Principles of Communication- Part I
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Module No 2
Lecture 08: Autocorrelation of Signal and Energy Spectral Density (ESD)

Hello welcome to another module in this massive open online course. So we are looking at the concept of auto correlation of a signal x(t), correct?

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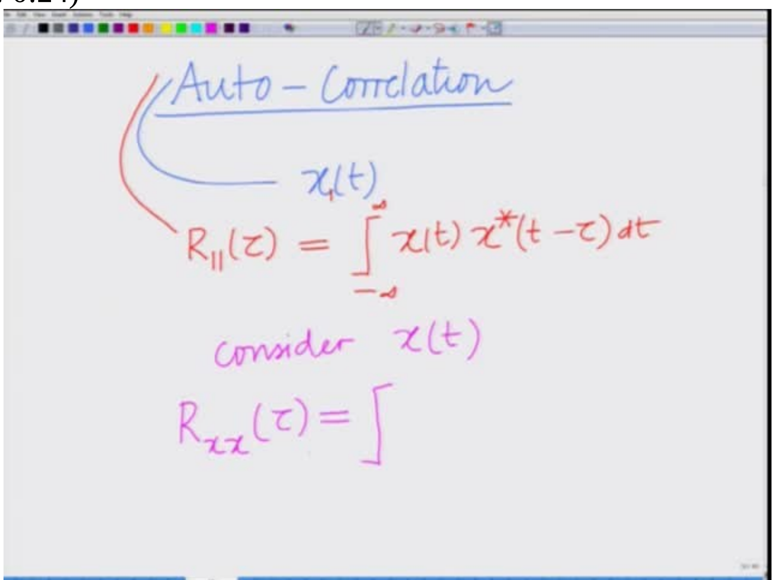

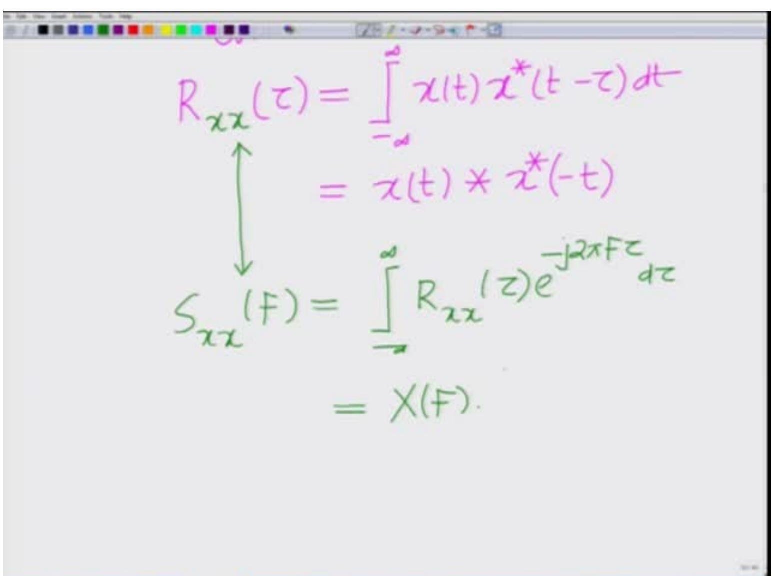

We are looking at the auto correlation of a signal. The auto correlation of a signal x(t) and this is R11 of tao or whatever relation x1 t is R11 of tao integral - infinity to infinity x t into x conjugate t - tao dt. Now since there is only1 signal x1 t instead of x1 t I can simply also consider also consider x1 t, alright.

So Uhh and for that the auto correlation for instances of x2 t consider x of t that is a single signal, we have single signal so no need for the notation x1 and x2. So x of t and for that the auto correlation can also be denoted by R x of tao, instead of R11 of tao you will also find this notation used for the auto correlation that is R x of tao or R x of tao and so on. That is - infinity to infinity, so here we have x1 of t x1 conjugate t of t.

Here I simply have x of t x conjugate t - tao dt and we have seen this is also equal to x(t) convolved with x conjugate of - t further if I take the Fourier transform of this that is remember the Fourier transform Sx x of F R x , if I look at the Fourier transform of R x tao I can denote that by S x of F, that is equal to integral - infinity to infinity R x of tao e to the power of - j 2 pi F tao d tao and we have shown that this is basically the Fourier transform.

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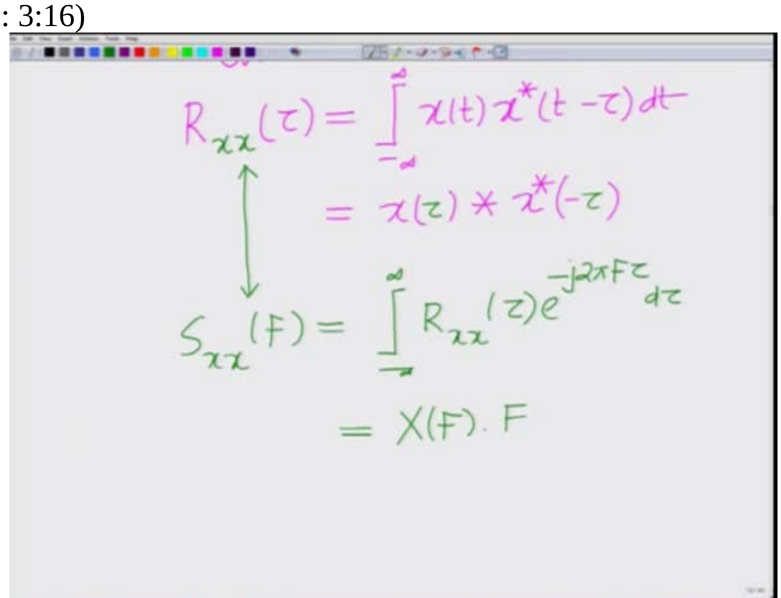

