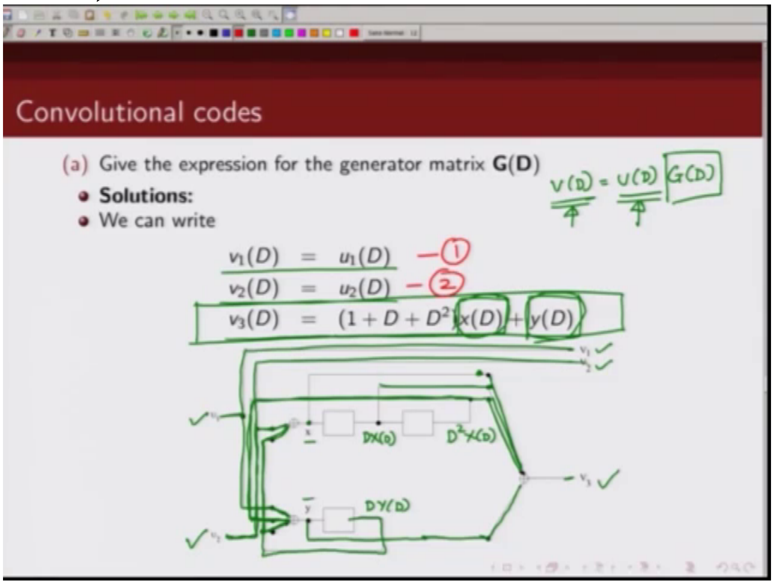

And the third

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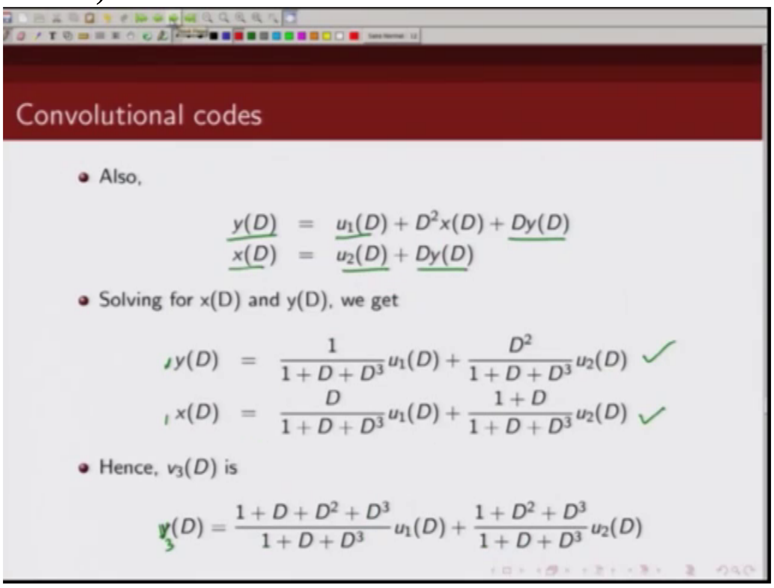

equation that we will require is this one,

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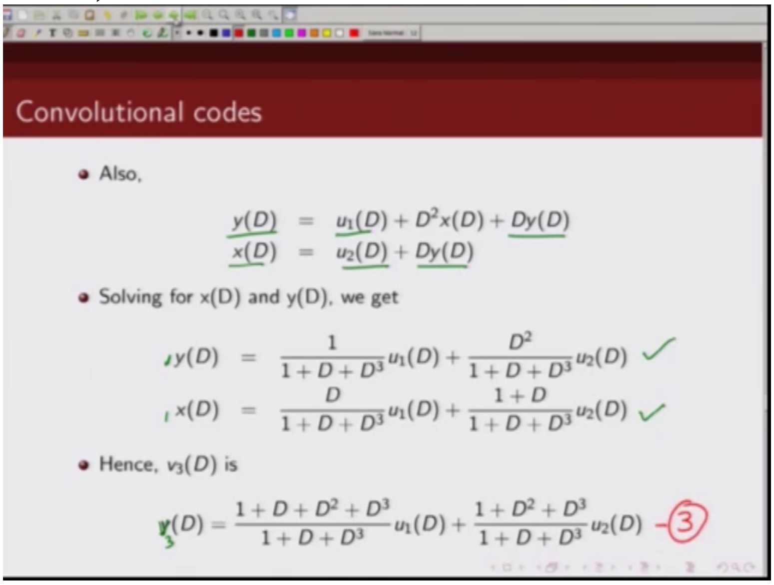

right. So

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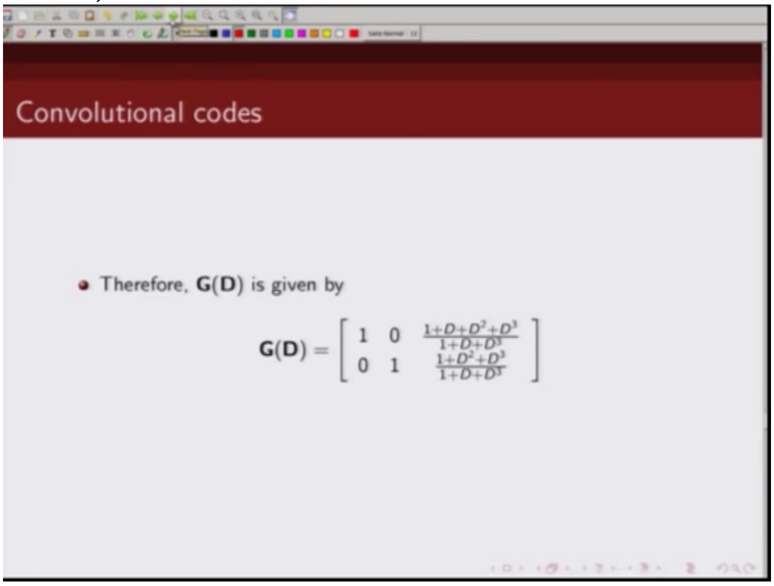

you can think of it as like this, so we have 3

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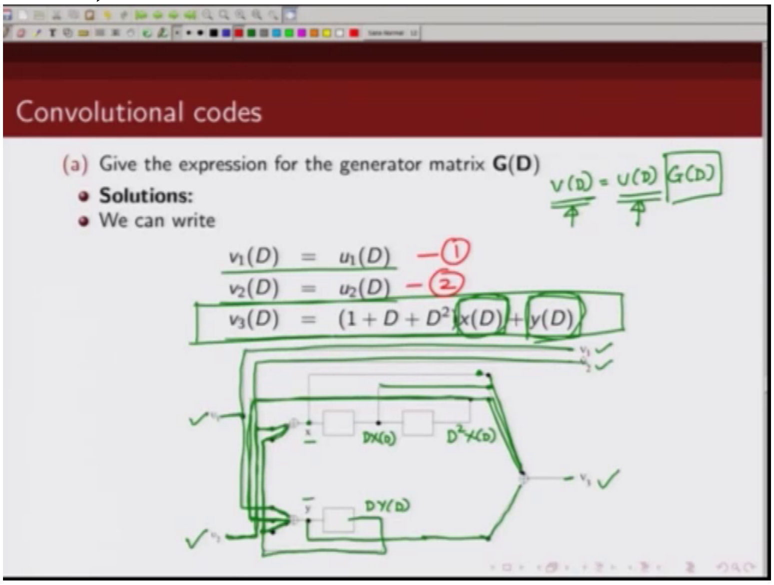

output, v 1, v 2, v 3, 2 input u 1, u 2, so we are writing

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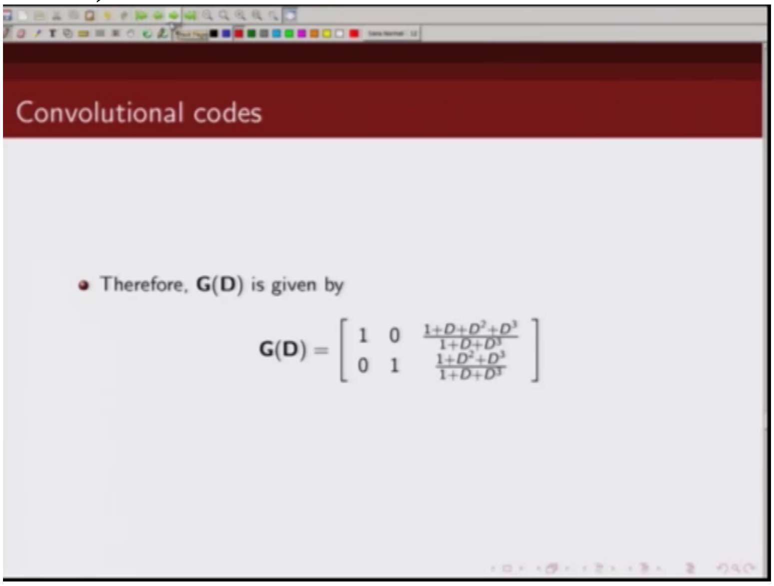

v 1 D, v 2 D, v 3 D in terms of u 1 D, u 2 D and this G matrix.

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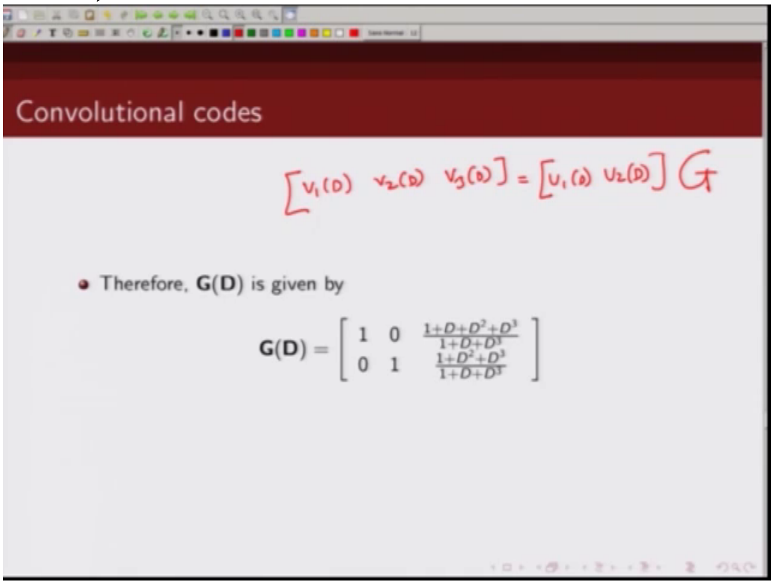

So what is v 1 D? v 1 D is u 1 of D. So then our G matrix here, again G matrix is 2 cross 3, so v 1 D is u 1 of D, so we get 1 0. v 2 of D is u 2 D, so we get 0 1, and what is v 3 of D, v 3 of D is this, this times

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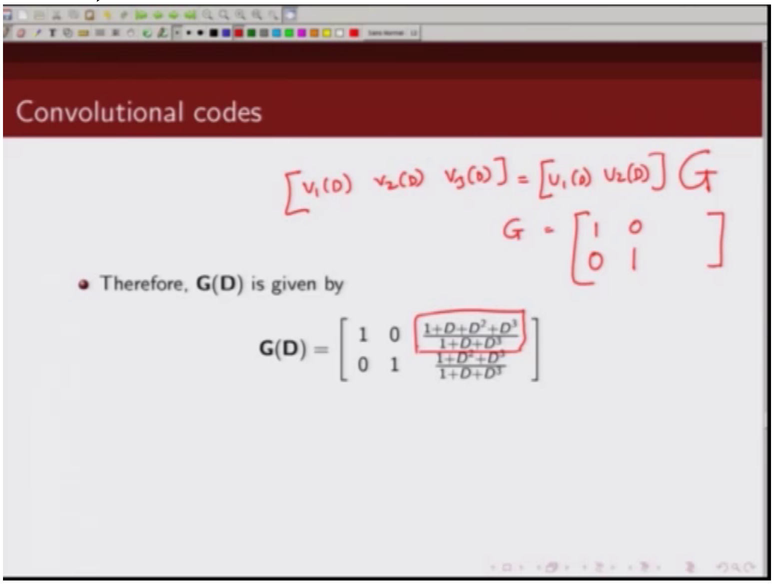

u 1 of D and this times u 2 of D.

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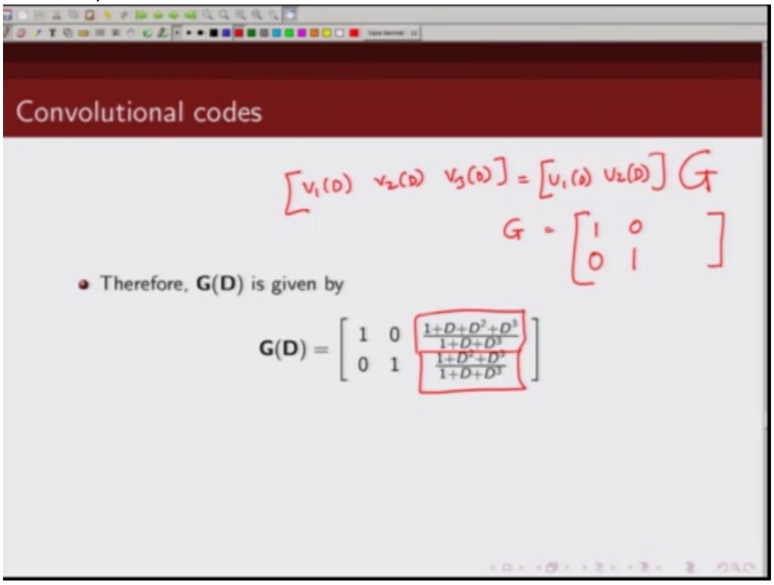

$$G(D) = \begin{bmatrix} 1 & 0 & \frac{1+D+D^2+D^3}{1+D+D^3} \\ 0 & 1 & \frac{1+D^2+D^3}{1+D+D^3} \end{bmatrix}$$

So this will be our final generator matrix corresponding to the encoder that is shown in this

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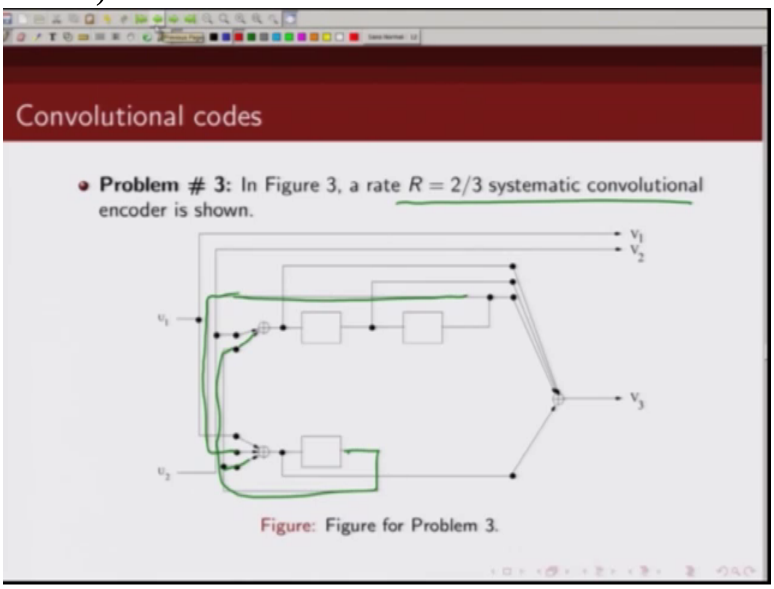

figure, Ok. Now

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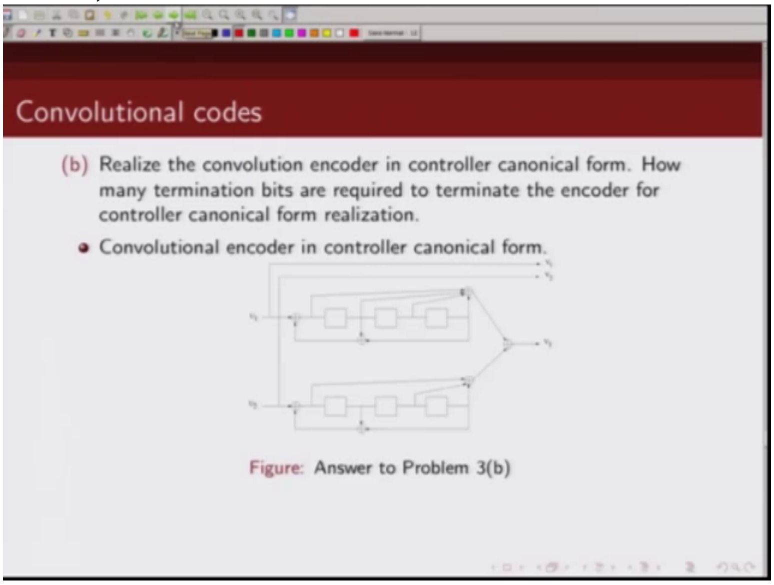

the next question is can we realize this encoder in the controller canonical form? So the answer is yes, we can realize it.

