Principles of Communication- Part I

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Module No 2

Lecture 09: Example for Auto-correlation of Signal and Energy Spectral Density (ESD)

Hello, welcome to another module in this massive open online course. So let us continue our discussion on the auto correlation on energy spectral density and we are looking at the auto correlation of this function e to the power of - a(t) u(t).

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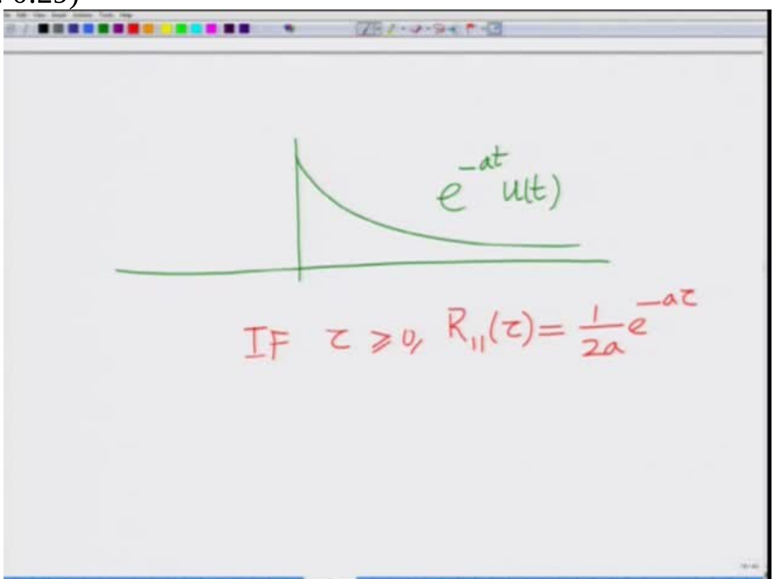

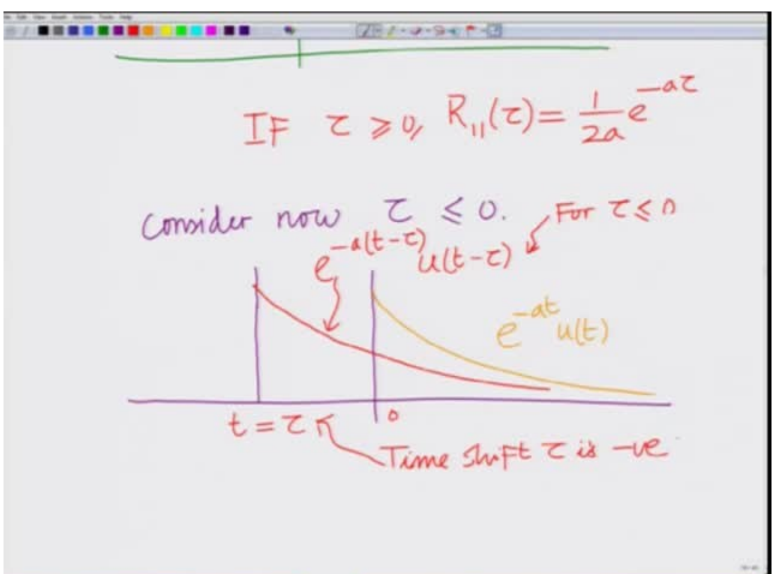

And this is the particular signal that we are looking at which is e to the power of - a(t) u(t) and we have already seen that if tao is greater than equal to zero than R11 tao equals we have seen that R11 tao equals 1 over root 2 a e to the power of - a tao. Let us now consider the scenario consider now tao less than or equal to zero, with tao less than or equal to zero we have e to the power of, this is your e to the power of - a this is e to the power of - a(t) u(t) and now if tao less than or equal to zero the shift is towards the left and therefore the resulting signal is going to look like this that is, it is going to look this is your e to the power of - a t - tao u t - tao for tao less than or equal to zero, that is the shift t equal to tao is negative, okay. So the time shift tao is negative.

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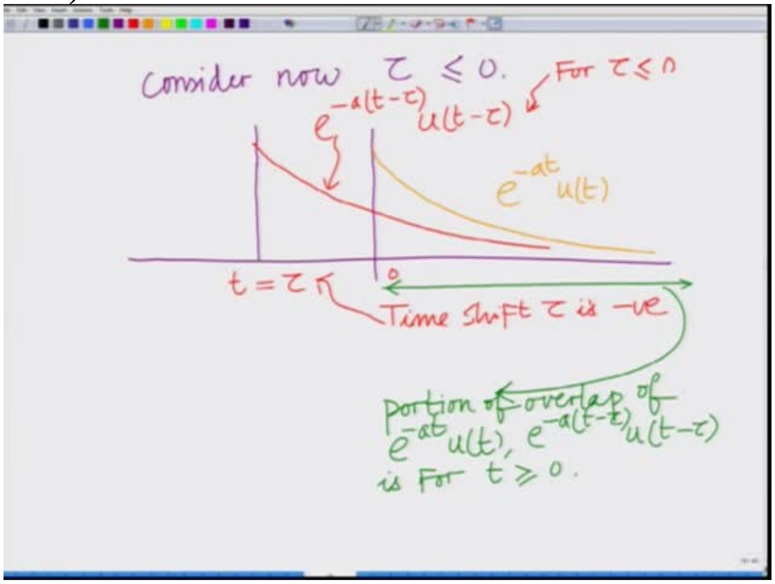

And now therefore if you can see the overlap that is this portion the overlap of the 2 exponentials portion of overlap of e to the power of - a(t) u(t) e to the power of - a t time t - tao is for t greater than equal to that is the portion where there is 2 signals that is e to the power of - a(t) u(t) and e to the power of - a(t) - tao u(t) - tao overlap is for all t greater than equal to zero that is the portion where this the period that is the the portion where they overlap implying the portion where the product of both these signals is nonzero.

