Principles of Communication- Part I
Professor Aditya K. Jagannathan
Department of Electrical Engineering
Indian Institute of Technology Kanpur
Module No 2
Lecture 07: Profession of Signal through Linear Time Invariant Systems (LTI) and Cross-Correlation of Signals

Hello welcome to another module in this massive open online course. So let us continue our discussion on linear time invariant systems and what happens when a signal x(t) is input to a linear time invariant system.

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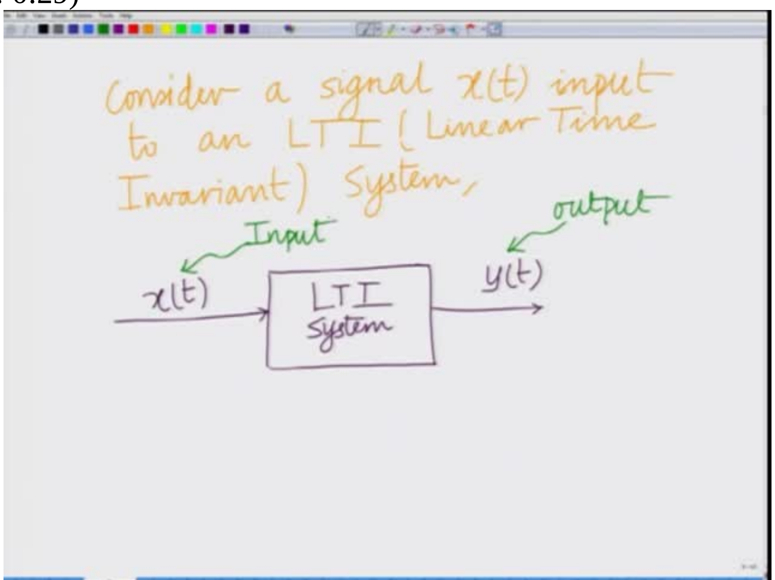

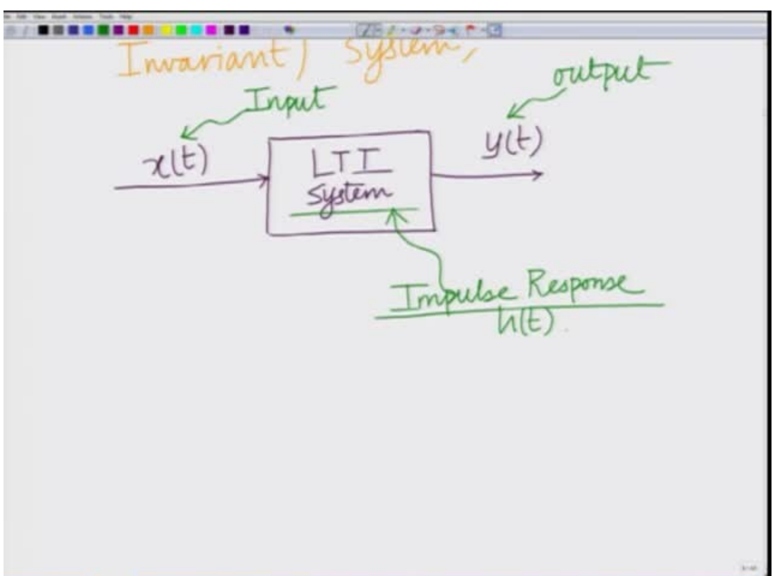

So consider a signal x(t) input to an LTI that is your linear time invariant system, remember yesterday we elaborated on what did, what a linear, the properties is of a linear time invariant system that is linearity and time invariance. So let us consider a signal x(t) that is given as an input to a linear time invariant system, okay. So we have our LTi system, okay. So x(t) and let the corresponding input output be y(t).

So this is our LTi system, so y(t) is the output of the LTi system, x(t) it is the input to the LTi system, correct? And any LTi system is characterized by an impulse response response h(t). What is the impulse response h(t)? That is if the impulse that is your direct Delta function delta t is given as input to the LTi system, the output is represented by h(t) which is the impulse which is the response to the direct Delta function delta t and this is known as the impulse response of the LTi system.