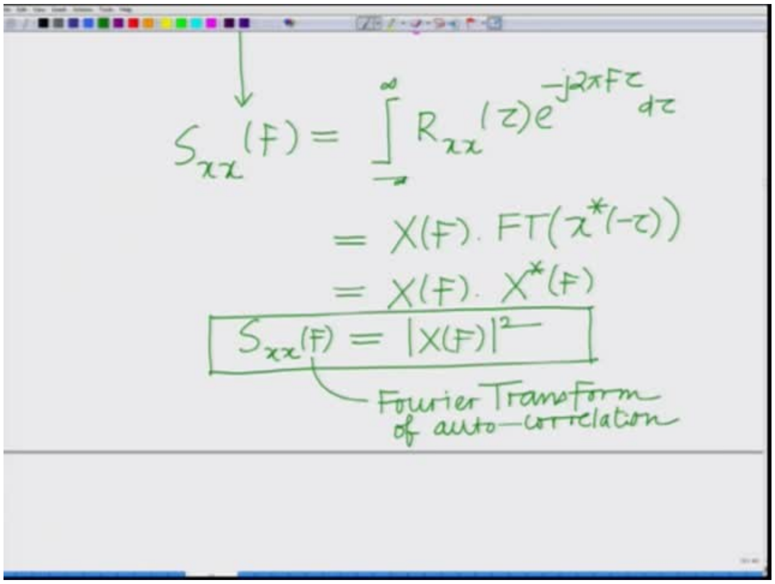

Remember convolution in the time domain is multiplication of the frequency domain so this x FF in fact, this is convolution of x of tao with x conjugate of - tao X(F) into the Fourier transform of x conjugate of - tao and we have shown Fourier transform of x conjugate of - tao is nothing but x conjugate of f. So therefore this is magnitude Sx X(F) square. So what we have showed is we have showed this interesting result where Sx x of F or this Fourier transform that is Fourier transform this has a name which will come to shortly, this is the Fourier transform of the auto correlation S x of F the Fourier transform of R x x of tao is nothing but magnitude X(F) square where X(F) is the Fourier transform of x of t.

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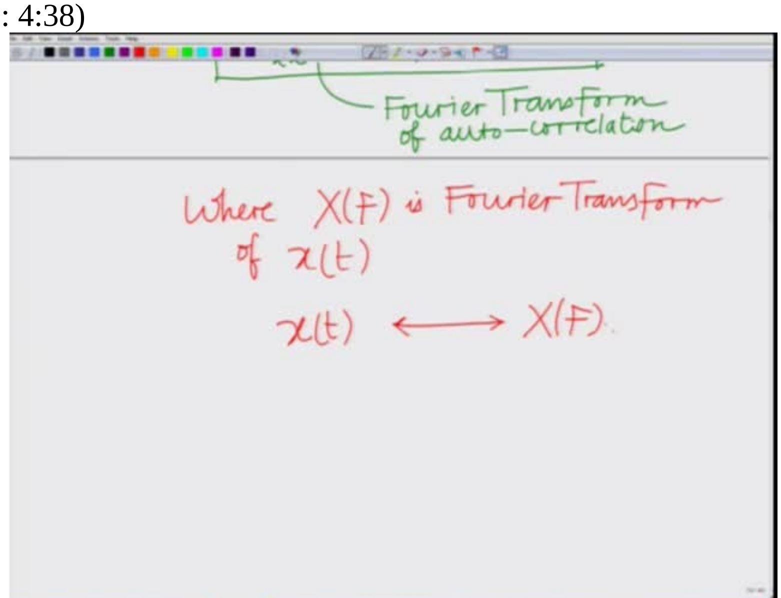

Okay, that is also important to keep in mind where X(F) is Fourier transform of x(t) that is x(t) and X(F) form a Fourier transform pair. Now let us look at another interesting property of this Uhh S x of F.

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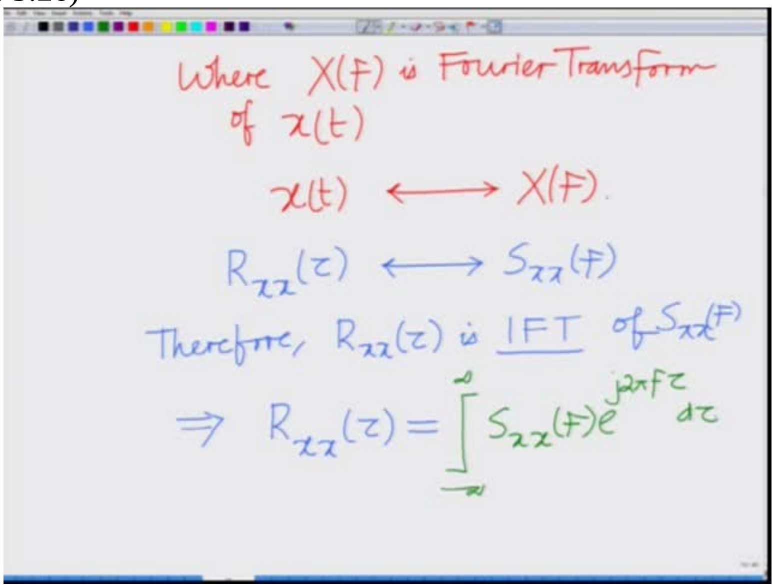

Now look at this we have S x x F equals Fourier transform that is if we look at R x x tao that has Fourier transform S x x of F therefore S or SR x x of tao is the inverse Fourier transform is IFT is the IFT of S x x of F, alright. This is the inverse Fourier transform Sx which implies R x x of tao can be expressed as integral - infinity to infinity S x x of F e power j 2 pi F tao d tao, correct?

So what we are saying is since S x x F is a Fourier transform of R x x of tao is the inverse Fourier transform of S x x of F. Now let us evaluate R x x tao at tao is equal to zero.