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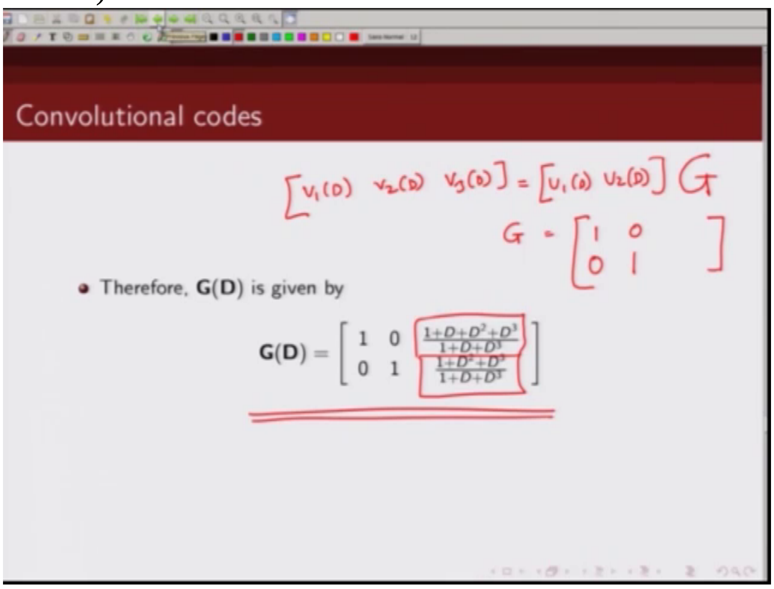

We can have the expression for generator matrix. So to realize it in controller

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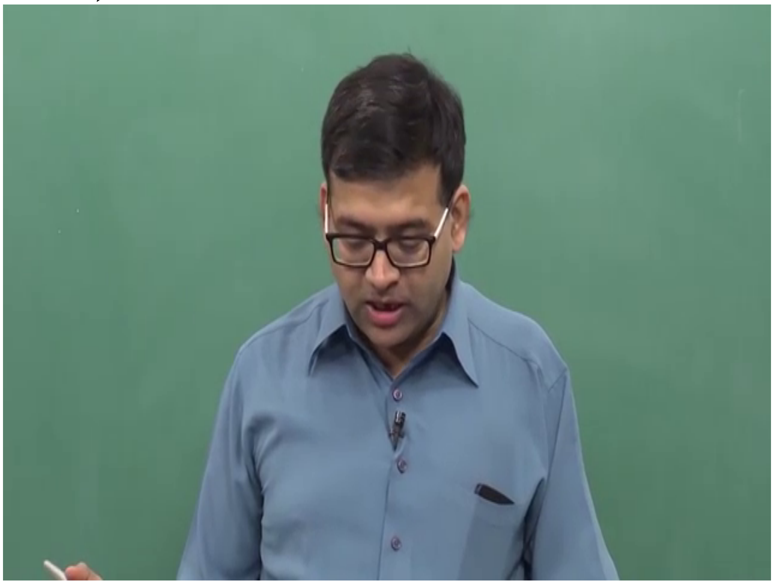

canonical form again, so there is one

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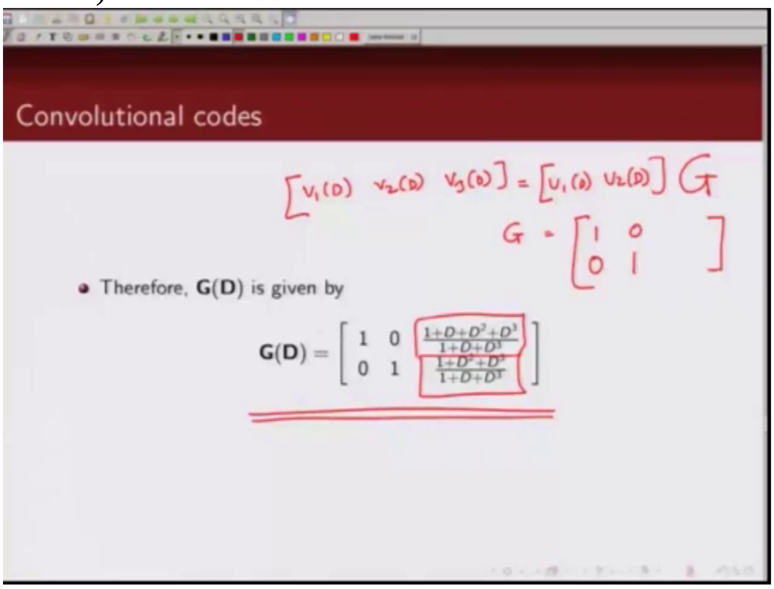

set of shift registers for each input, so this is one input, this is second input right. Please note this is a feedback polynomial so we would require a feedback polynomial and now maximum degree here is 3, maximum degree here also is 3,so

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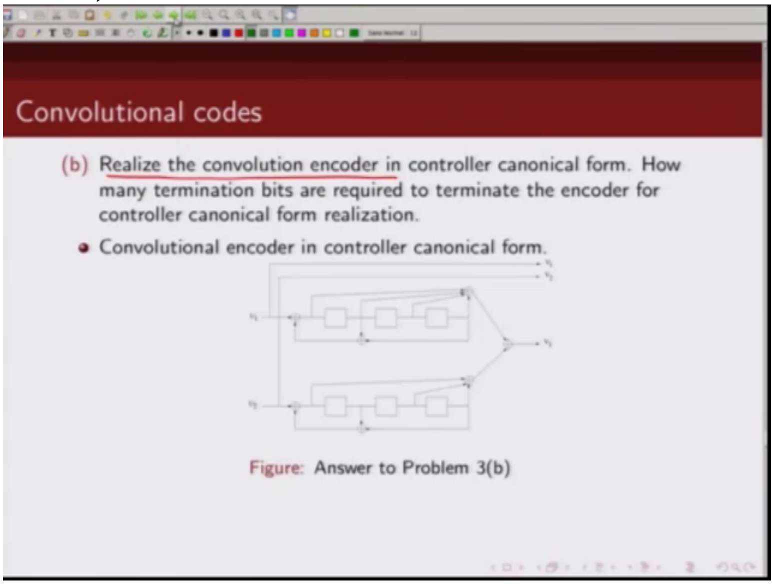

we will require 2 set of shift registers, first one is this one. Please note this as

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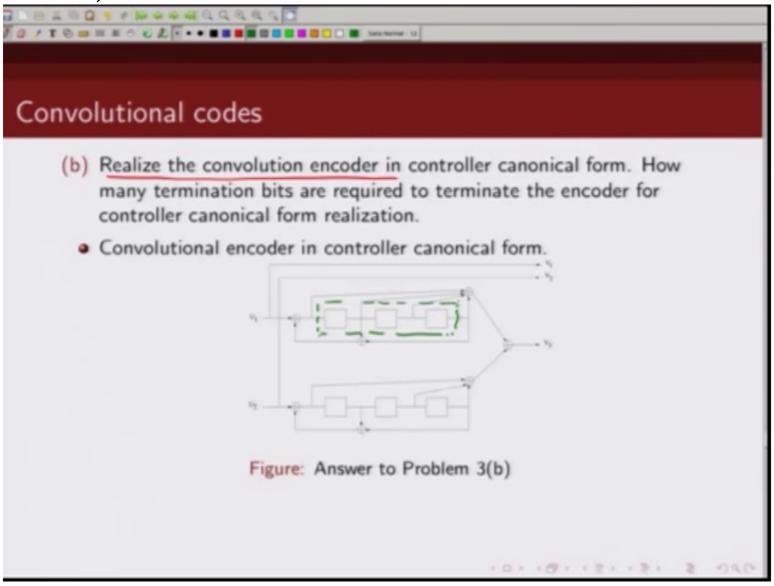

3 memory elements and similarly second shift register, this also has 3 memory elements.

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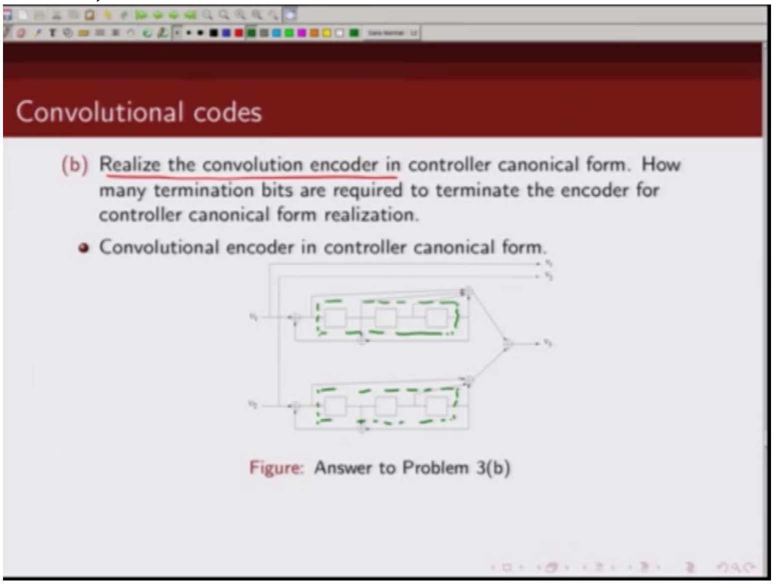

That is because the maximum degree

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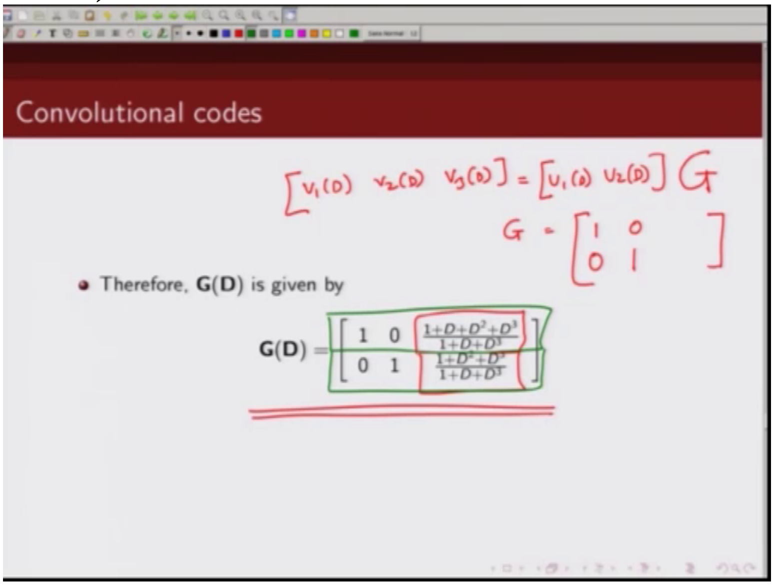

of this rational function is 3 and similarly maximum degree of this rational function is 3. And we just implement this.

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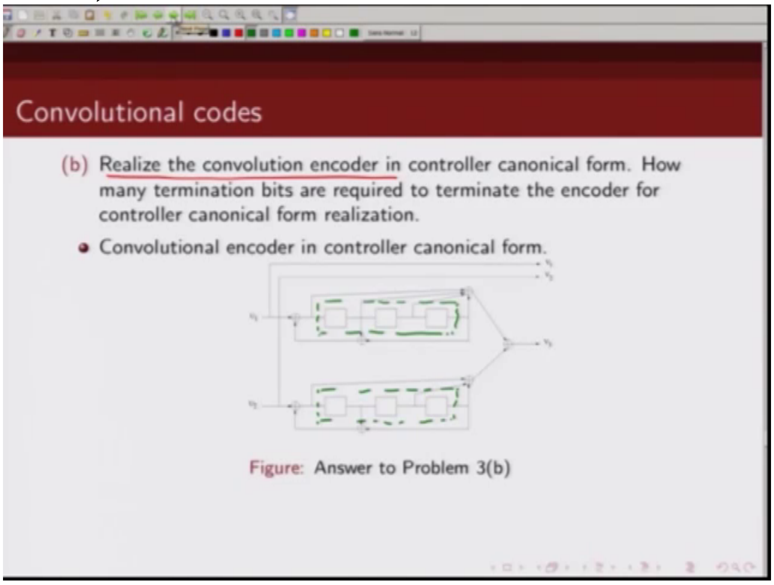

So v 1, v 1 D is just u 1 D, so that's just this.

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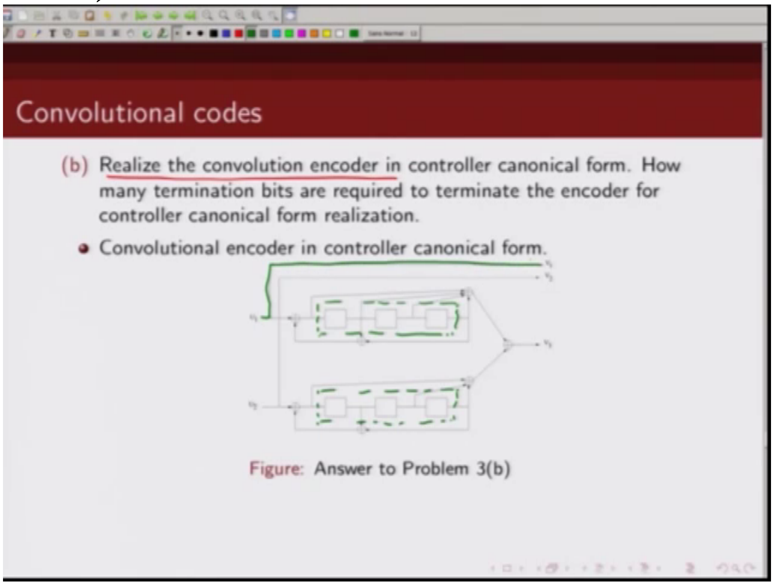

v 2 D is u 2 D, that's just this.