And this fundamentally characterizes the properties in the behavior of the LTi system. And the output corresponding to any signal any input signal x(t) can be described in terms of the input x(t) and the impulse response h(t) of the LTi system as follows.

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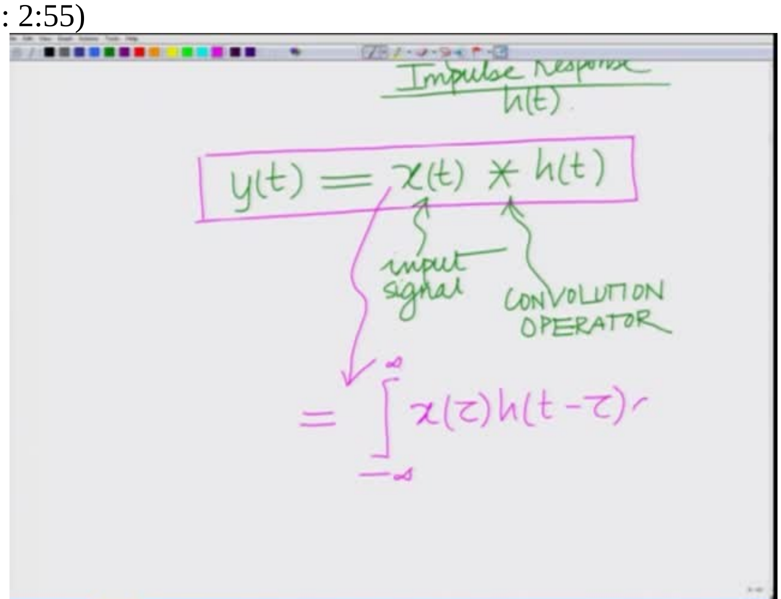

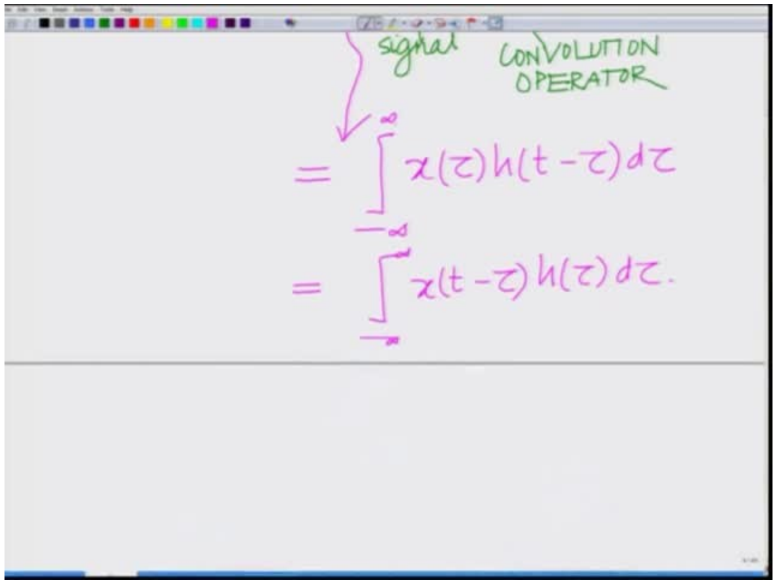

y(t) that is the output corresponding to any signal x(t) is x(t) convolved with h(t) where x(t) is your input signal, this is the convolution operator y(t) is x(t) convolved with h(t), therefore if I know the impulse response h(t) of the LTi system that completely characterises the LTi system for any given in other arbitrary input signal x(t) the output y(t) can be derived by can convolving the input x(t) with the impulse response h(t).

And this convolution operation grease represented as this is equal to basically integral - infinity to infinity xt - tao or that is your x tao h t - tao d tao which is also equal to integral - infinity to infinity x t - tao h tao d tao. So the output y(t) is the input x(t) convolved with the impulse response h(t) and this has a very interesting, the convolution has a very interesting has a convolution follows a very interesting property.