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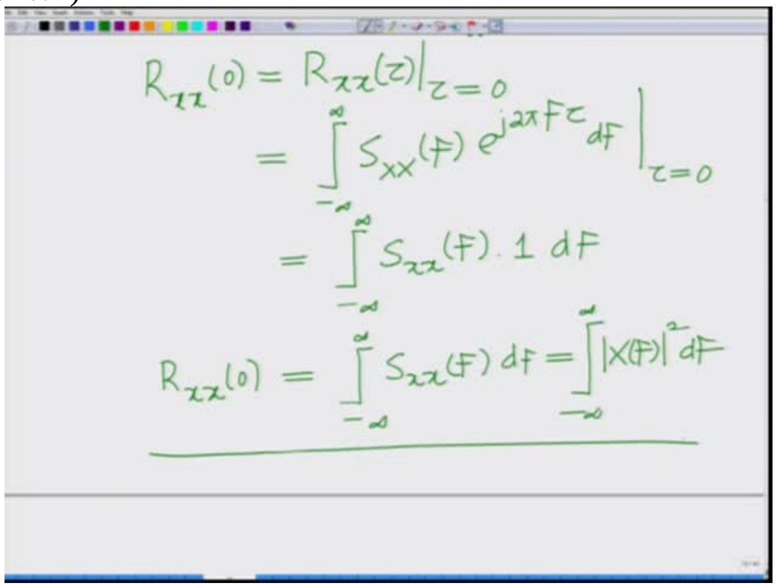

So R x x tao R x x zero is nothing but R x x tao evaluated at tao equal to zero which is nothing but this inverse Fourier transform S x x of F e to the power of j 2 pi F tao d tao evaluated, correct? I am sorry this has to be dF this is the inverse Fourier transform dF evaluated at tao is equal to zero this has to be, now if you substitute tao is equal to zero what I get is - infinity to infinity S x x of F e to the power of j 2 pi F tao equal to zero this is 1 into 1 dF which is basically - infinity to infinity integral of S x x F d F which is basically but remember S x x of F is magnitude X(F) square. So this is simply integral - infinity to infinity magnitude X(F) square and this is equal to R x x of zero. So we have this relation.

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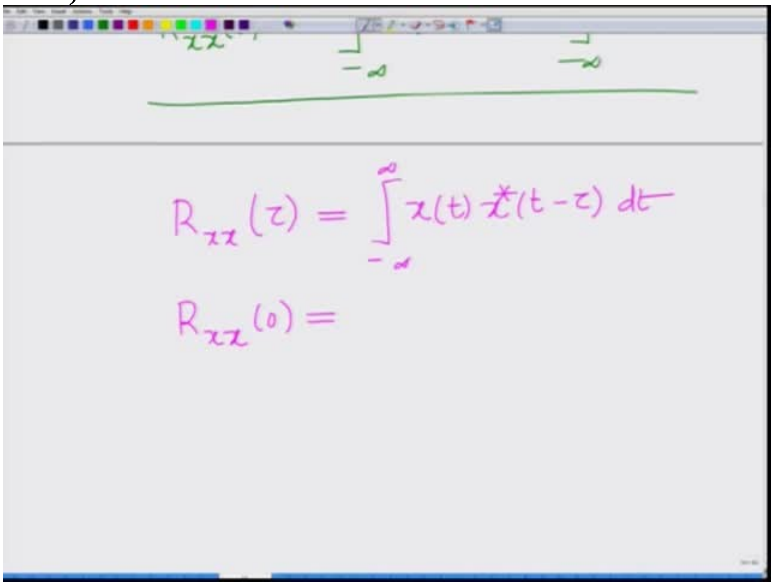

But now also realize that R x x tao equals integral - infinity to infinity x(t) x conjugate t - tao dt. Now tao equal to zero therefore R x x of zero equals that is now I have the auto correlation expression for the auto correlation that is integral - infinity to infinity x(t) into x conjugate t - tao dt integral from - infinity to infinity set tao equal to zero in this and what I have now set tao equal to zero.

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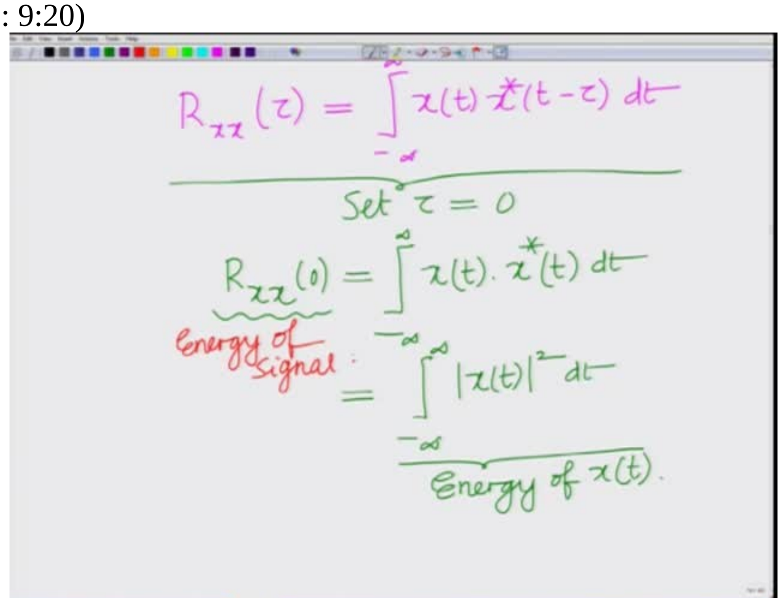

So what we are going to do is set tao equal to zero R x x zero equals to - infinity to infinity x(t) into x conjugate t since tao equal to zero which is nothing but basically which is nothing but - infinity to infinity magnitude x(t) square dt. Therefore this is the, look at this is integral - infinity to infinity magnitude x(t) square dt which is nothing but the energy of the signal x(t). Therefore R x x zero is the energy of the signal. So therefore R x x zero is nothing but this represents the energy.