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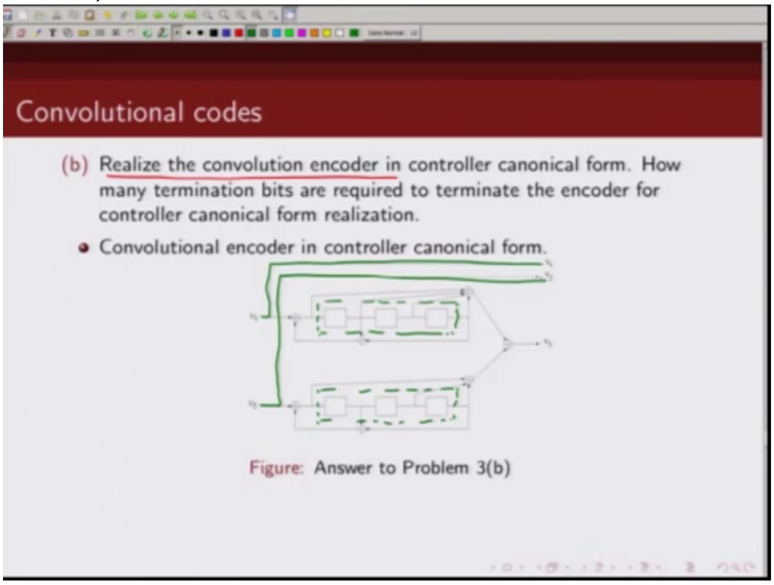

What is

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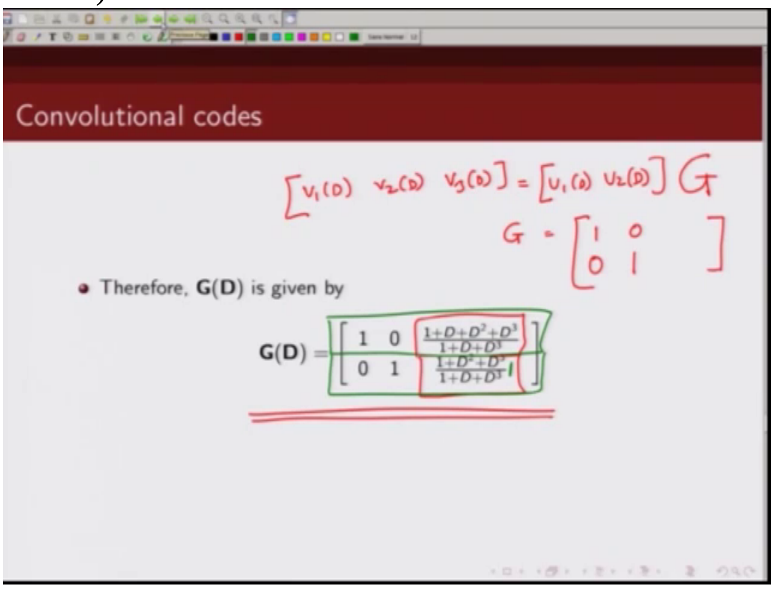

v 3 D? v 3 D is 1 plus D plus D square plus D cube divided by 1 plus D plus D 3 u 1 D, plus 1 plus D square plus D cube 1 plus D plus D cube u 2 D,

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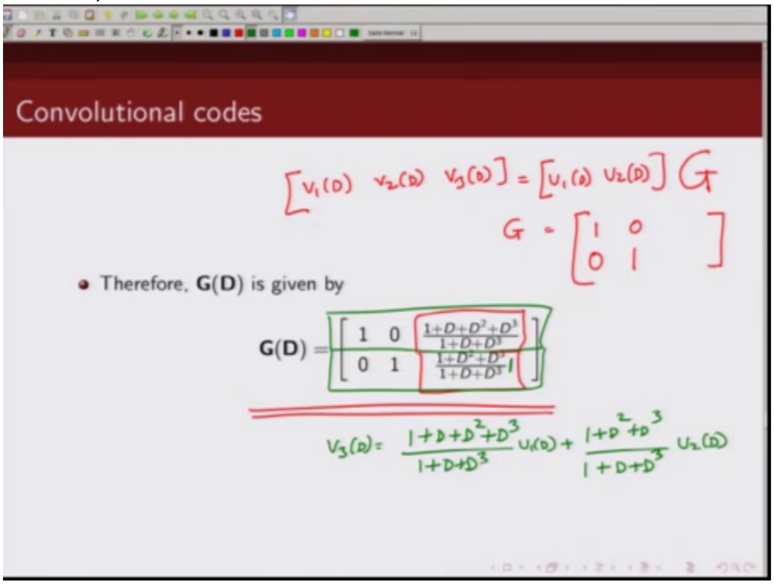

right? So relationship between v 3 and u 1 D is given by this. So let's

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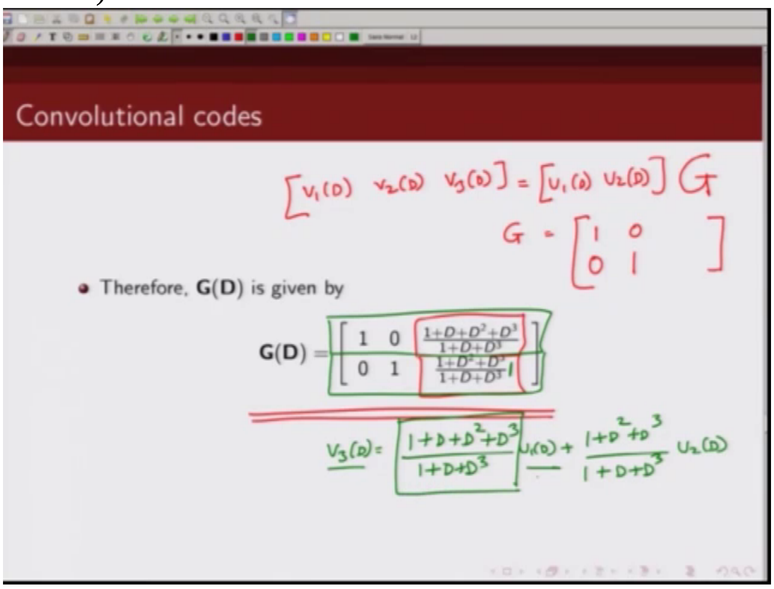

implement this. So numerator has 1 plus D plus D square plus D cube. So you can see here,

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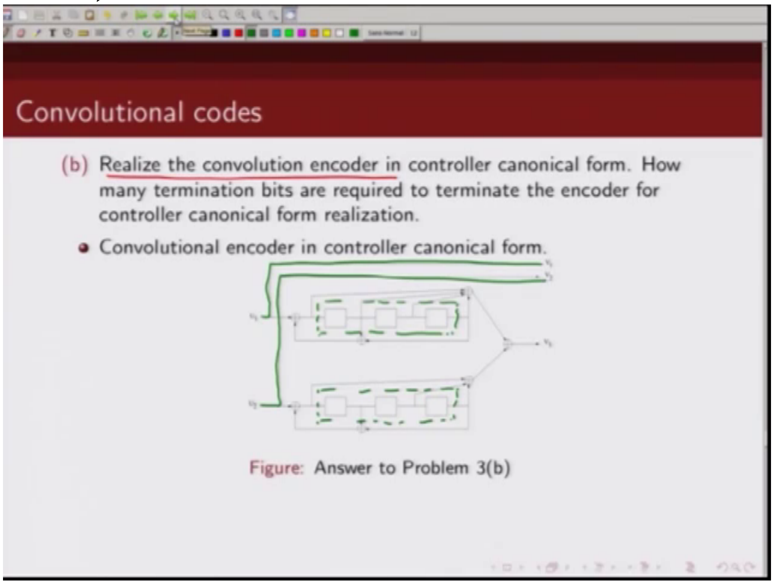

this is my 1, this is my D, this is my D square, this is my D cube.

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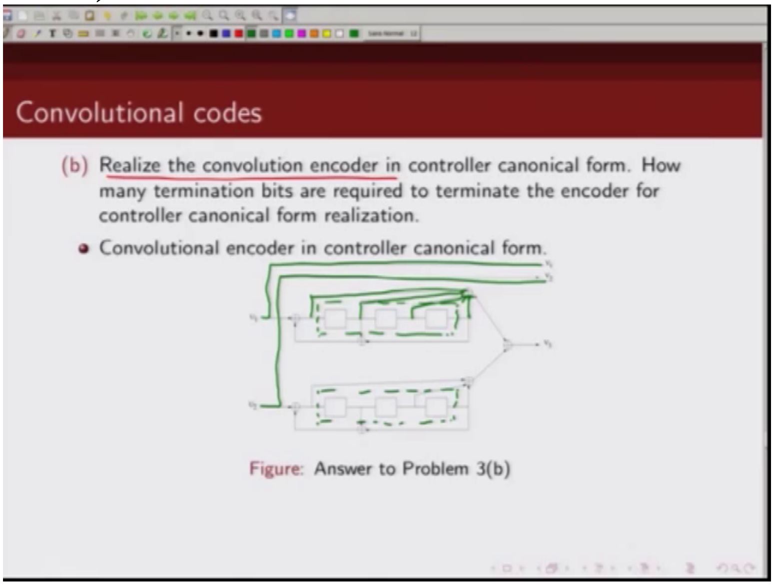

And similarly the denominator has 1

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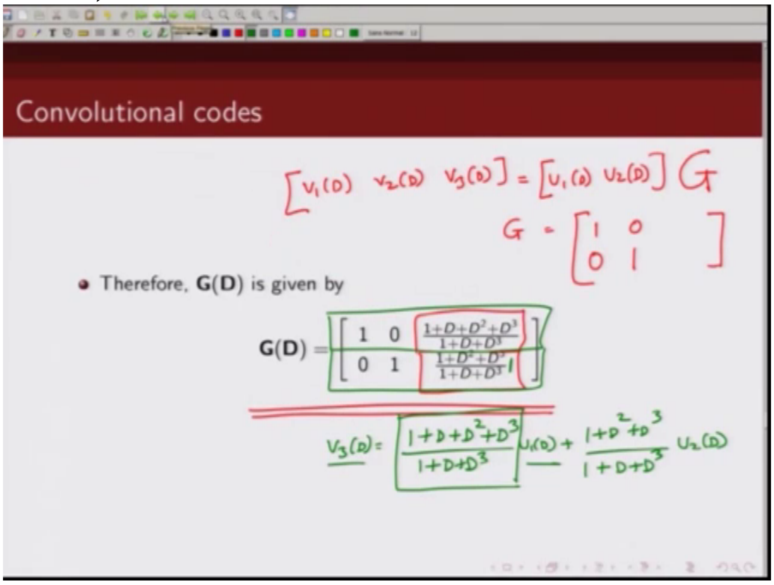

plus D plus D cube. So

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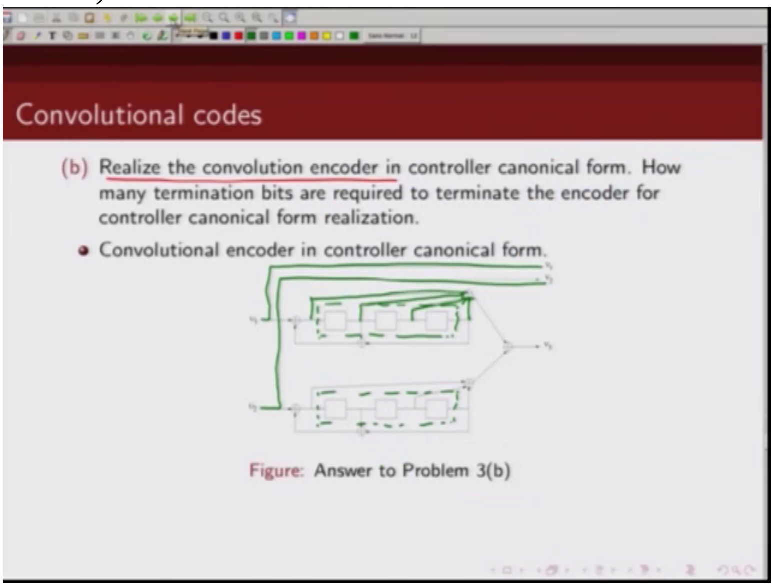

the denominator, this is the 1 term, this is the D term; this is the D cube term. So this part is implemented.