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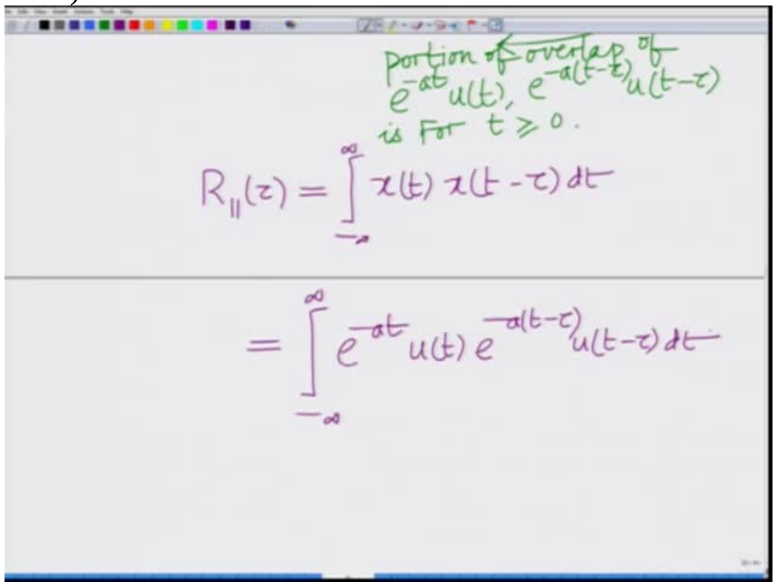

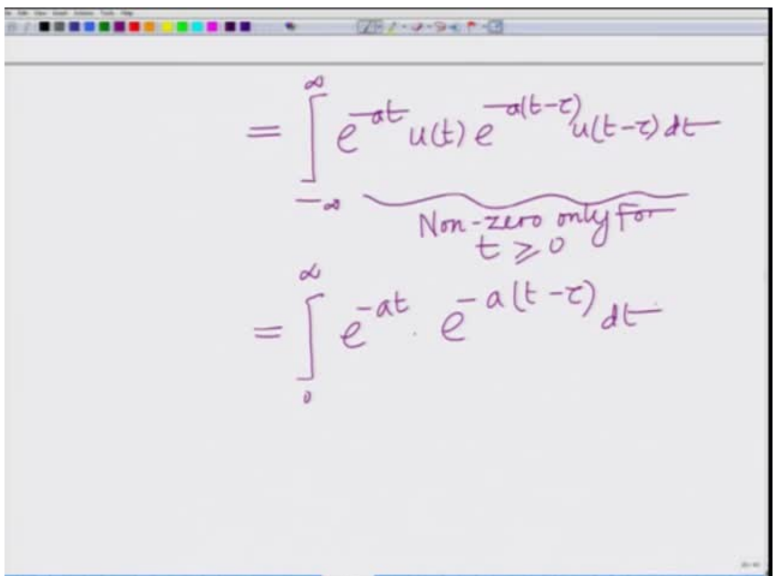

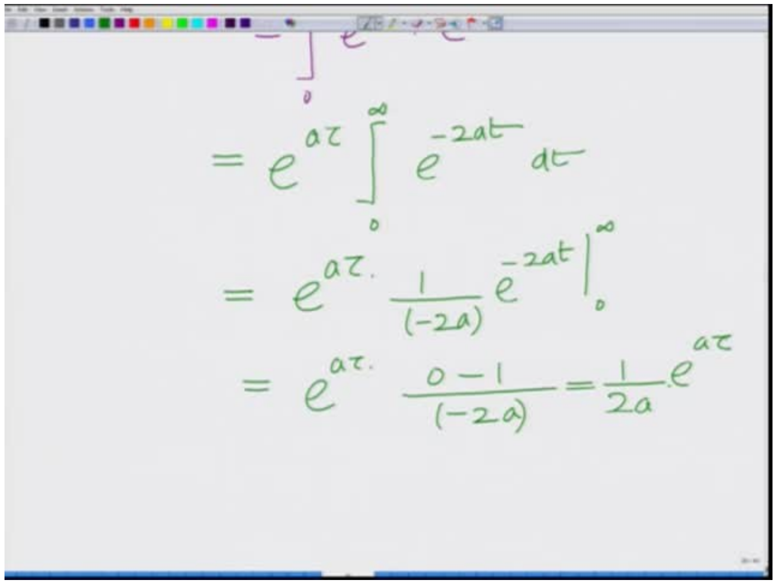

Therefore R11 tao remember, our definition is R11 tao again - infinity to infinity x(t) x(t) - tao d tao, sorry dt and this is equal to - infinity to infinity e to the power of - a(t) u(t) e to the power of - a t - tao u(t) - tao dt. This is nonzero only for t greater than product is nonzero only for t greater than equal to zero. So therefore this is equal to zero to infinity e to the power of - a(t) e to the power of - a e to the power of - a t - tao d t which is equal to again e to the power of a tao comes out common.

So this is, this does not depend on t so this comes out of the integral this is integral zero to infinity e to the power of - 2 a(t) dt which is equal to e to the power of a tao 1 over - 2 a e to the power of - 2 a(t) evaluated between the limit zero to infinity equals e to the power of a tao times zero - 1 divided by - 2 a equals 1 over 2 a e to the power of a tao equals 1 over this is going to be 1 over 2 a 1 over 2 a.

I am sorry this has to be e to the power of - a, e to the power of a tao so this has to be e to the power of a tao, okay.

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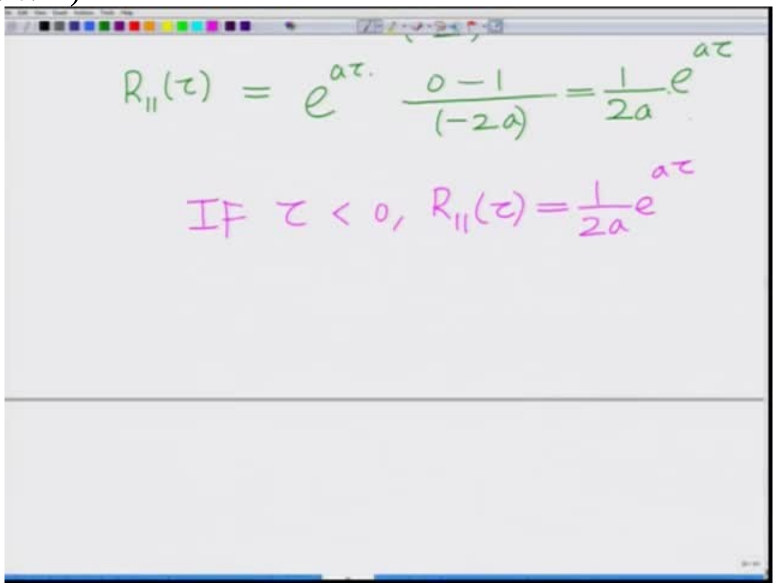

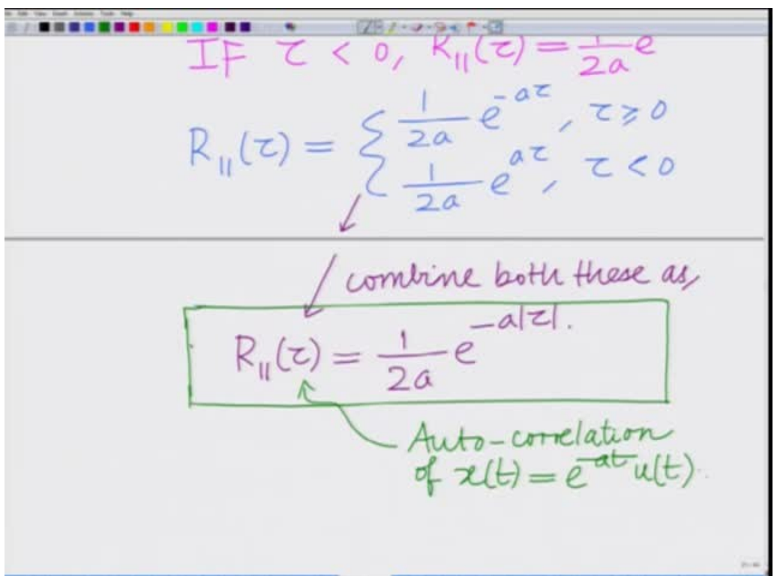

So this is you can see R11 tao if so this is basically your R11 tao so if tao less than equal to zero or tao less than zero R11 tao equals 1 over 2 a e to the power of a tao, okay. So therefore now we have derived R11 tao corresponding to both the regions of tao that is if tao is greater than or equal to zero it is 1 over 2 a e to the power of - a tao, if tao is less than zero than it is 1 over 2 a e to the power of tao.