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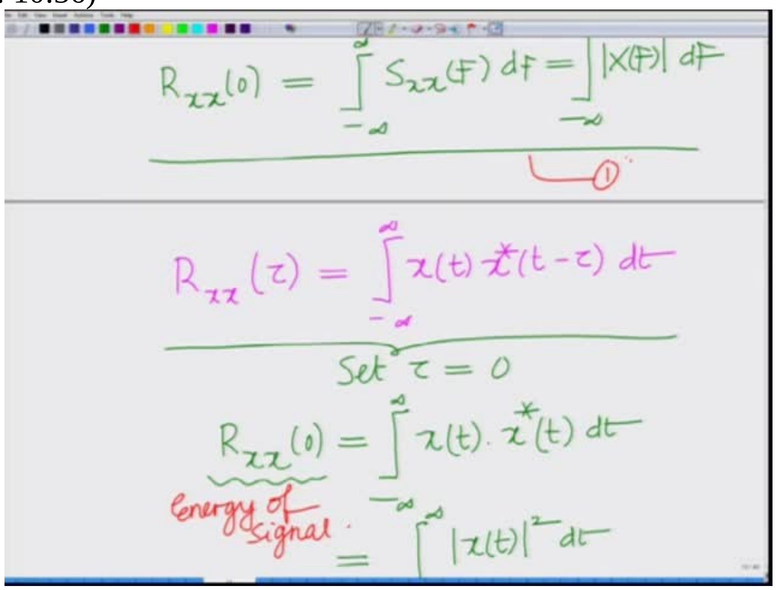

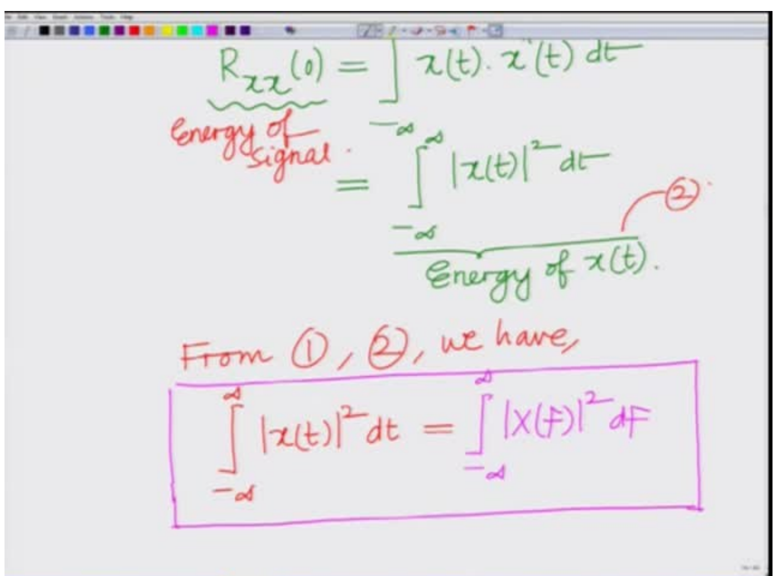

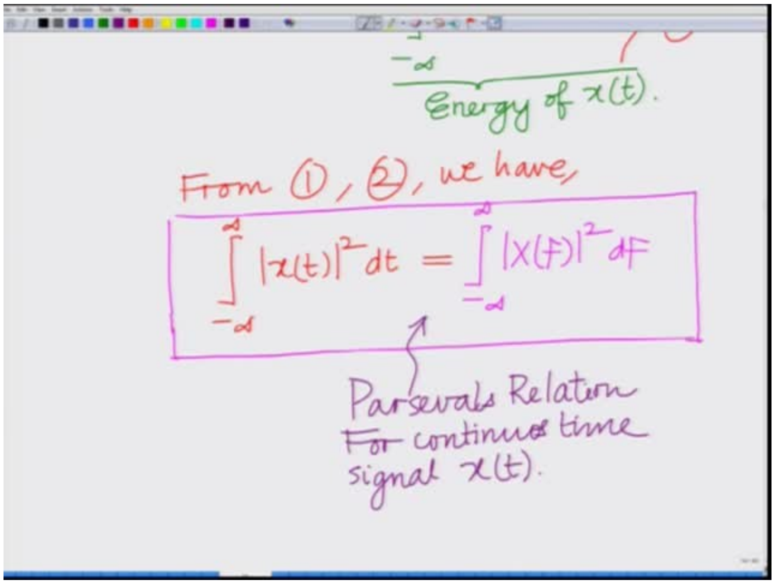

And from here we have R x x zero is basically integral - infinity to infinity magnitude X(F) square d F. So from this result let us call this result as 1 and let us call this result as 2. From 1, 2 we have that is R x x zero from results 1, 2 we have integral - infinity to infinity magnitude x(t) square dt equals integral - infinity to infinity magnitude X(F) square dF. And this is nothing but this is the Persavals relation.

That is if you take the total energy in the time domain that is integral - infinity to infinity magnitude x(t) square d F that is basically equal to - infinity to infinity magnitude X(F) square dF that is the total energy of the spectrum that is total energy of the time domain signal is total is equal to the total energy in the frequency domain, alright. This is known as this is the Parseval's relation. First, this gives this is known as the Parseval's relation, correct?

Okay, so this is your Parseval's relation for a continuous time signal or Parseval's theorem for continuous time signal x(t), correct?