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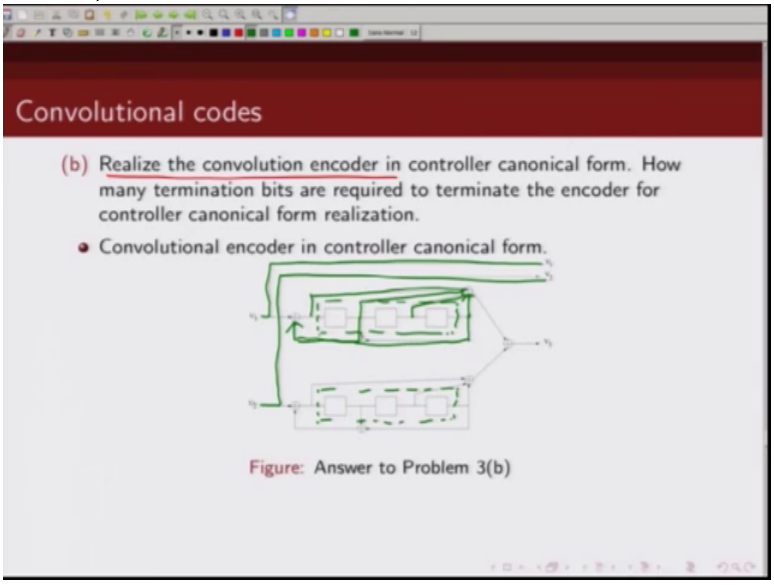

Next is this. Following the same procedure

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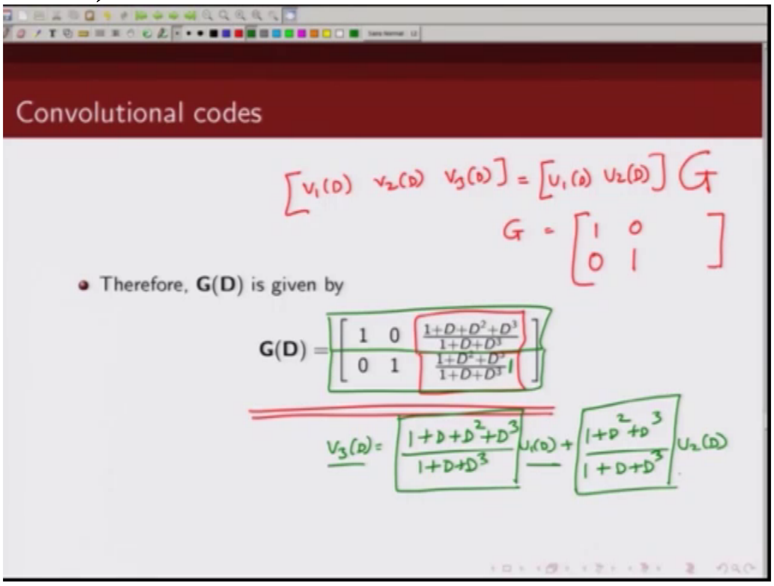

we can find out the mapping between u 2 D and v 3 D. The feed forward connections are 1, D square and D cube. So then this is

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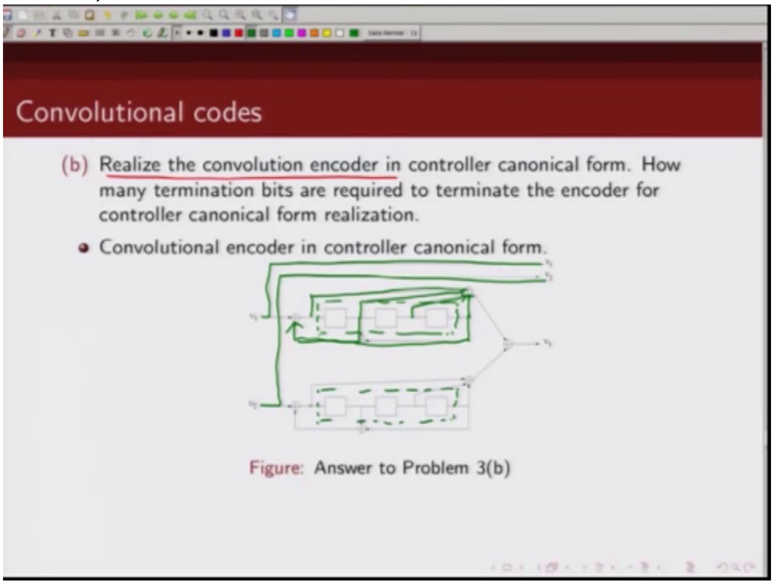

1, this is D no connection,D square is this and D cube is this.

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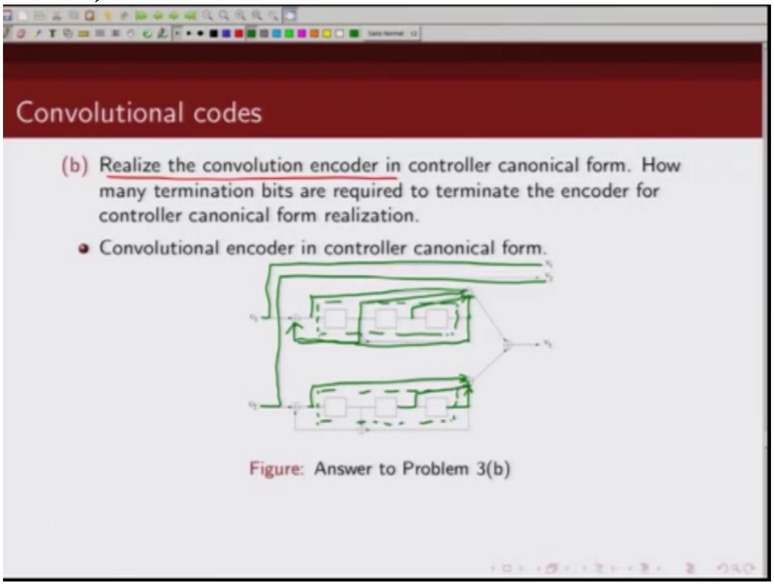

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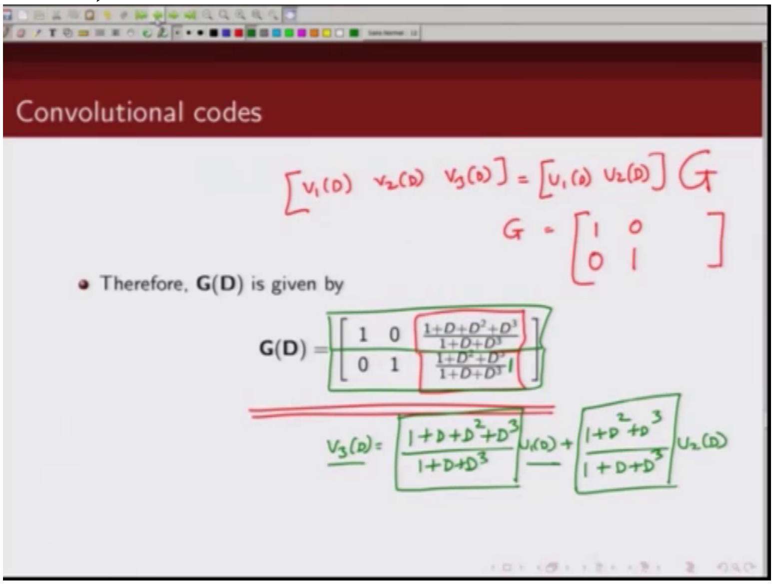

Similarly the feedback connections are 1, D and D cube. So

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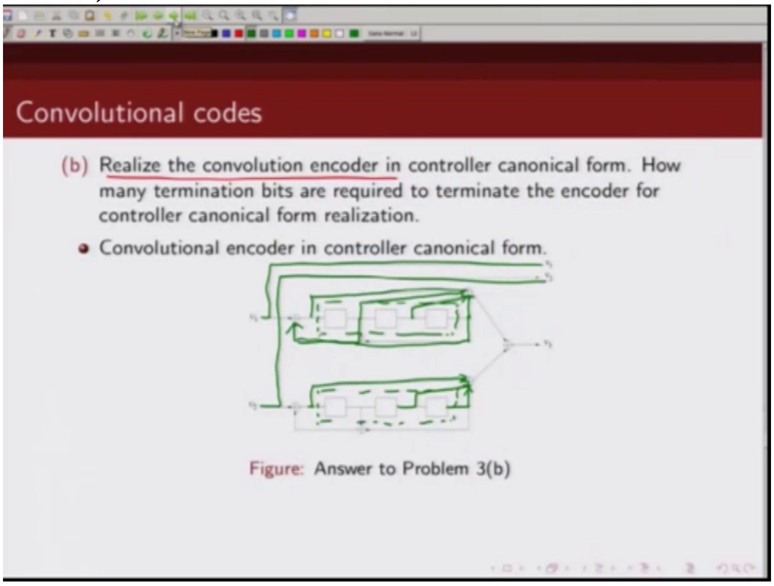

the feedback connections; this is the 1, this is D and this is D cube. And

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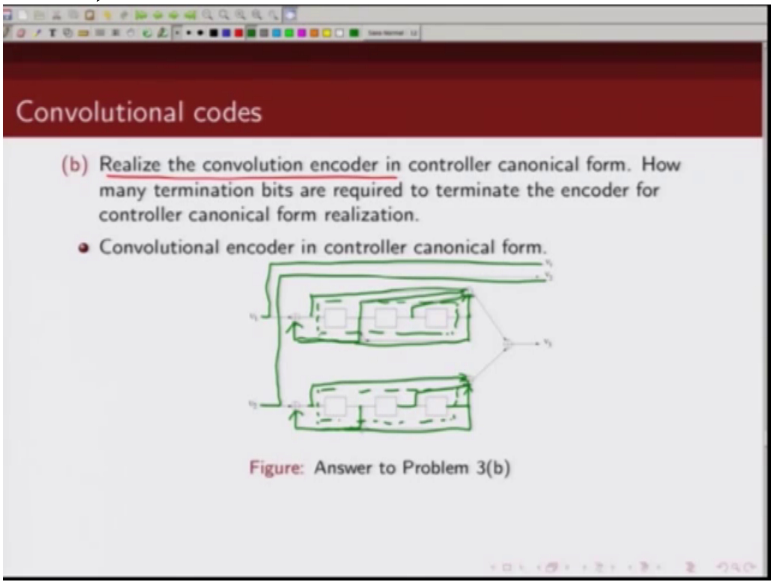

v 3 is the combination of these 2. So this is my v 3.

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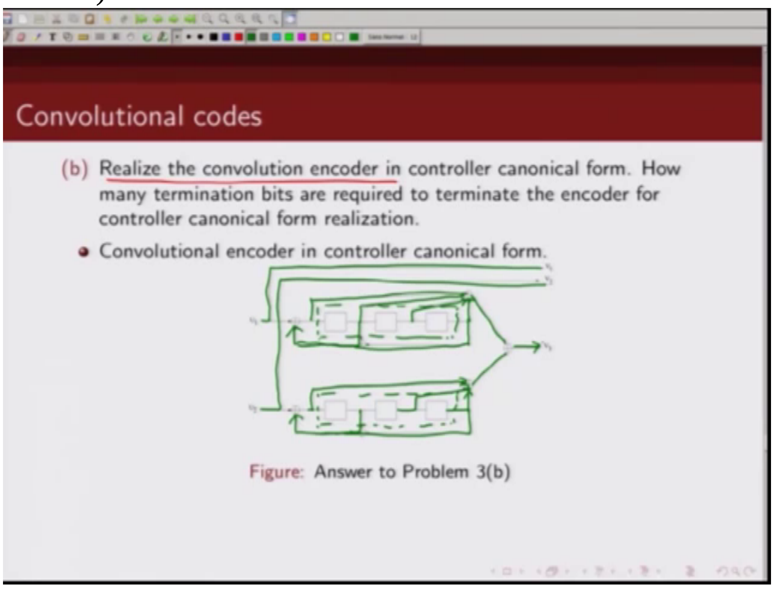

So I hope this is clear how we can realize

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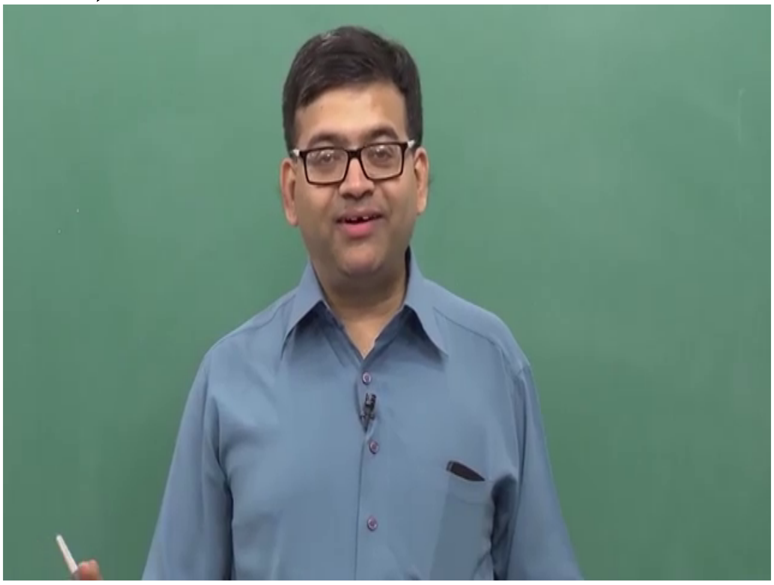

this encoder using controller canonical form realization.

Now the next

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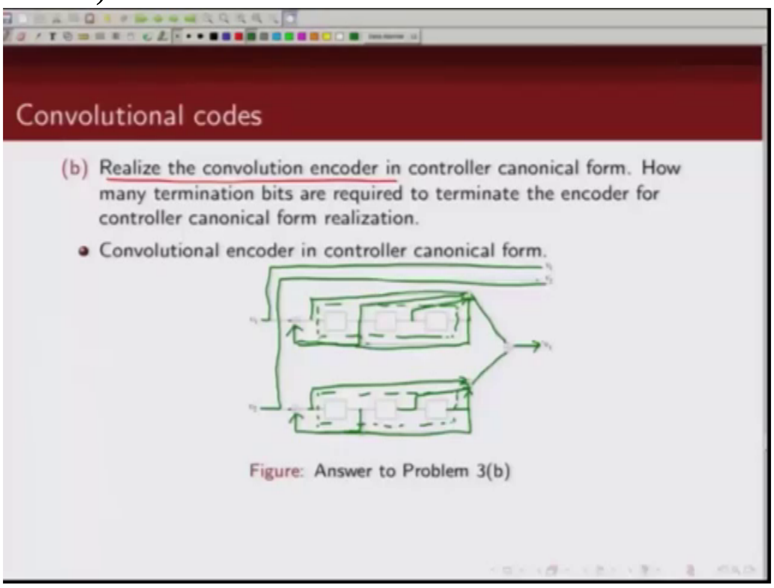

question is how many termination bits are required to bring this encoder back to all zero state? Now what does termination means? Termination means

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we are bringing this encoder back to all zero state. So no matter what the state is, if you want to bring them back

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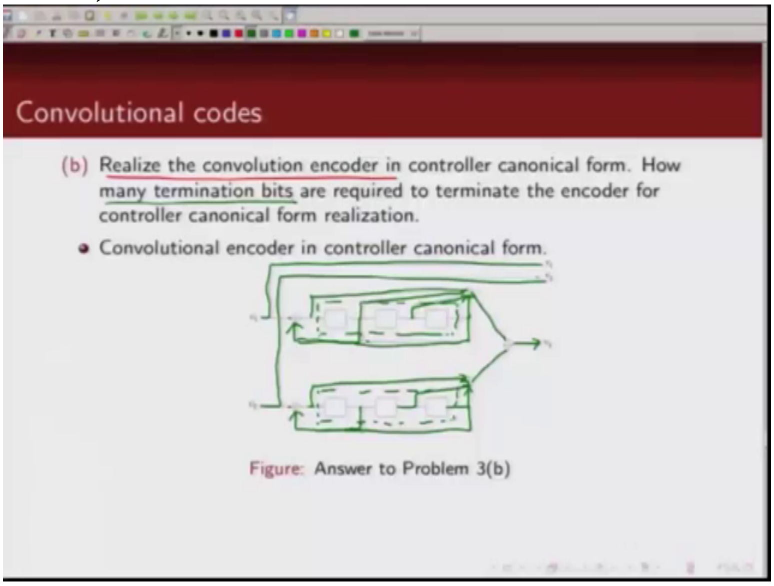

to the all zero state, the number of termination bits required is equal to how many memory elements we have. So in the controller canonical form realization to bring this shift register, the first shift register you want to bring it to all zero state, we would require 3 bits because we have 3 memory elements here, 1, 2 and 3. Similarly