Therefore we have, so we have to summarize R11 tao equals 1 over 2 a, e to the power of - tao a tao if tao is greater than or equal to zero this is 1 over 2 a, e to the power of a tao if tao is less than zero and now you can see I can combine both these I can combine both these as R11 tao equals 1 over 2 a, e to the power of - a mod of tao magnitude of tao.

So the I can combine this I can combine both these as 1 over 2 a, e to the power of - a magnitude of tao, that is the auto correlation function of this signal e to the power of - a(t) u(t), okay. So this is the auto correlation this is the auto correlation of the signal x(t) equals e to the power of - a(t) u(t).

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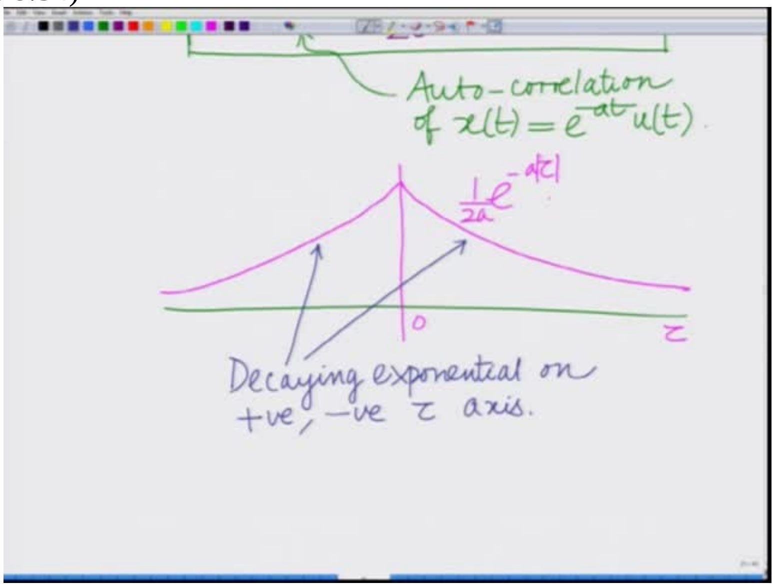

And it looks as follows, it is e to the power of 1 over 2 a, e to the power of - a magnitude tao so it is basically nonzero, this is nonzero and decaying exponential that is this is basically, let me draw it with the same colour this is basically decaying exponential on both axis that is this is your e to the power of - a tao - a magnitude of tao 1 over 2 a. So this is a decaying (expo) so this is on the tao axis.

This is tao equal to zero, so this is 1 over 2 a, e to the power of - a magnitude of tao. So this is a decaying exponential on both sides decaying exponential on positive and also the negative tao axis, okay. Okay, 2 sided exponential which is decaying on both the positive tao axis and also the negative tao axis. So this is the auto correlation functions of e power - a a(t) u(t), okay. This is the auto correlation function.

Now let us find the energy spectral density, alright. So to find the energy spectral density, let us first start by finding the Fourier transform, okay.

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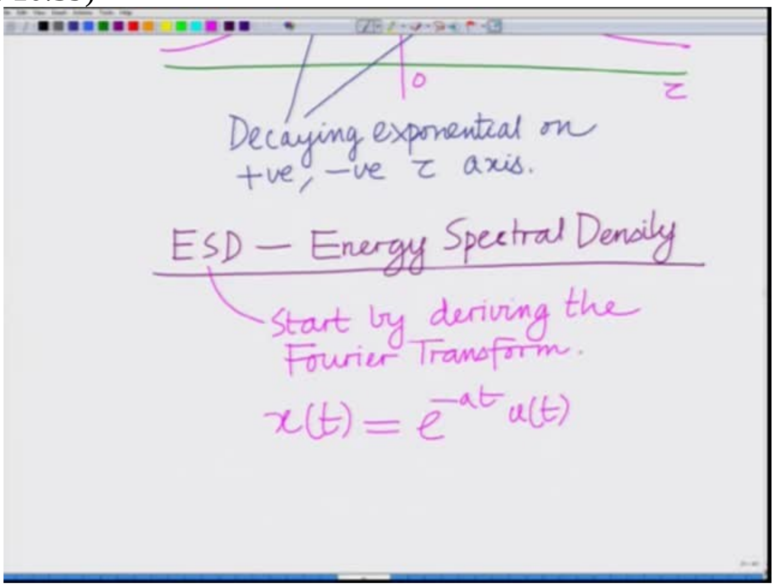

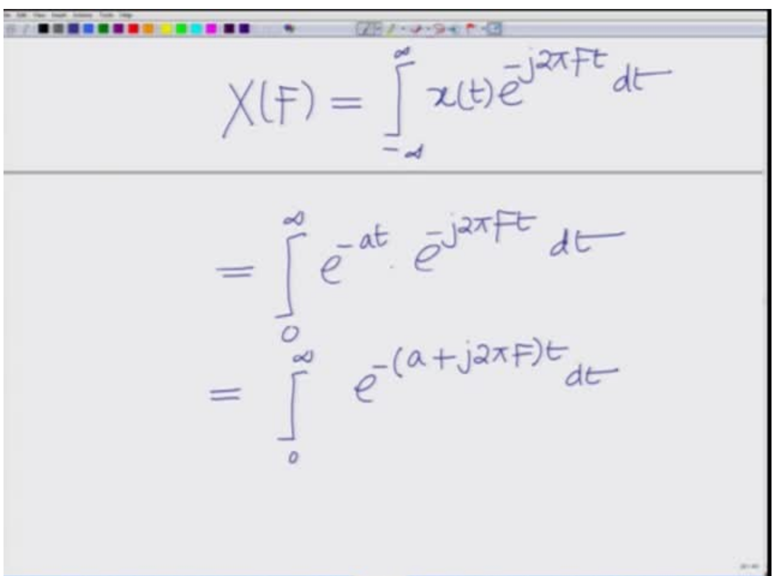

So to find the energy spectral density, let us now find the energy spectral density , so we start by finding start by deriving the Fourier transform. We have x(t) equals e to the power of - a(t) u(t). The Fourier transform X(F) equals integral - infinity to infinity x(t) e power - j 2 pi Ft dt which is equal to integral - infinity to infinity.