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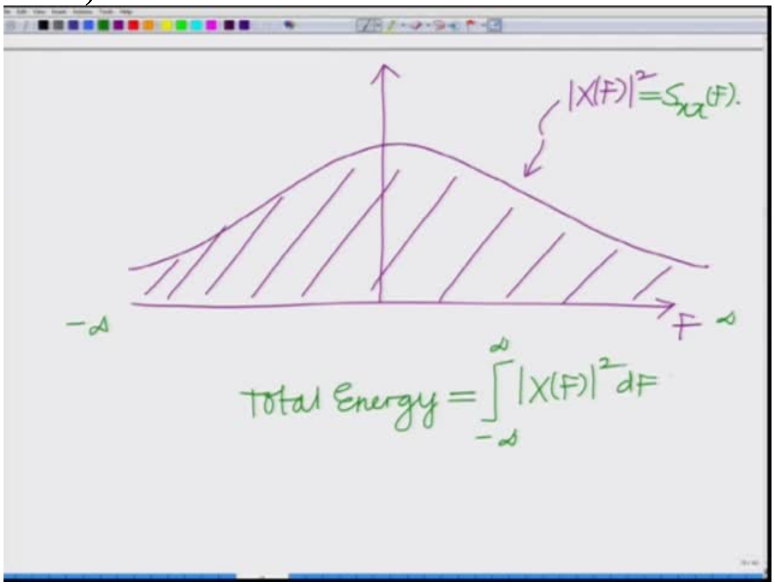

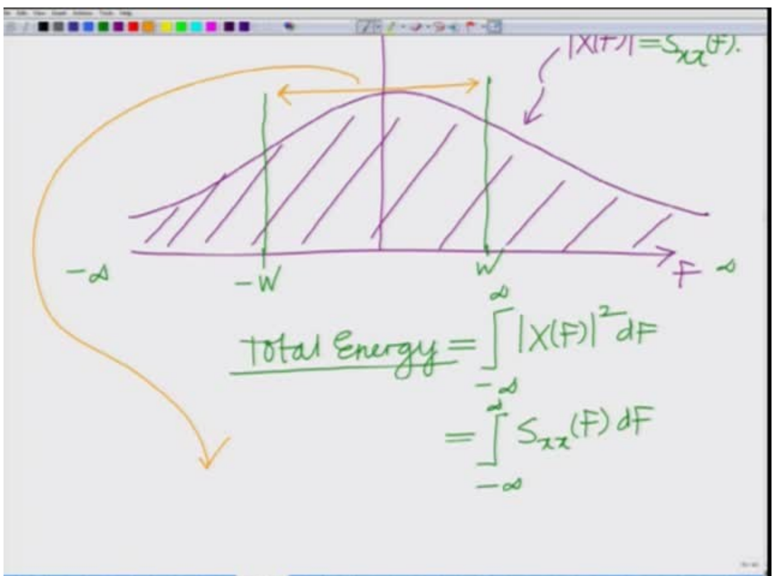

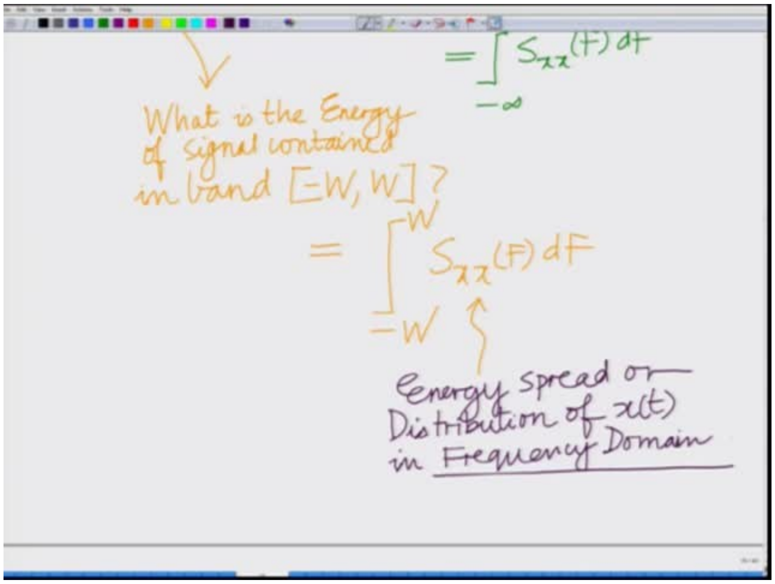

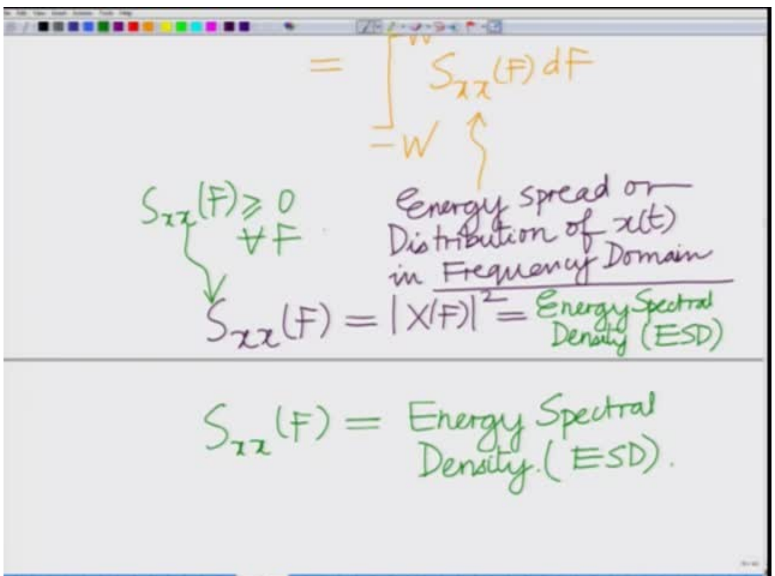

Now what you can see is basically if you look at magnitude X(F) square, let us look at magnitude X(F) square, okay. In the frequency domain, let us say this is your magnitude X(F) square. If you integrate this, correct from - infinity, so this is your frequency axis. So if you integrate this from - infinity to infinity we get the total energy, correct?

So the area under this from - infinity to infinity that is area under Uhh magnitude X(F) square, which is basically S x x F, right from - infinity to infinity gives us the total energy. So this is magnitude X(F) square which is S x x F. So total energy integral - infinity to infinity magnitude

X(F) square d F equals - infinity to infinity S X(F) S x x F d F, okay. So this is the total energy of this the total energy of the signal, alright.