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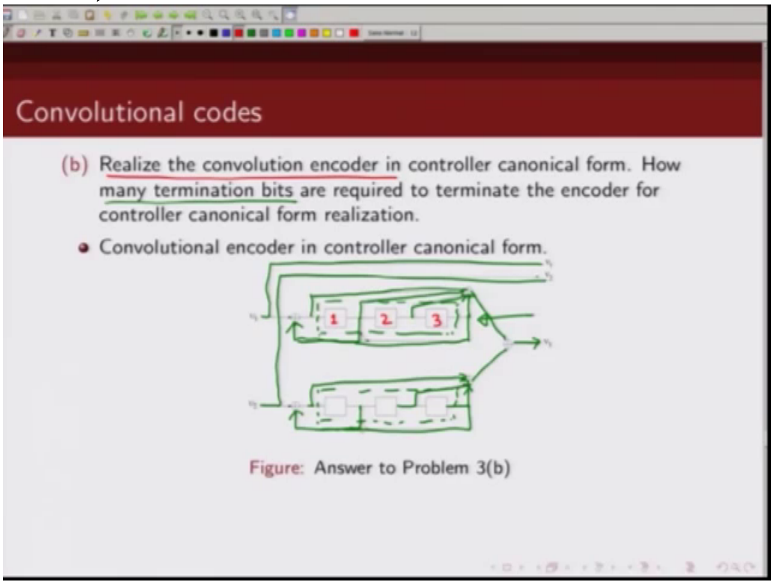

for this shift register we require additional 3 bits, so 4, 5, 6 so

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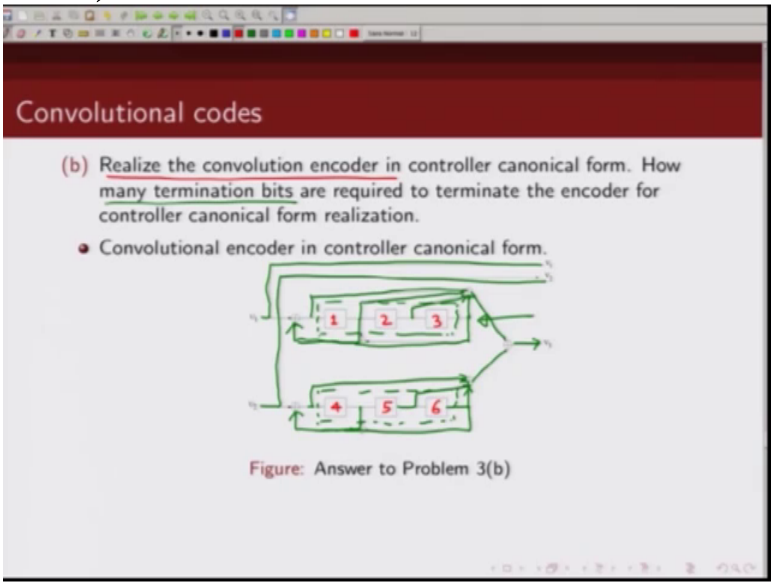

total we require 6 termination bits, 3 to terminate this encoder and 3 to terminate this encoder.

So we require

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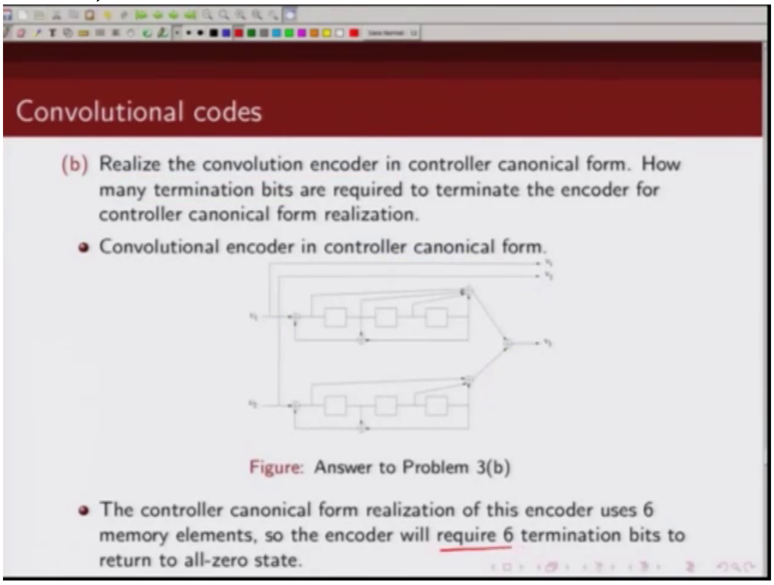

6 termination bits.

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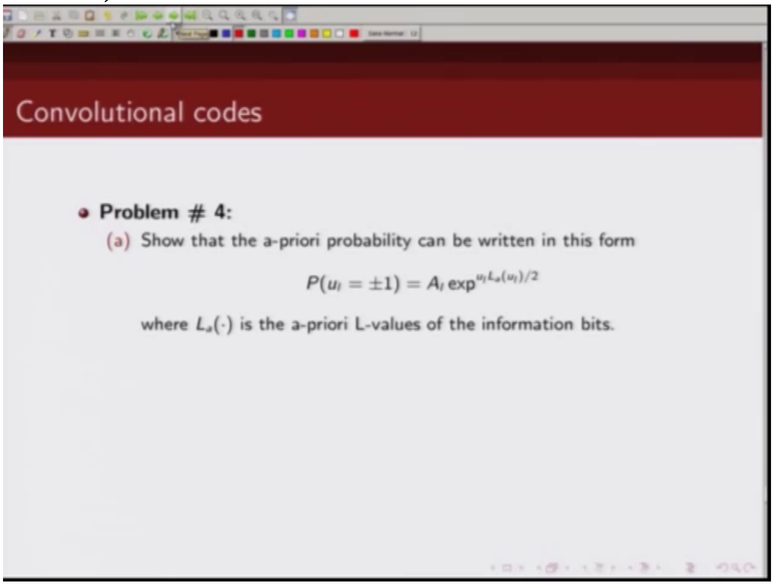

Finally let's come to the B C J R algorithm that we talked about. So the first question is can you write

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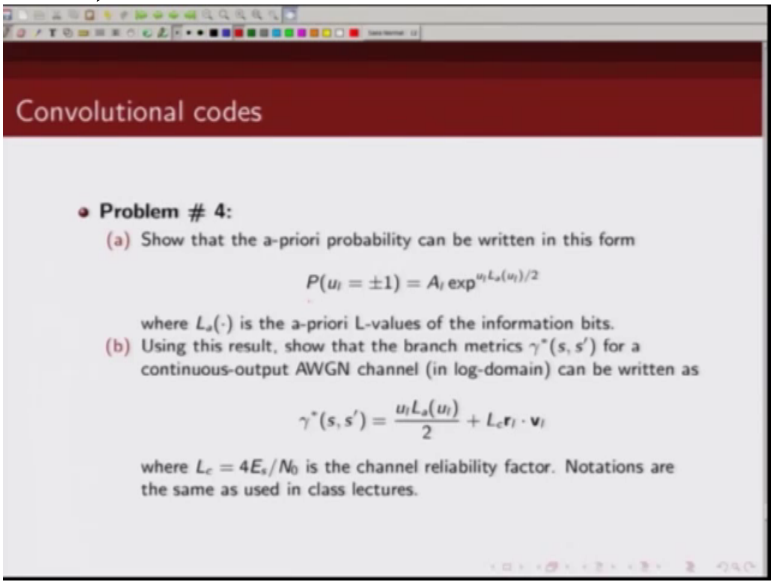

the a priori probability in this particular form, and also the branch metric in log domain, can it be written in this particular form? Now u l is my input, L a is the a p p value for the a priori inputs, L c is the reliability factor which is given by 4 times E s by N naught,

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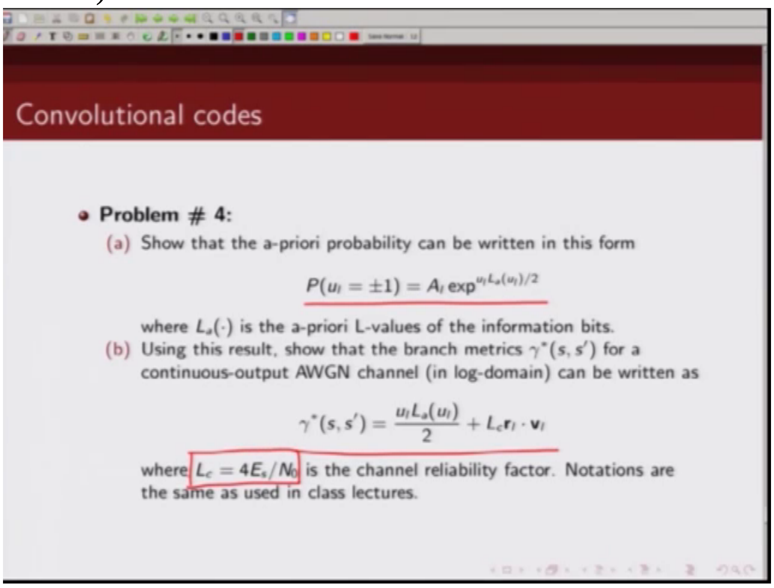

other notations are same as which are used in the lecture. v is codeword, r is received sequence so can we write these in terms like this. So let's look at it.

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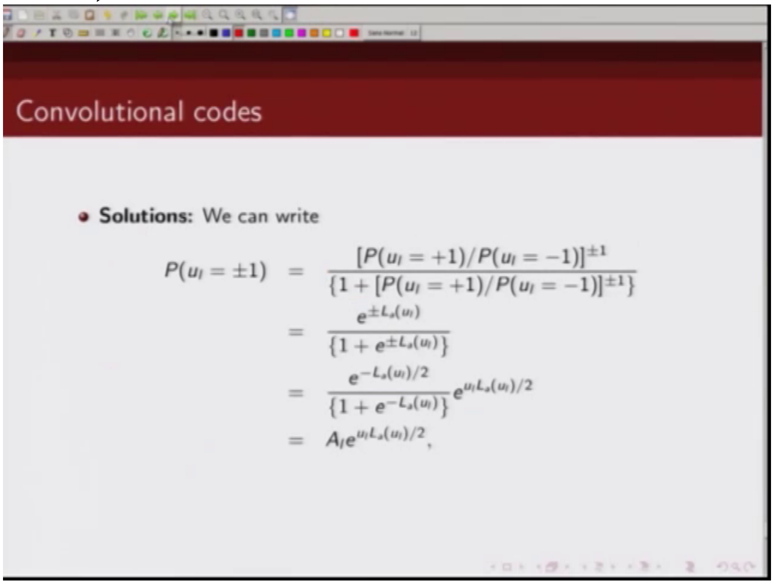

So what's the probability of u being plus 1 or minus 1? Let's take like plus 1, let's say what's the prob, what's the probability that u l is plus 1? Now this can be written as this by 1.

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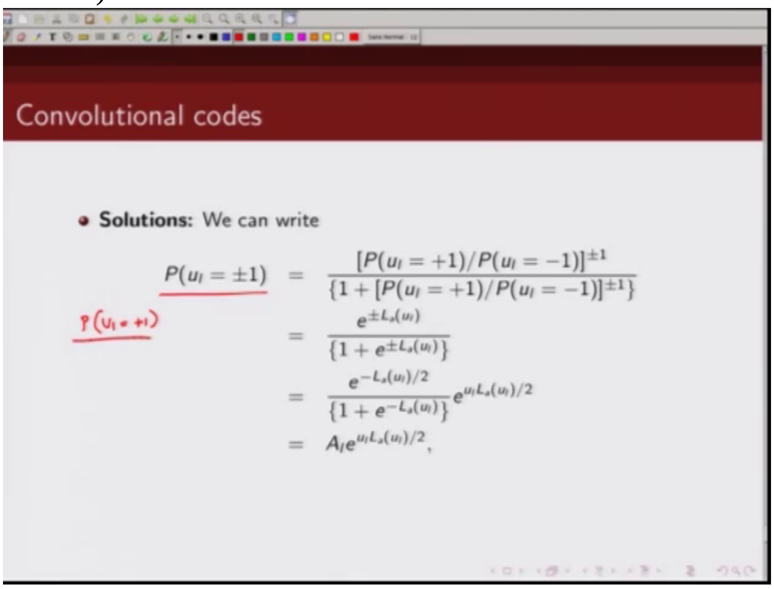

So I can write as probability of u l being plus 1 divided by probability u l is plus 1 plus probability u l is minus 1, I can write it

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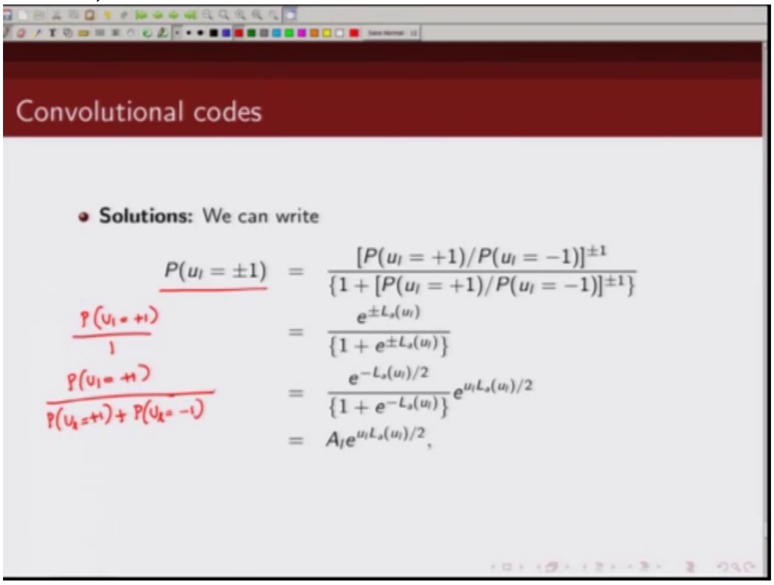

this way, right? And if I divide by probability of u l being minus 1, then what I get is probability being plus 1 by probability of u l being minus 1, 1 plus probability of u l being plus 1, probability of u l minus 1.