Well, e power - a(t) u(t) is nonzero only for t greater than equal to zero. So this is e power - a(t) u(t) e power - j 2 pi Ft dt which is integral zero to infinity e power - a plus j 2 pi Ft dt, correct?

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Now therefore this integral can be evaluated as e to the power of - a plus j 2 pi Ft divided by - a plus j 2 pi F substitute the limits zero to infinity equals zero - 1 divided by - a plus j 2 pi f equals 1 over a plus j 2 pi F and therefore what we have is the Fourier transform X(F) is 1 over a plus j remember this is the Fourier transform, this is your Fourier transform of the signal is, this is the Fourier transform of the signal x(t). And now energy spectral density is given by the magnitude square of the Fourier transform.

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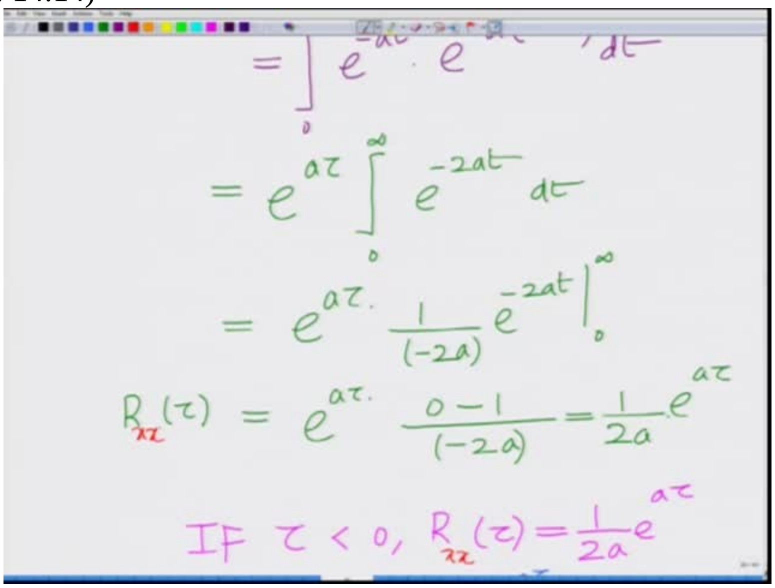

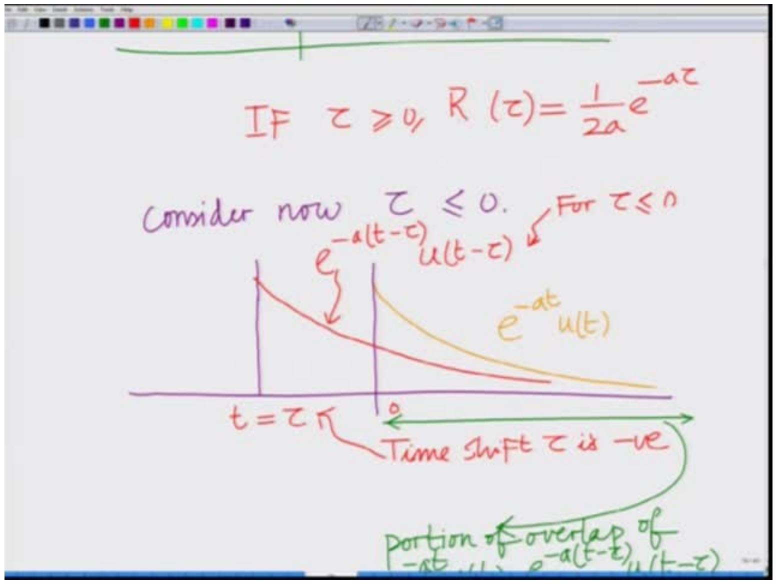

So the ESD that is S 11 of F or S of s x x of F equals magnitude x x X(F) square, alright. Here also I can denote all this auto correlation functions by R x x of tao, this is R x x of tao rather than R11 of tao, these can be denoted by R x x of tao and so on and these things can be corrected at all the other places that is we have here for instance R x x of tao, okay.

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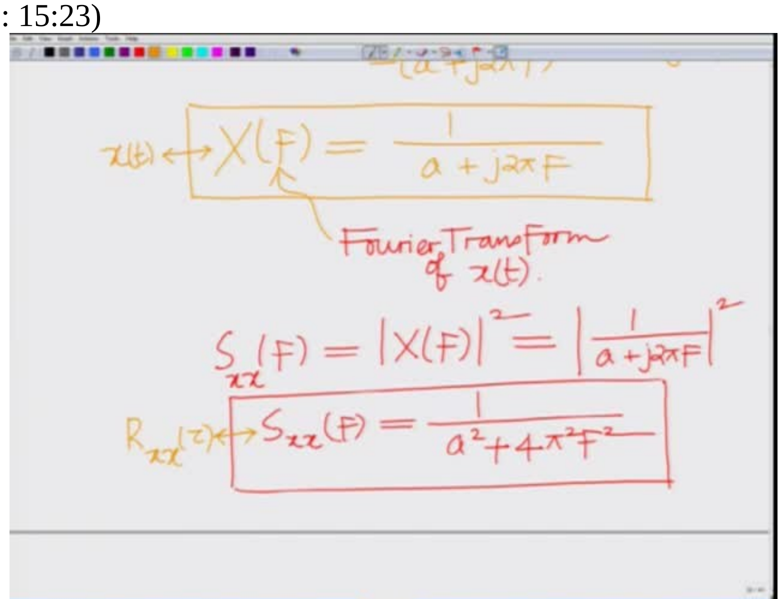

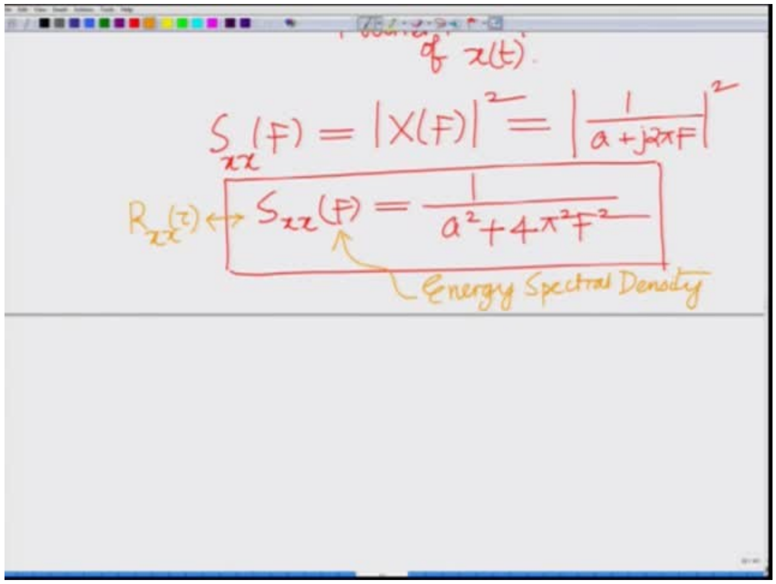

And therefore now the Fourier transform S x x of F that is magnitude X(F) square that is magnitude 1 over a plus that is magnitude 1 over a plus j 2 pi F that is 1 over a square plus 4 pi square S x x of F, this is magnitude X(F) square and remember this is also one can also, now remember that this is also the Fourier transform of the auto correlation that is R x x tao, that is what we have over here that is if you remember that S x x F that is energy spectral density is also the Fourier transform of the auto correlation function R x x tao which is magnitude X(F) square.