So the total energy of the signal is given by the area under this S x x F under this function S x x F from - infinity to infinity, alright. So that is integrated over the entire frequency band, alright. So one can make the argument that is the energy corresponding to a band from - W to W is therefore given by the integration of this function S x x F over that frequency band - W to W. So one can ask the question what is the energy of the signal contained in this band from - W to W?

That is if I look at this band, correct and ask the question what is the energy? That is if I look at this band of bandwidth W and ask the question what is energy of signal contained in the band - W to W? then that can be obtained by integrating this quantity S x x F over the band of - W to W, alright. So therefore energy in band - W to W this is nothing but this is the integral of S x x F over the band - W to W.

Therefore what is this function S x x giving this S x x F gives us what is the spread of the energy over the frequency domain. That is, what is the spread? How is the energy of the signal x(t) distributed over the frequency x or distributed in the frequency domain, alright. And if I am looking at a particular band, the energy of the signal x(t) corresponding to that band is therefore given by the integration of this quantity S x x F over that band - W to W. This is therefore known as the energy spectral density, right?

This gives us the distribution of the energy in the frequency domain, correct? And therefore if I want to look at the energy in a particular band I have to integrate this energy spectral density or consider the area under this energy spectral density in that particular band - W to W. So this S x x of F gives me energy the energy spread or distribution of x(t) in the frequency domain. This is therefore S x x of F which is therefore giving me the density of energy over the frequency axis, this is known as the energy spectral density.

S x x of F equals magnitude X(F) square, this is known as the energy spectral, so this is known as the energy spectral density or the ESD. So S just let me write it clearly S x x of F is the energy, this is the energy spectral density or the ESD this is the energy spectral density of x(t), correct? It is giving that density of the energy of x(t) in the frequency domain.

So the density, correct multiplied by the area or basically density multiplied by the length or basically compute the area under this density curve in the frequency domain pertaining to a band of interest, correct? That gives us the energy of the signal x(t) contained in that particular band of interest, alright. So that is the interesting interpretation we have regarding the energy spectral density. So this is a very important and therefore naturally if you integrated over the entire frequency domain - infinity to infinity, that gives us the total energy that is contained in signal that is integral - infinity to infinity S x x F d F is the total energy contained in the signal which is nothing but - infinity to infinity magnitude x(t) square dt, alright.

So this is the energy spectral density and also observe another interesting property is is magnitude X(F) square. So naturally this is greater than or equal to zero the energy spectral density is nonnegative this is greater than or equal to zero for all F this is magnitude X(F) square. So this naturally has to be greater than or equal to zero for all frequencies, energy spectral density is a nonnegative quantity, alright.

Okay, so this, what we have is the auto correlation R x x of tao, alright. R x x tao which is a measure, alright. Which is the Uhh which is the measure of the degree of self similarity signal x(t)? That is R x x tao, take the Fourier transform of that that gives us S x x of F, which for which we now have a very interesting interpretation that is the energy spectral density of x(t) and that gives characterizes how the energy of x(t) is spread in the various frequency bands is spread across the frequency axis in the frequency domain, alright?

Let us now look at an example to understand this concept of auto correlation and energy spectral density, okay. So let us look at an example to understand this that is example of auto correlation. Let us look at a simple example of the auto correlation function.

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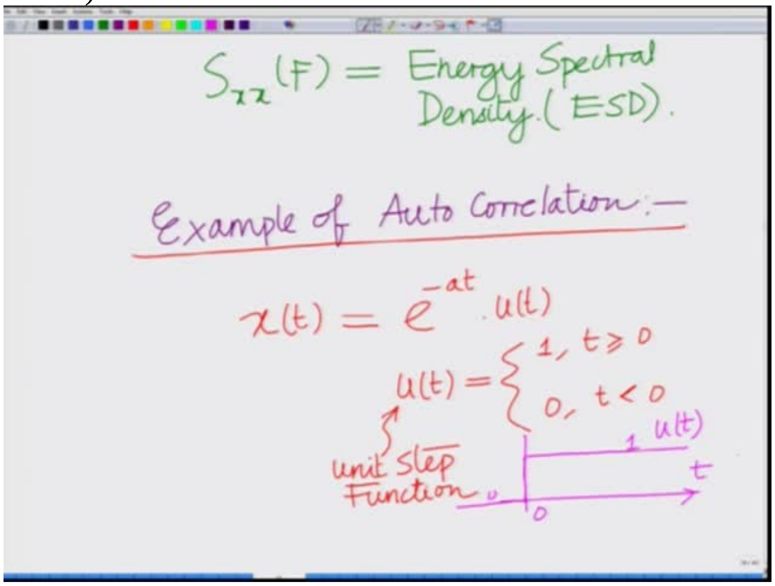

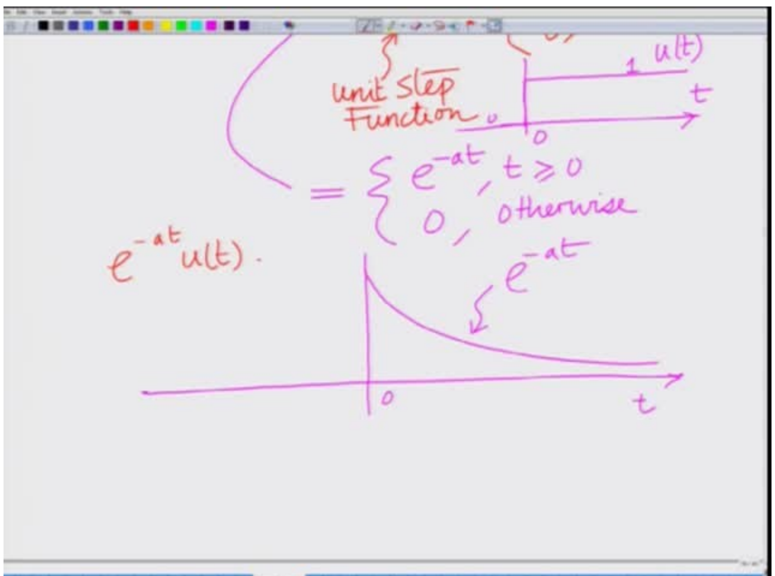

Let us consider this function or let us consider this signal x(t) equals to e power - at u(t) where u(t) as we are all familiar is the unit step function which is 1 for t greater than equal to zero, for t less than equal to zero this is known as the unit step. This is known as the unit step function, correct, this is zero for t less than equal to zero, correct? And for t greater than or equal to zero this is one.