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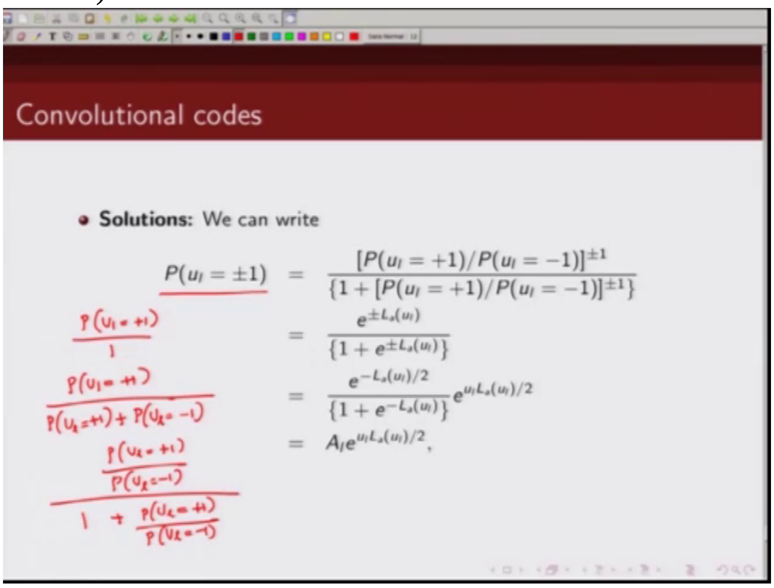

So this is what I will get of the form here. You can see here, the form for, when u l is plus 1, I get probability of u l being plus 1, I get in this particular form. Now let's look at what's the probability that u l is minus 1. Again I can follow the same procedure. I can write this same as this by 1 or I can write the probability of u l is equal to minus 1 divided by probability of u l being plus 1 plus probability of u l being minus 1 and

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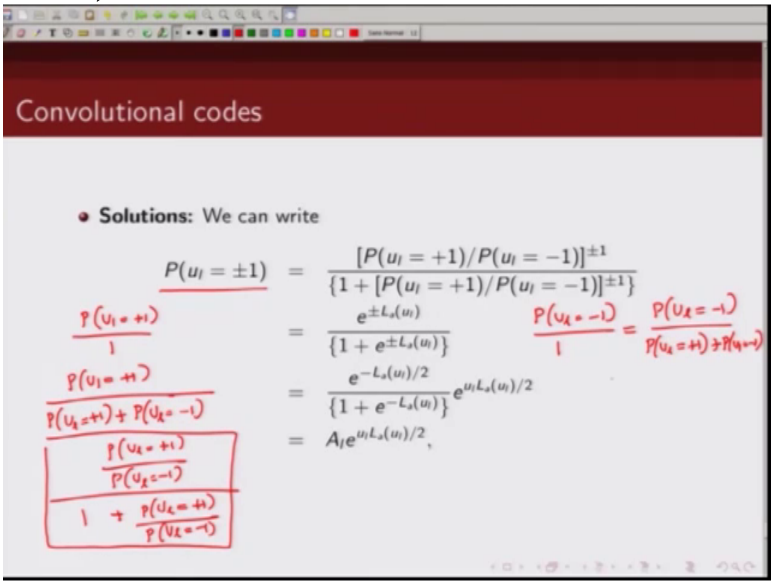

I can divide this by probability of u l being 1, so this will be probability of u l equal to minus 1 by probability of u l being plus 1, 1 plus probability of u l being minus 1, probability of u l being plus 1, right?

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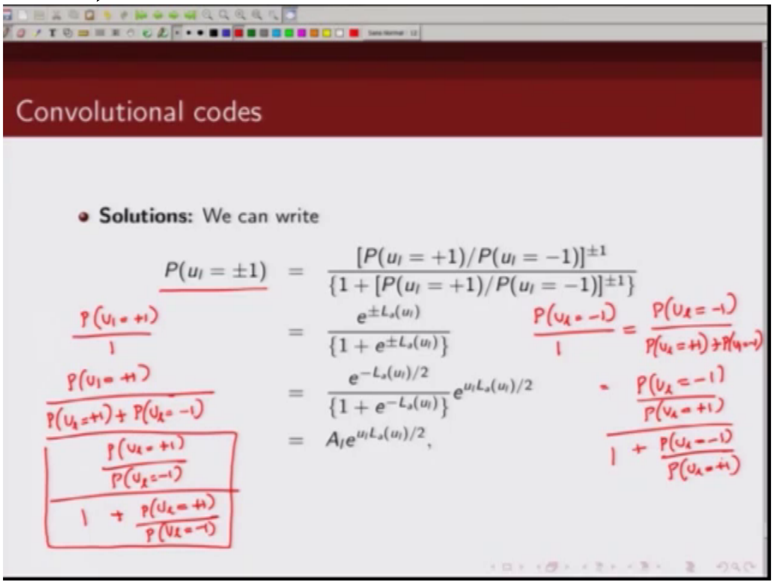

So this I can also write as this is equal to probability of u l being plus 1 by probability of u l being minus 1 raised to power minus 1 and this is 1 plus the same thing, raised to power minus 1. So if I combine this and this what I get

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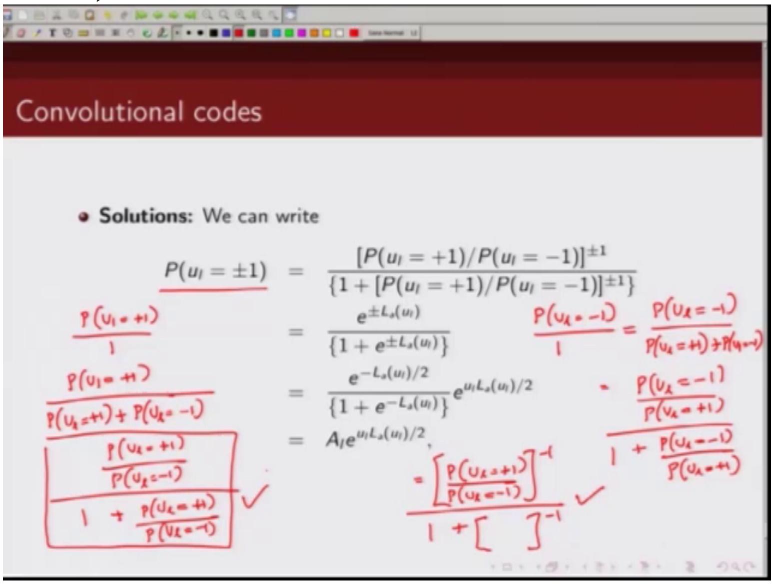

is the first step here, Ok.

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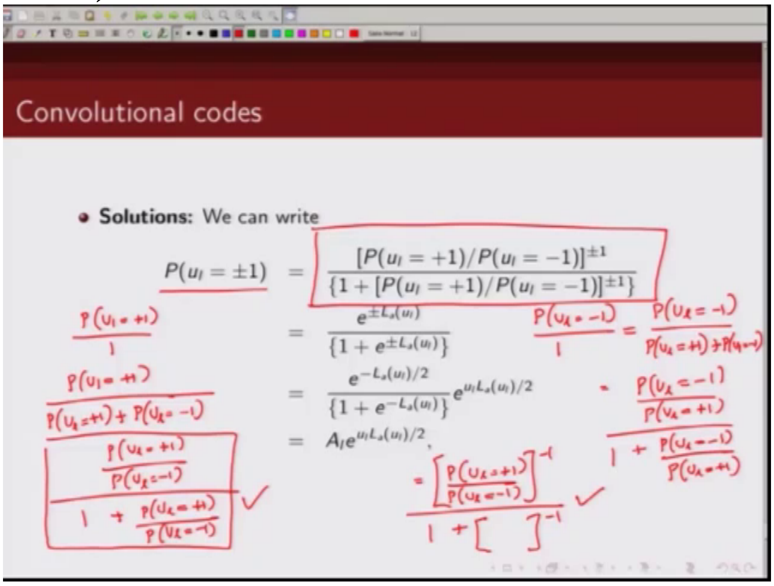

I can write by combining this and this, I will get this. Now if I write these ratios of probabilities in terms of L values, so what is this L value of u l, this is log of probability of u l being plus 1 by probability of u l being minus 1. So this can be then written as

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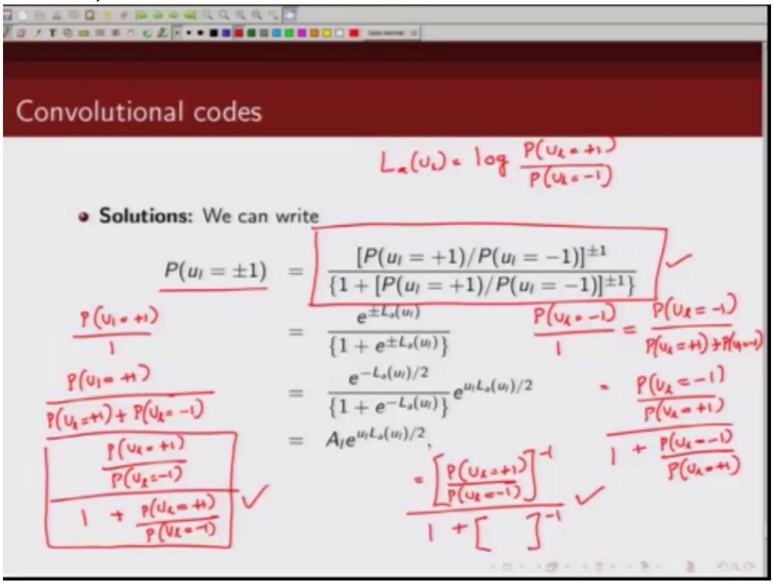

e raised to power L a u l. If I do that, if I plug this in

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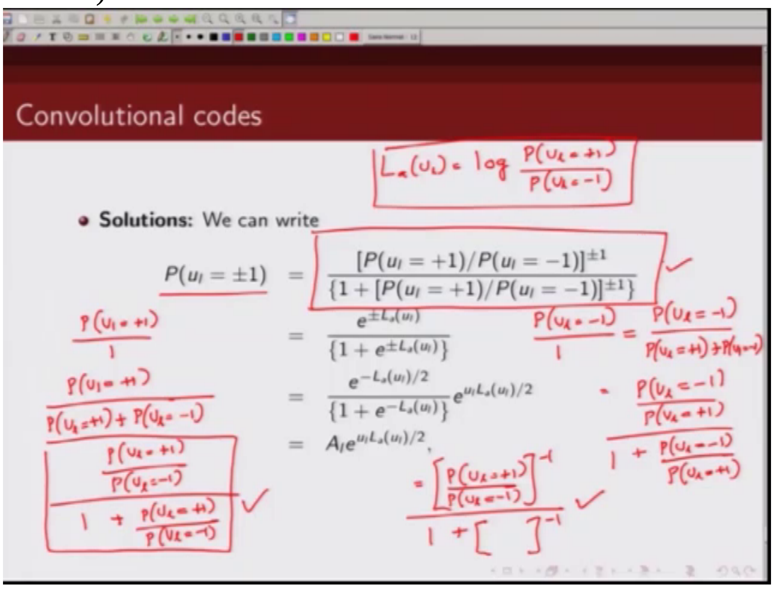

first line what I get here is this term, Ok.

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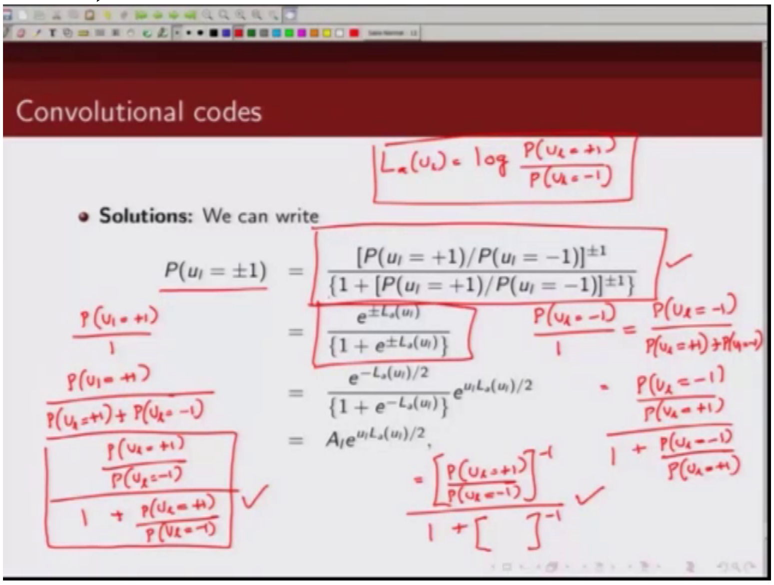

Now note I can further simplify this into this expression.

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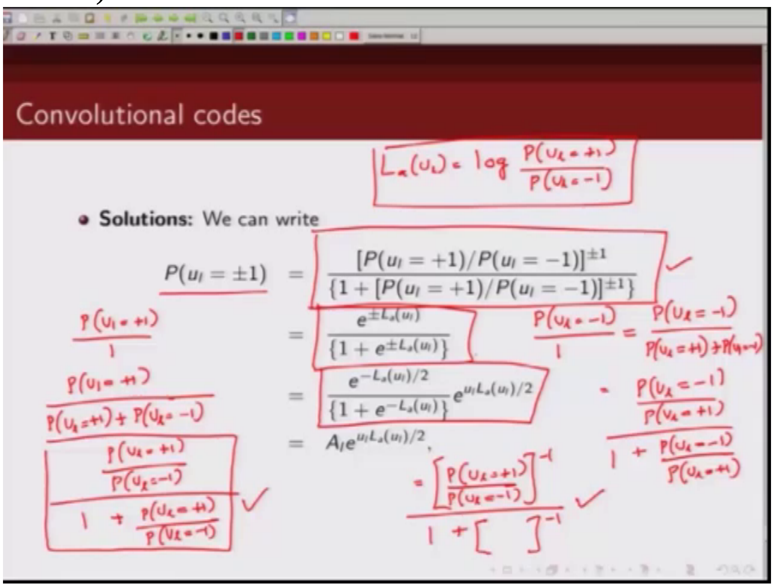

You can see when u l is plus 1, when u l is plus 1, what do we get? When u l is plus 1,

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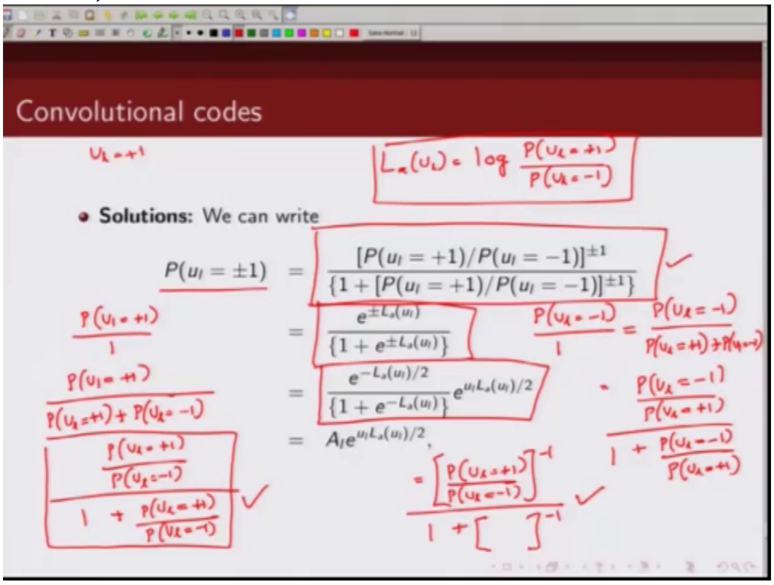

this is e raised to power L a u l by 2 and e raised to power minus L a by 2. So this will be basically 1 so this will be 1 times 1 plus e raised to power minus L a u which can be