Therefore R x x tao the Fourier transform is 1 over, so R x x tao Fourier transform of R x x tao is S x x F which is 1 over a square plus 4 pi square F square, this is also your energy spectral density. This is also the energy. Now we have already seen that this energy spectral density characterizes the distribution of energy in the frequency. So there is a spread of energy in the frequency domain.

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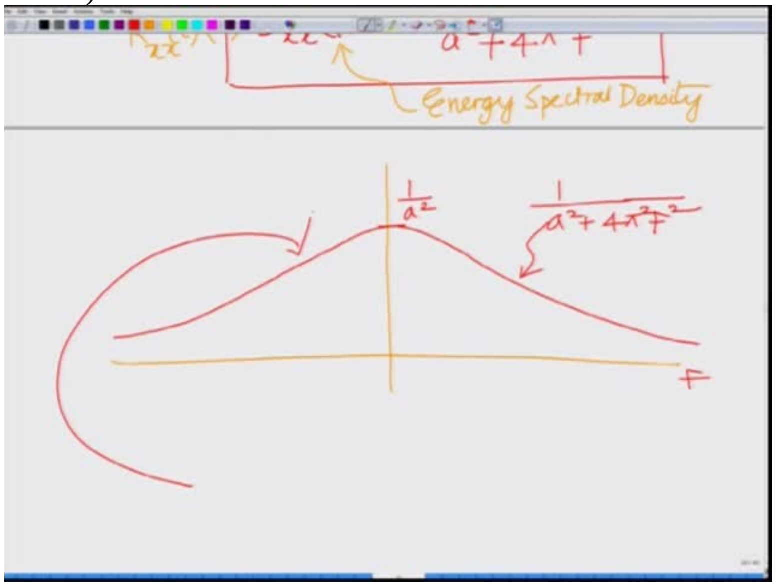

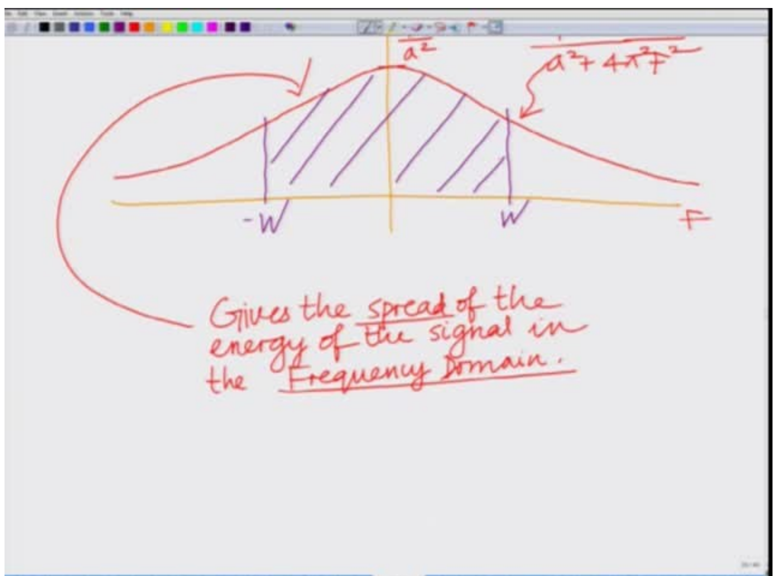

So if you plot this thing S x x of F at F equal to zero, this is 1 over a square and it looks it decays as F increases towards infinity it decreases, so this is at zero it is 1 over a square and this is 1 over a square plus 4 pi square F square and this characterizes this gives the spread of the energy in the frequency domain. Spread of the energy of the signal in the spread of the energy of the signal in the frequency domain.

For instance one can ask the question, what is the energy of the signal between in the band - W? Remember we can ask the question what is the, what energy how much energy is contained in the band that is one can ask this questions how much energy, we would be interested in knowing, how much energy of this signal, right? Is considered in the band - W to - W to W, alright. And that can be derived by integrating this energy spectral density in this frequency band.

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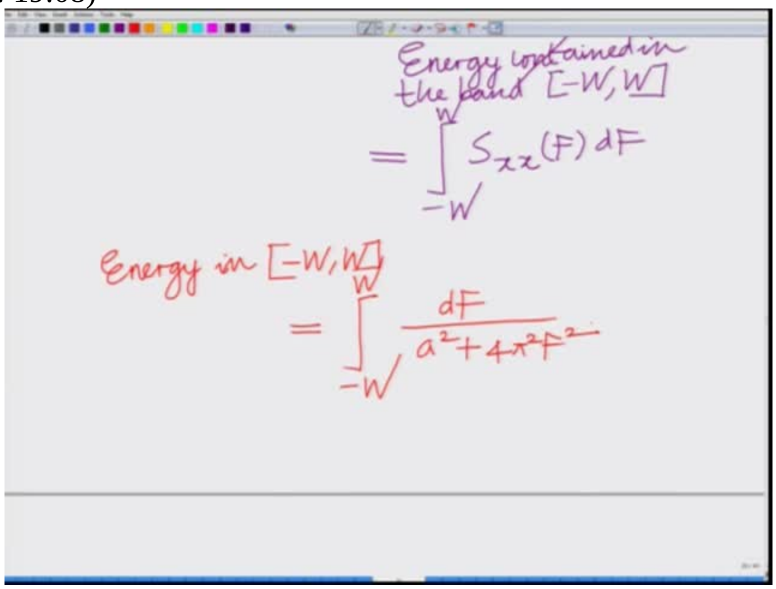