This is our unit step function, this is zero and this is for t equal to this is basically zero and this is the time axis, alright. Therefore e to the power of - a(t) u(t) this is equal to e to the power of - a(t) t greater than or equal to zero this is equal to zero otherwise, correct? And therefore this looks as follows. This looks as e power - a(t), for t greater than or equal to zero this is e power - a(t) and for t less than zero it is obviously zero. So this is basically your function e power - a(t) into u(t), okay. This is basically your signal e power - a(t) into u(t).

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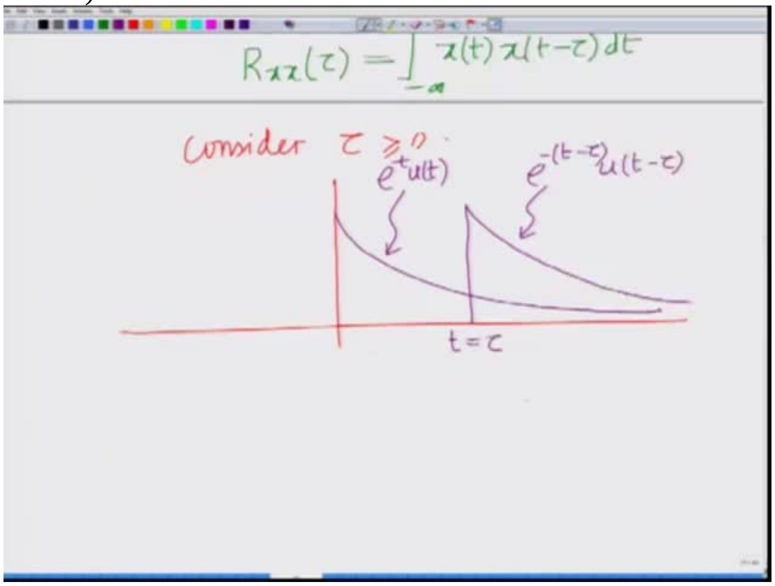

Now let us compute the auto correlation of this signal e power - a(t) u(t). Now the auto correlation is nothing but R x x tao. Consider tao Uhh R x x tao equals - infinity to infinity x(t) well here the signal is a real signal, so no need for the conjugate this is x(t) - tao d tao.

Now let tao greater than or equal to consider tao greater than or equal to zero, for this case to consider tao greater than or equal to zero. Now if tao greater than zero or tao is greater than equal to zero let us look at how the functions look we have x(t) which is e to the power of - a(t), correct u(t), we have x(t) - tao which is x(t) - tao which is so this is your e to the power of - t u(t), this is your e to the power of - u(t) e to the power - t - tao u(t) - tao, correct?

So this is shifted to t - tao, remember x(t) - tao is nothing but x(t) shifted or delayed by tao, if tao is greater than or equal to zero and therefore if tao is greater than or equal to zero it is delay which means which means the signal is shifted to the right, alright. So I have the signal x(t) which is e to the power of - t u(t) and the shifted or delayed version which is e to the power of - t - tao into u(t) - tao, alright.

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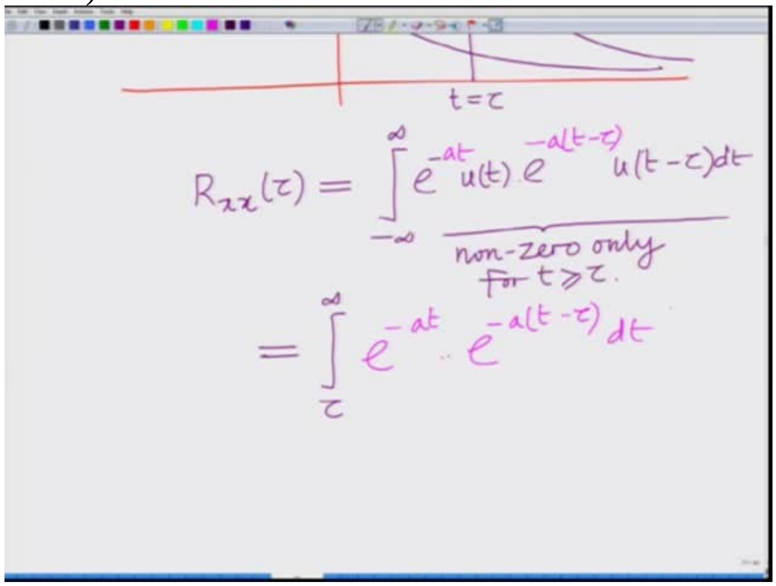