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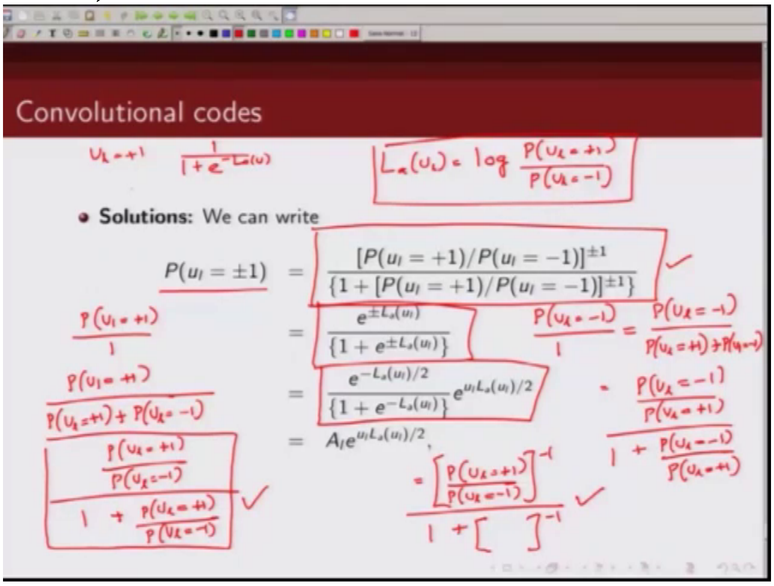

written as e L a u 1 plus e L a u. This is

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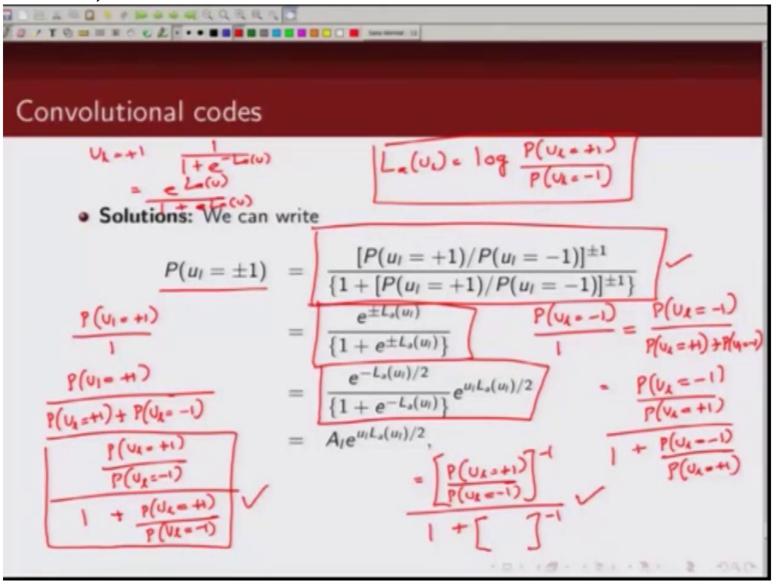

precisely what I have written here. And if u l is minus 1, this will be e raised to power minus L a u by 2 and e raised to power L a u by 2. So this term will become, in that case, e raised to power minus L a u 1 plus e minus La u l. So this term can be written in terms of this, right? And what is this term? What is this term? This I can simplify this term, let's make some space. I can simplify this term as e raised to power minus L a u l by 2 and I have e raised to power minus L a u l by 2, this is e raised to power L a u l by 2 plus e raised to power minus L a u l by 2.

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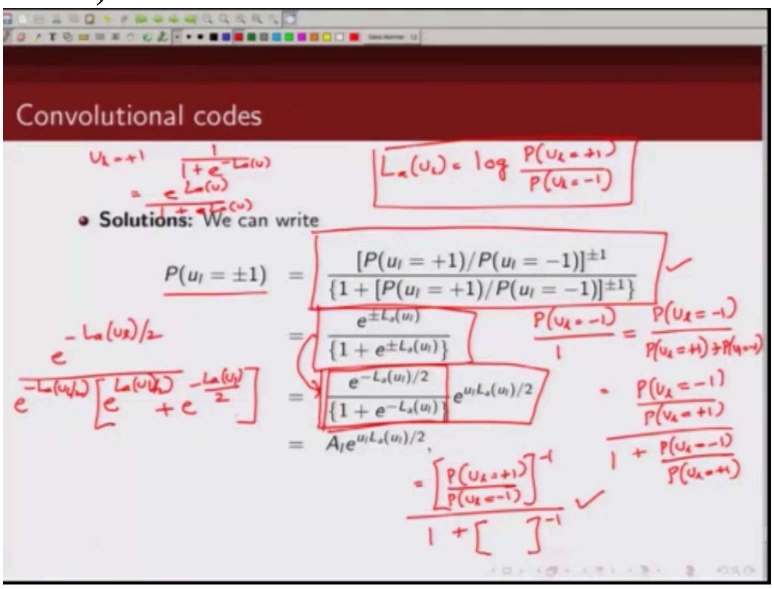

So what I am doing here is I am writing this particular term. So this I can write as this and this. So this cancels out.

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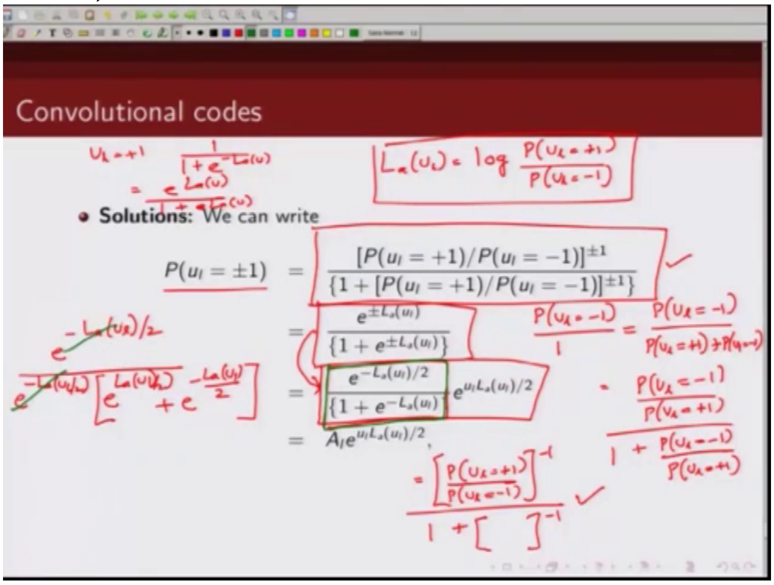

And this is e raised to power x plus e raised to power minus x. This will be cosh of x and that's the symmetric function, so that does not depend on sign of u l whether u l is plus 1 or minus 1, it does not depend on that. I can write this in terms of this expression.

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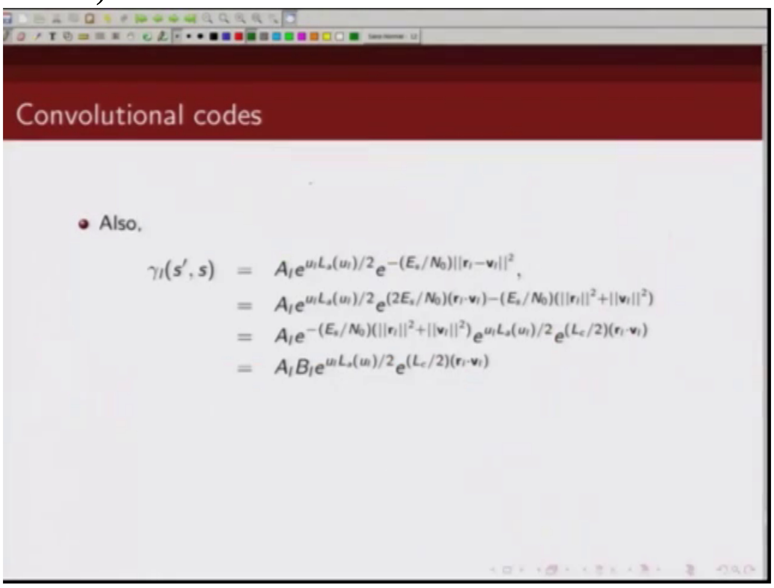

So we will use the expression that we derived in the previous slide for a priori value which was u l being plus 1 or minus 1 as a L e raised to power u l L a u l by 2. We will use this

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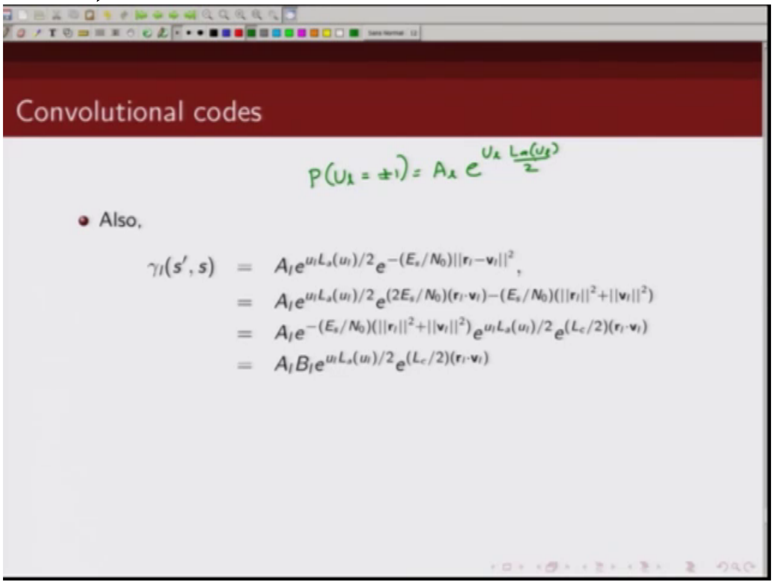

expression to simplify the expression for our branch metric for our b c j r algorithm. Now note if you recall we have written the expression for branch metrics as a priori probability u l and then we had, for a w g n channel we had this expression and

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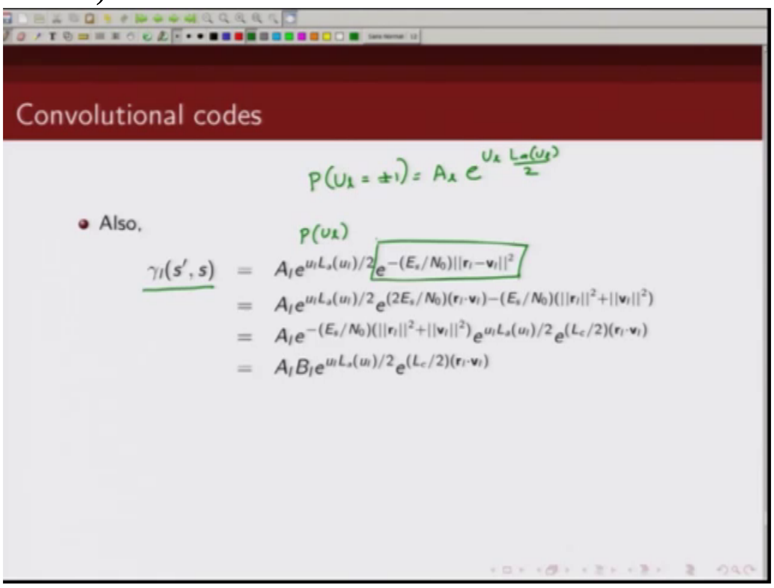

of course there was some constant factor which did not depend on u of l,

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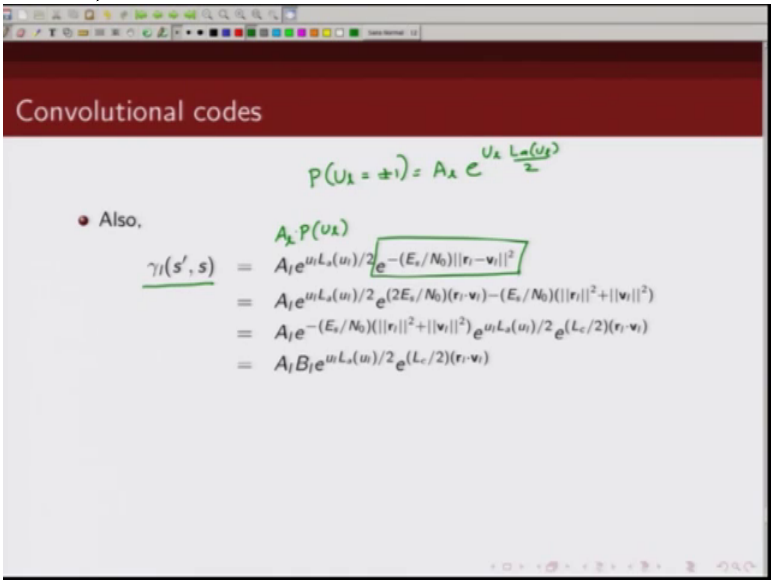

right. So what we did just now was we derived that this a priori probability can be written in this particular fashion, right. Let's further simplify the expression for branch metric. So this we can expand as r square plus v l square plus 2 times dot product or r and v l. Now this does not depend on choice of v of l. And if v of l is mapped to plus 1 and minus 1, v l square will be 1. So this also will be a constant term. So then this term will then not depend on choice of v of l. So what then will we be left with is, so this term we can just take out as some sort of constant which does not