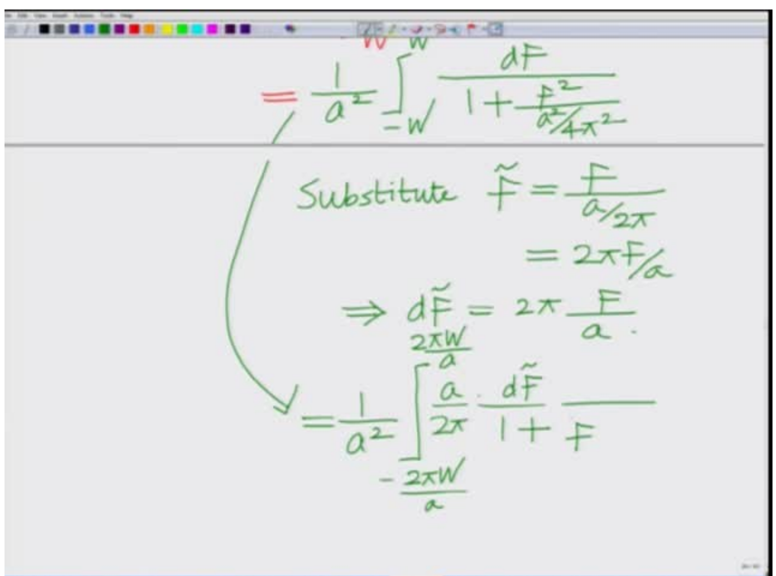

So if I ask the question, what is the energy contained in the band - W to W? That is equal to the integral - W to W of the energy spectral density S x x of F into d of F we can evaluate this, in fact so the energy in the band - W to W this is equal to - W to W integral of the energy spectral density that is dF by1 over a square plus 4 pi square F square. Now I am going to do some manipulations, I am going to bring the a square outside, so this can be written as 1 over a square - W to W d F by1 plus F square divided by a square divided by 4 pi square, okay.

I have brought the 4 pi square into the denominator, now let us set or use the substitution substitute till F tilde equals F divided by a over 2 pi, correct? Which implies or F is equal to F tilde equals F over a over 2 pi which is basically 2 pi F divided by a which implies d F tilde equals 2 pi d F by a and therefore this integral here can now be modified with this substitution as 1 over a square the limits become - 2 pi W over a to 2 pi W over a times d F tilde into 2 pi over a or d F tilde equals d F equals d F tilde by 2 pi over a. So this is into a over 2 pi by1 plus F square divided by a square by 4 pi square is nothing but F tilde square.

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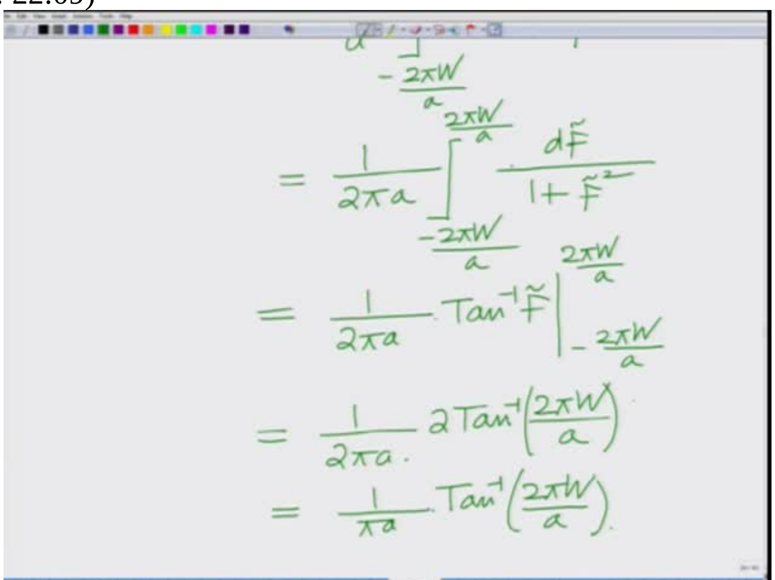

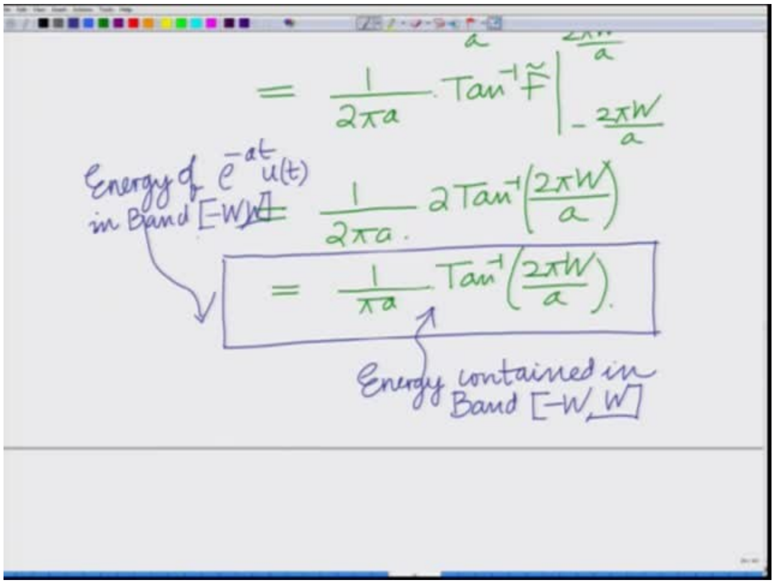

And therefore what I have over here is the energy contained in the band - W to W is 1 over 2 pi a - 2 pi W by a to 2 pi W by a d F tilde divided by1 over F tilde square and this you can see is 1 over 2 pi a Tan inverse F tilde between the limits 2 pi W by a - - 2 pi W by a. So this is simply1 over 2 pi a into 2 Tan inverse 2 pi W over a which is basically1 over pi a Tan inverse 2 pi W over a.

And this is a very interesting result, what is this? This is the energy contained in the band, this is the very this is the energy contained, energy of what? Energy of x(t) that is e that is the signal e to the power of - a(t) u(t) contained in the band - W. So what we are now able to achieve is being we are able to we are able to compute not only what is the total energy of the signal, of course that is what we have done before, right?

Much before when we defined the energy of the signal but we are also able to compute what is the precise amount of energy of, what is the precise amount of energy of a particular signal which is concentrated in a certain band of interest that is - W to W? And that we are able to achieve because we are able to compute the energy spectral density of this signal and from the energy spectral density we integrated over the frequency band of interest to derive the energy of the signal contained in that band of interest - W to W.

So, alright so this is the energy again of your signal e to the power of - a(t) u(t) in band. Naturally now if I set W equal to infinity that is from - infinity to infinity I should get the total energy of the signal.