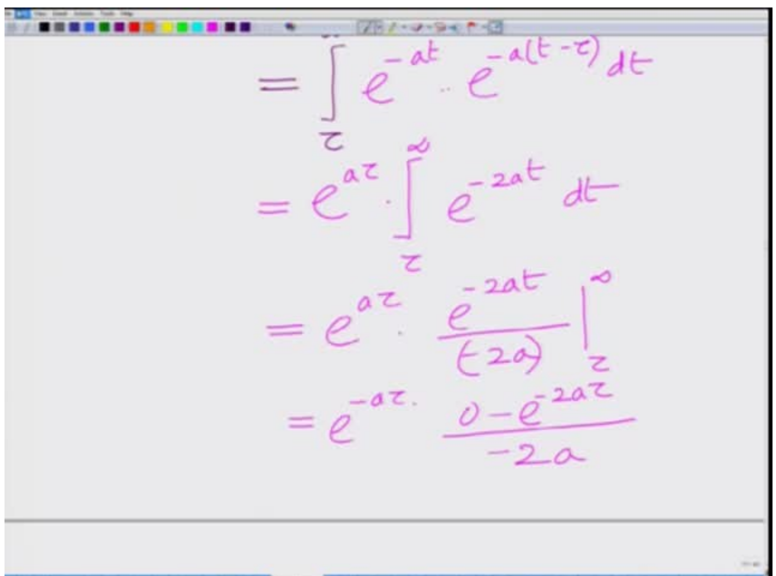

And now R1 R x x of tao is integral, well it is integral - infinity to infinity e to the power of - t u(t) e to the power of - t - tao u u(t) - tao d tao. Observe that this is nonzero only for t greater; remember look at this if you look at this plot the overlap is only after for t greater than or equal to tao. So therefore the product is nonzero only for t greater than or equal to tao and therefore I can ride this as e to the power of - infinity to infinity, correct?

Or e to the power of I can simplify this as since the product is nonzero only for t greater than or equal to tao I can simplify e this as tao infinity e to the power of - t e to the power of - t, correct? E to the power of - Uhh I have e to the power of - a(t) over here, so this is e to the power of, let me just insert e to the power of - a(t) over here, e to the power of - a t - tao.

So here I have e to the power of - a(t) e to the power of - a t - tao the product is nonzero only for t greater than or equal to tao. So this is e to the power of - a(t) e to the power of - a(t) - tao dt. And now I can simplify I can remove e to the power of a tao that is a constant it comes out of the integration integral tao to infinity e to the power of - 2 a(t) dt which is basically equal to e to the power of a(t) e to the power of - 2 a(t) by - 2 a integral from tao to infinity e to the power of a(t) e to the power of substitute the limits this becomes 1 - or this becomes at t equal to infinity this is zero - e to the power of - 2 a tao divided by - 2 a which is basically e to the power of - a tao into 1 over 2 a e to the power of - 2 a tao which is equal to e to 1 over 2 which is basically equal to, I am sorry this is e to the power of a tao . So this will be, well this will be 1 over 2 a e to the power of - a tao.

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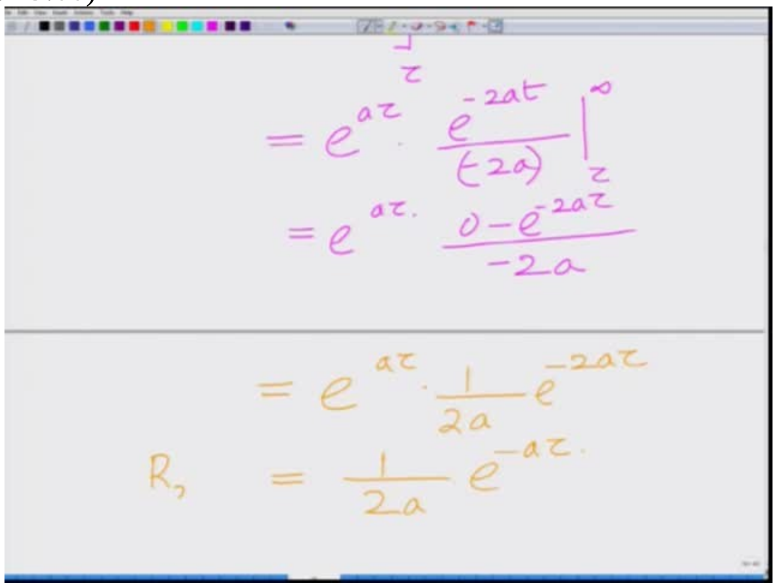

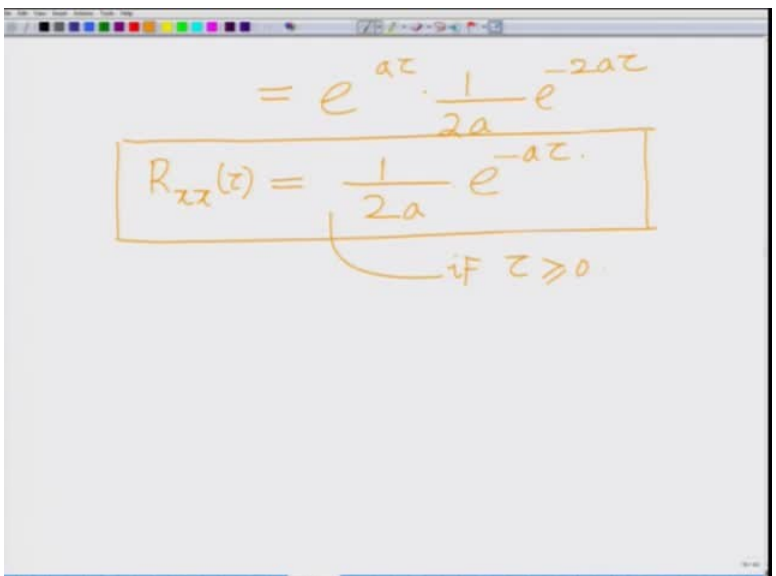

So this is your Rx R x x of tao, if tao is greater than so this is R x x of tao but keep in mind this is only if tao is greater than equal to zero. So this is only if tao is greater than equal to if there is a caveat if tao is greater than equal to zero R x x tao equals 1 over 2 a e to the power of - a tao. So alright, so what we are doing in this is basically we are solving an example to compute the auto correlation of this function e to the power of - a(t) into u(t), alright.

So let us stop this module here where we have looked at the auto correlation, alright. The Fourier transform of auto correlation S x x of F which is which we have interpreted as the energy spectral density the distribution of energy of x(t) over the frequency domain and now we are doing a simple example to compute the auto correlation function. Let us stop here and continue in the subsequent modules, thank you.