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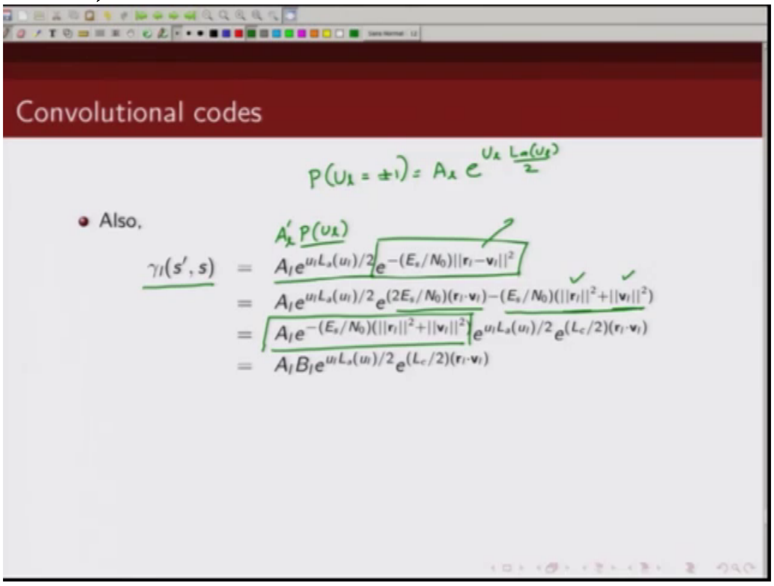

depend on choice of v of l and what will be left is this term which we are

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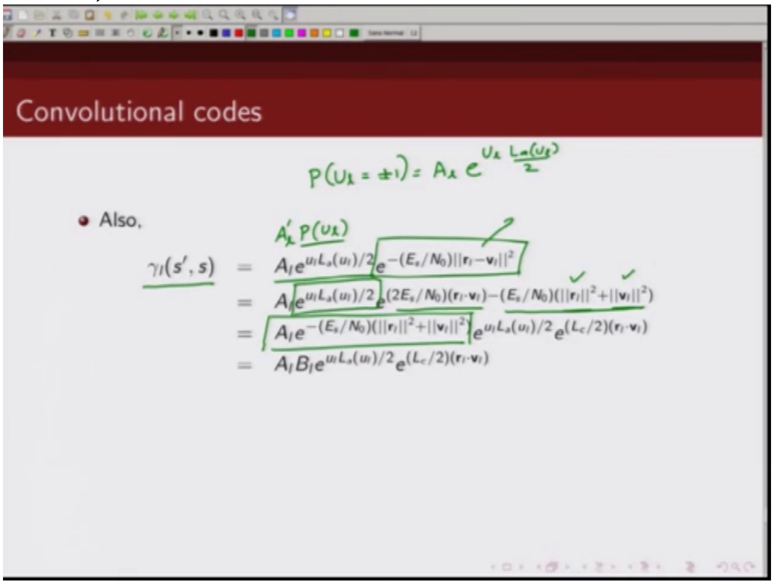

writing here, which we are writing here and

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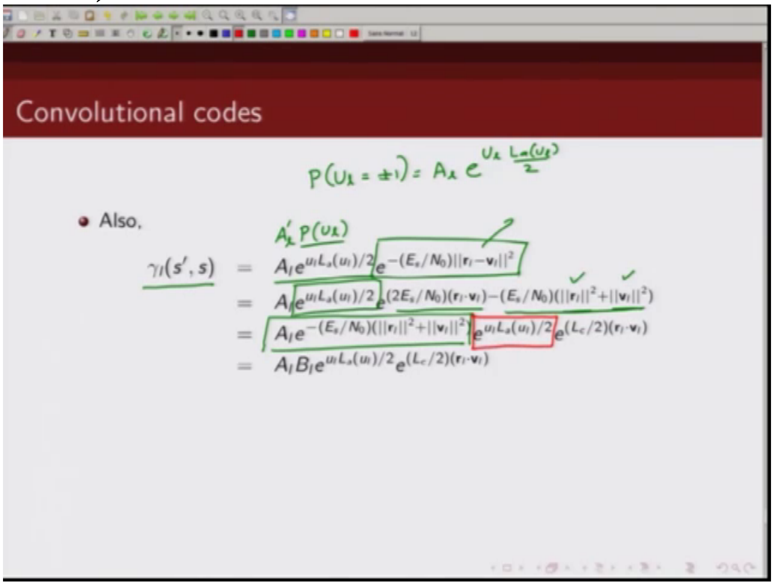

the next term that will be left is this term which we are writing

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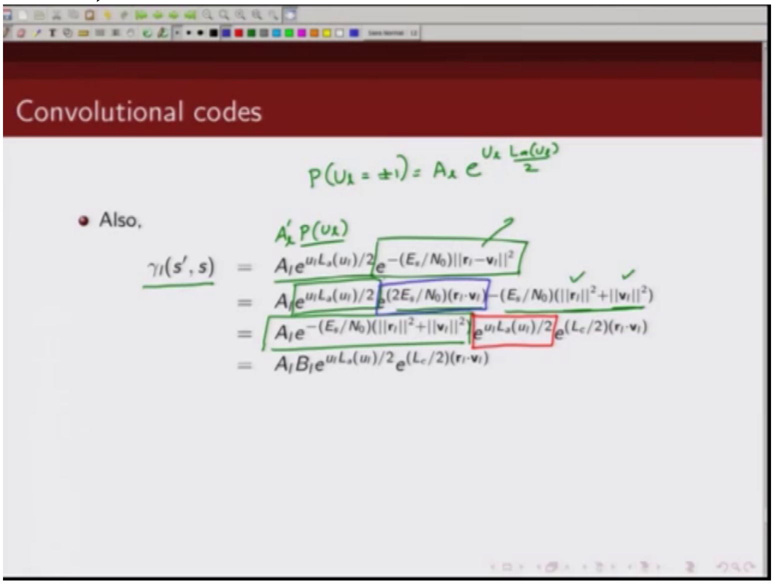

here. Please note

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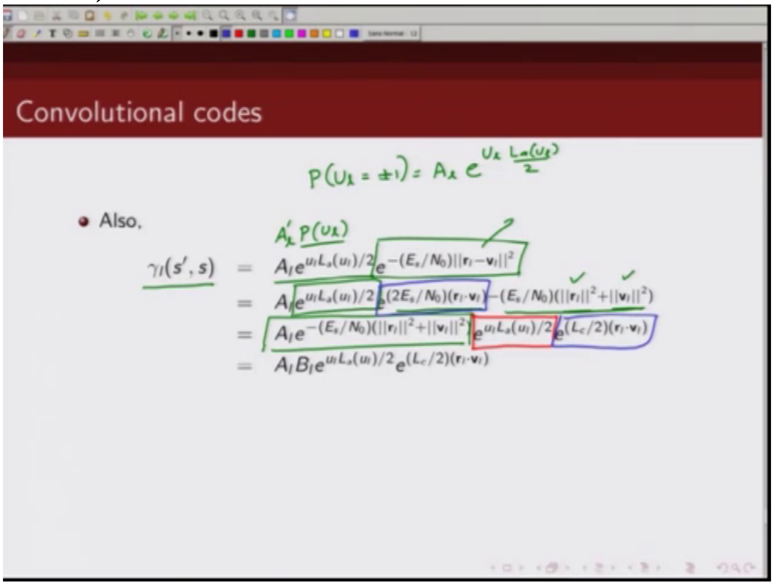

L c is 4 times E s by N naught. So that's why we are writing it as E raised to power L c by 2 and dot product

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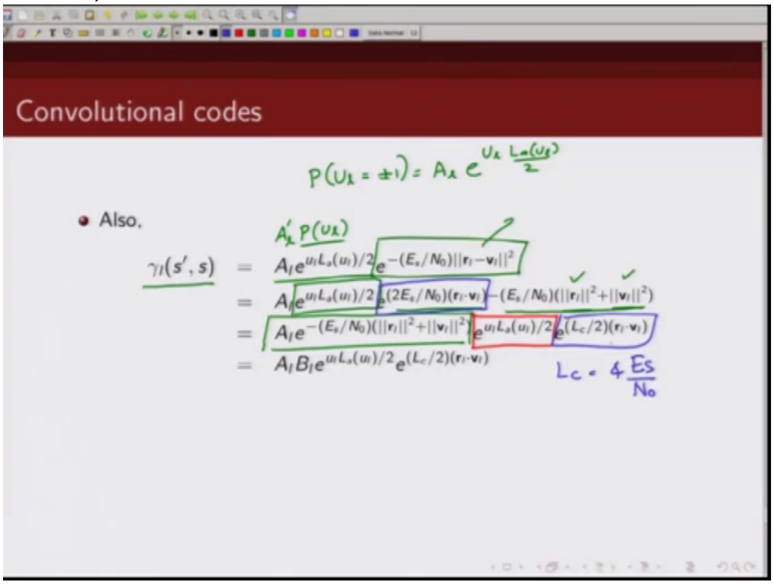

between r and v l. So then we can just simplify this expression as some constant terms multiplied by this a priori, this a priori term and this is the term which depends on received channel values.

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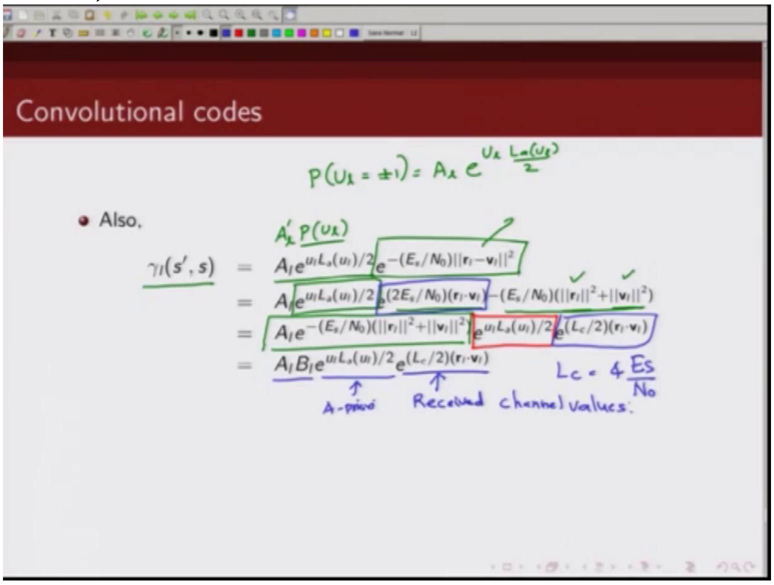

So this is the simplified expression for branch

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computation for our B C J R

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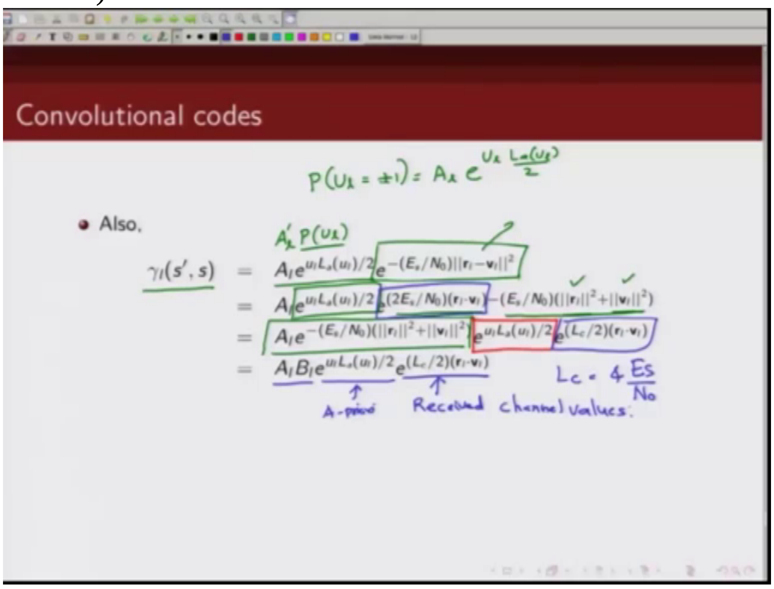

algorithm if we are considering additive white Gaussian noise

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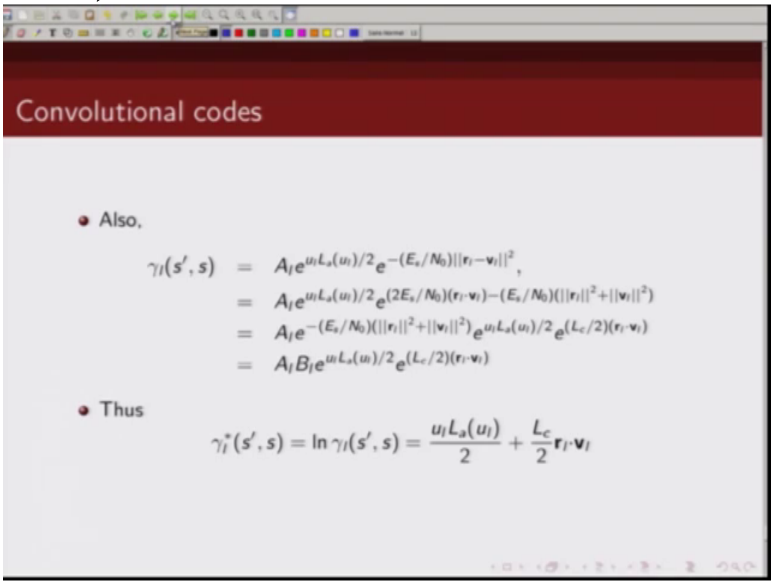

channel. Now if we consider branch metric in the log domain, then log of this term will be some sort of constant, we just

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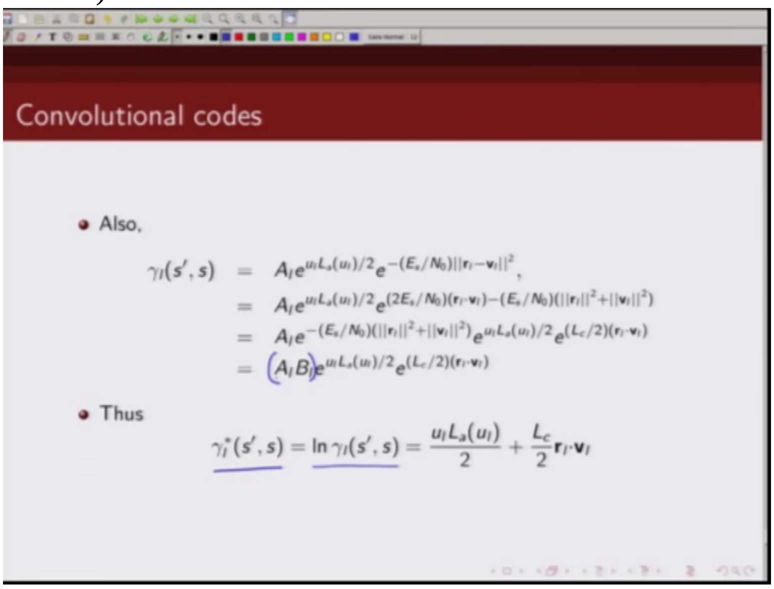

ignore it because this does not depend on choice of v of l, u of l so this will become u l, L value, a priori L value by 2 plus L c by 2 and dot product between the received sequence and the transmitted codeword. So this will be then our simplified expression for

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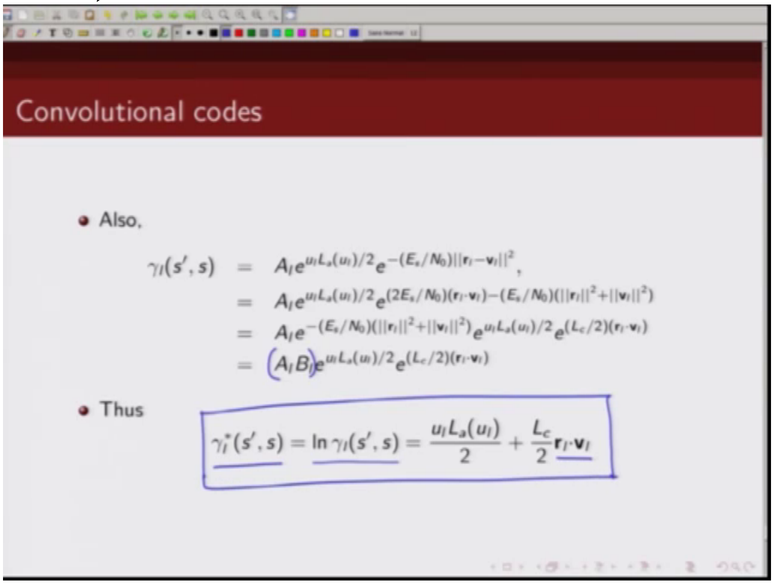

for branch metric computation

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for B C J R algorithm over additive white Gaussian noise channel, thank you.