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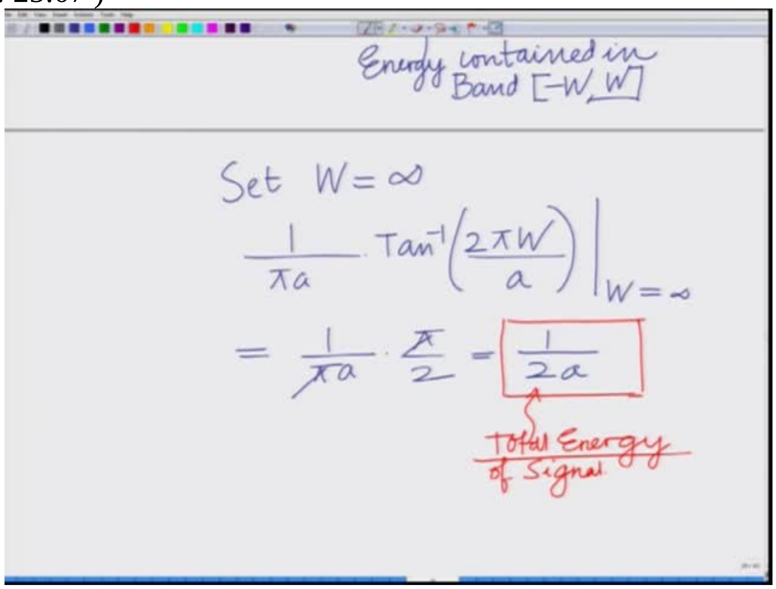

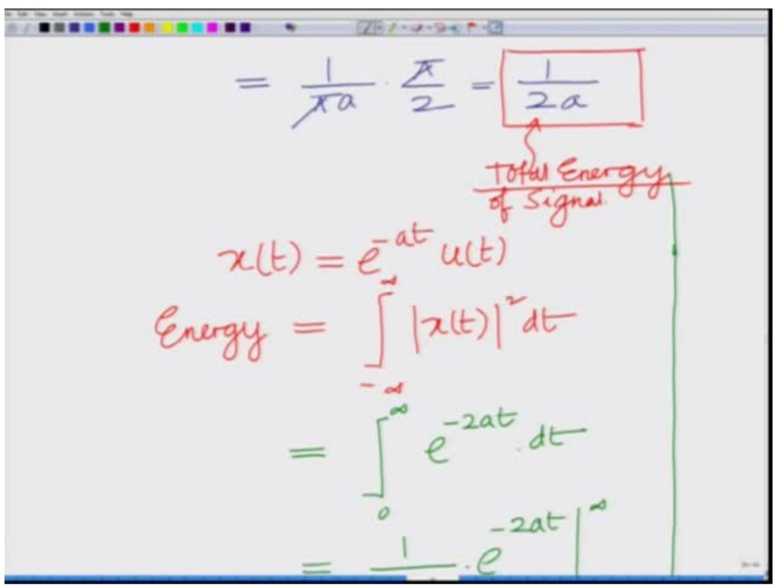

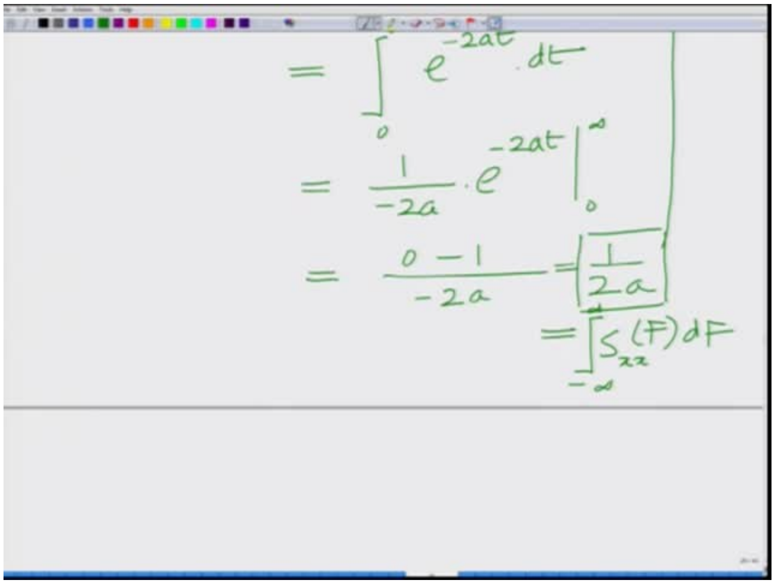

Now set, let this is an interesting exercise set W equals infinity1 by pi a Tan inverse 2 pi W by a for W equal to infinity this becomes 1 over pi a Tan inverse infinity is pi over 2 so the pis go away and this will become 1 over 2 a. And this should be the total energy and this should give us technically this should give us the total energy of the, technically this should give us the total energy of the signal that is contained over the entire. This should technically be total energy of the, signal 1 over 2 a.

Is it the total energy of the signal? Let us check. Our signal is x to the power of x(t) equals e power - a(t) u(t). It is energy is integral - infinity to infinity x(t) whole square that is magnitude x(t) square dt. This is from our definition of energy, now remember this is this function is nonzero only for t greater than equal to zero. So this is zero to - infinity e to the power of - 2 a(t) dt which is 1 over - 2 a e to the power of - 2 a(t) evaluated between the limits zero to infinity and therefore this is zero - 1 divided by - 2 a which is 1 over 2 a and this is indeed if you can look above this is indeed if you can go above this is indeed the total energy of the signal.

This is indeed the total energy of the signal, this is also equal to what we have shown is basically this is also this quantity is indeed the total energy and this is also integral - infinity to infinity S x x of F d of F. By setting W equal to infinity that is the band of interest to be from - infinity to infinity we are integrating the energy spectral density over the entire frequency axis that is the area under the entire energy hat is the area under the energy spectral density over the entire frequency domain that is from - infinity to infinity that yields the total energy of the signal. And not only that something even more interesting let us look at our auto correlation function R x x tao.

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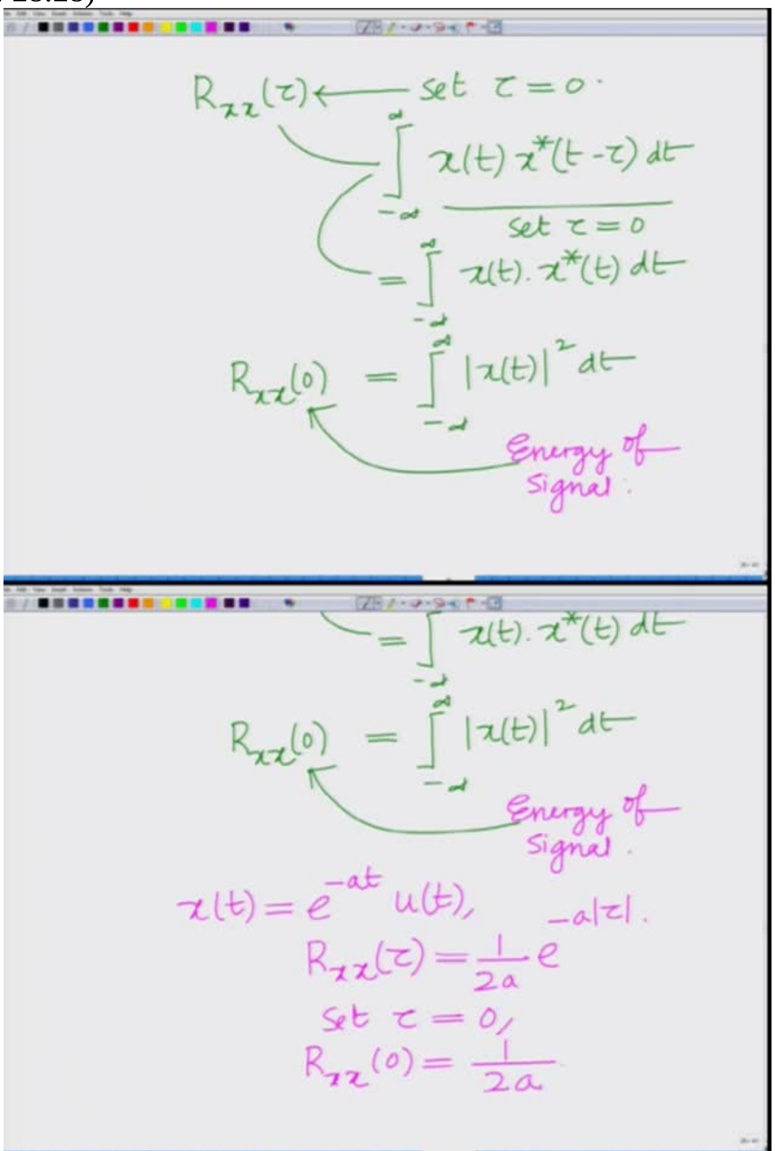

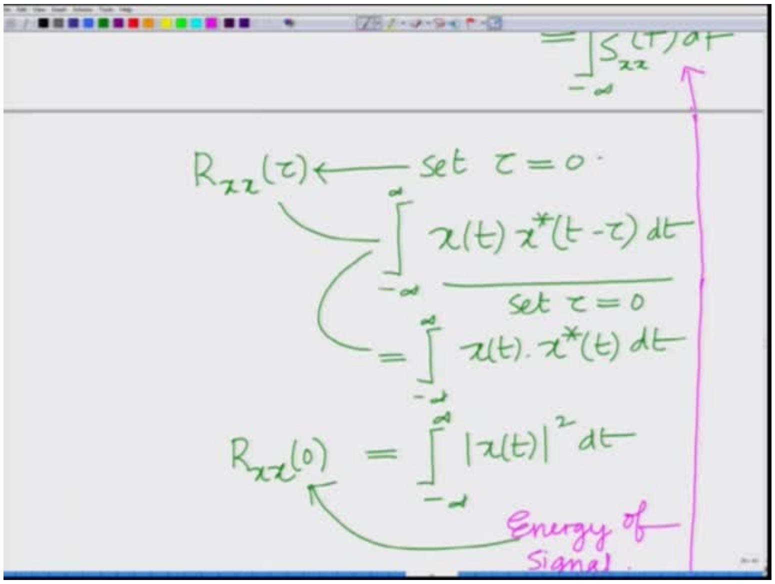

Now set tao equal to zero. Remember auto correlation function is - infinity to infinity x(t) into x conjugate t - tao d tao. Now set tao equal to zero in this and this becomes integral - infinity to infinity x(t) once you set Tao equal to zero its conjugate of t dt which is basically equal to magnitude x(t) square dt.

So R x x of xero that is if you set Tao equal to zero R x x of zero equals again R x x of zero again gives us the energy of the signal, it again gives us the energy of the. Now for our signal x(t) equal to e to the power of - a(t) u(t). We have shown that R x x tao equals 1 over 2 a, e to the power of - a magnitude of Tao. Now set Tao equal to zero and R x x of zero, you can now once again see equals 1 over 2 a, e to the power of - a zero, e to the power of zero which is again simply1 over 2 a.

And now you can once again see this is also equal to that is if you go again this quantity is again you can see this quantity is once again equal to the total energy this quantity is once again equal to the total energy of the signal that is 1 over 2 a. So again R x x tao the auto correlation function the auto correlation function of the signal x(t) evaluated at Tao equal to zero gives us the total energy of the signal that is 1 over 2 a.

Therefore all these things are equivalent and that is what we have shown, if you go back and look at thet theory you will see that all these things are equivalent and yield the total energy of the signal.

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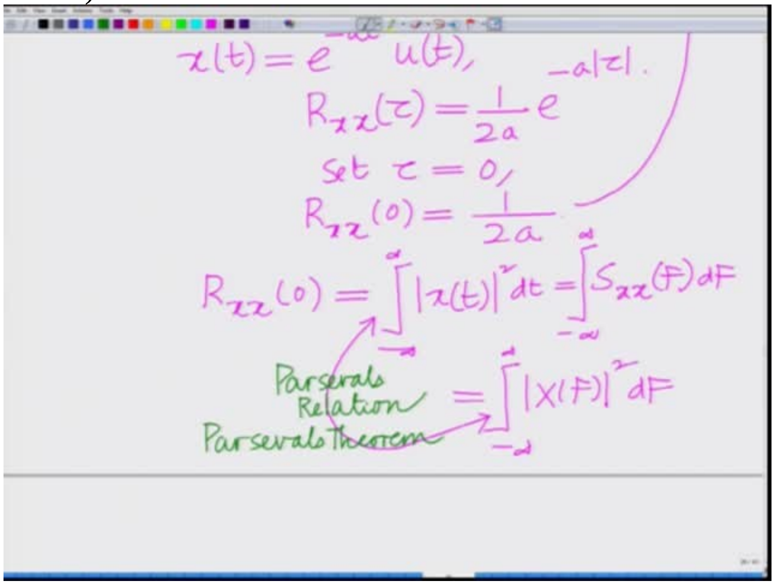

That is your R x x auto correlation function evaluated at zero equals - infinity to infinity magnitude x(t) square dt which is - infinity to infinity total area under the energy spectral density which is equal to - infinity to infinity magnitude X(F) square d F from the Parseval's , remember this part that is this is what is traditionally known as your Parseval's relation or Parseval's theorem for that matter. This is also known as Parseval's that is integral - infinity to infinity magnitude x(t) square dt equals integral - infinity to infinity magnitude X(F) square d F, alright.

So this completes our example of the auto correlation function and energy spectral density where we have shown and explored several interesting things, we have considered a signal, derived its auto correlation function derived the corresponding energy spectral density, alright, realized that this is the distribution of the energy of the signal in the frequency domain, derived the portion of energy of the signal contained in the band of interest that is - W to W, demonstrated that if W is set to infinity than it gives the total energy of the signal which in this case is 1 over 2 a.

That is also equal to the energy of the that is integral - infinity to infinity magnitude x(t) square dt, it is also equal to R x x at Tao at Tao equal to zero that is R x x zero the auto correlation function of the signal evaluated at Tao equal to zero, alright. So this comprehensively completes the signal analysis and the spectral analysis that is the discrete Fourier series and the Fourier transform and the various properties of the Fourier transform of the signal, thank you.