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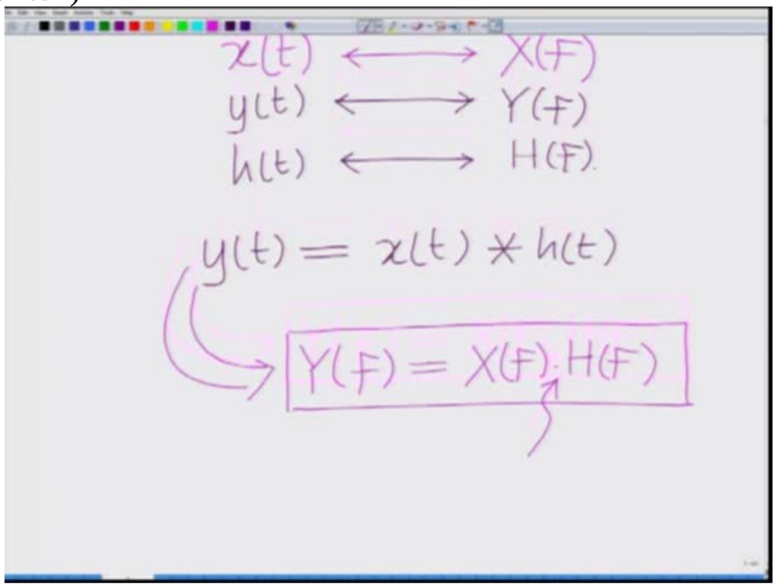

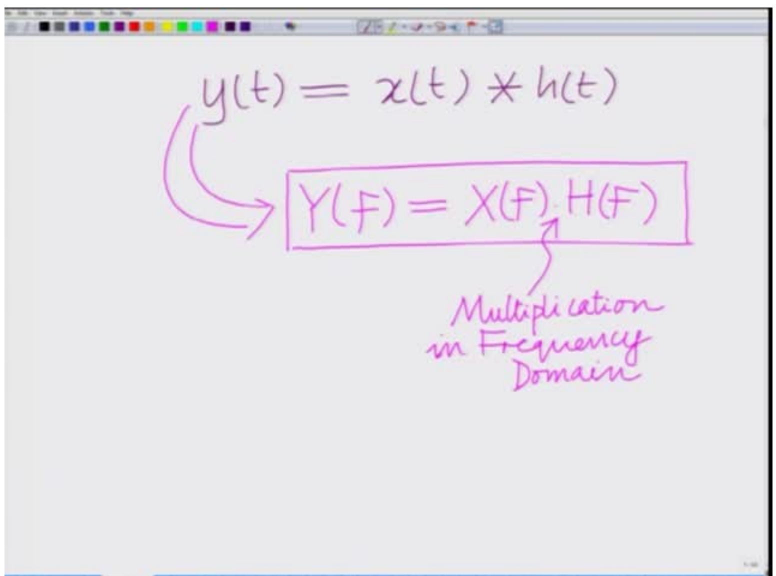

If I consider the Fourier transform of the input and output, let x(t) have a Fourier transform our x(t) be a Fourier transform pair with X(F) the output y(t) we Fourier transform pair with Y(F) that is Fourier transform of y(t) is given by Y(F). The Fourier transform of the impulse response h(t) be given by H of F then we have in the time domain y(t) equals x(t) convolved with h(t). This implies in the frequency domain the Fourier transform Y(F) equals the input Fourier transform X(F) times that is the product H(f). That is convolution in the time domain becomes multiplication in the frequency domain.

This becomes multiplication. This is a way convolution satisfies a very interesting property that is if 2 signals are convolved in the time domain that is y(t) is the convolution of x(t) with h(t) then the Fourier transform of y(t) that is Y(F) is obtained by the multiplication of the Fourier transform that is the Fourier transform of x(t) that is X(F) with the Fourier transform of h(t) that is HF.

So Y(F) equals X(F) times H(F) that is convolution in the time domain is multiplication is equivalent to multiplication of the 2 signals in a of the Fourier transform of that is 2 respective signals in the frequency domain and this is 1 of the fundamental properties of linear systems and it is also very applicable, 1 of the fundamental principles of, communication because we are going to look at several instances where input signal is passed through an LTi system and we derive an output signal and therefore it is very important to remember that the Fourier transform, right?

Or spectrum of the output signal is is derived by the multiplication of the spectra of the Fourier transform. So the input signal and the impulse response alright. Now let us look at another concept that is the cross correlation between 2 signals.

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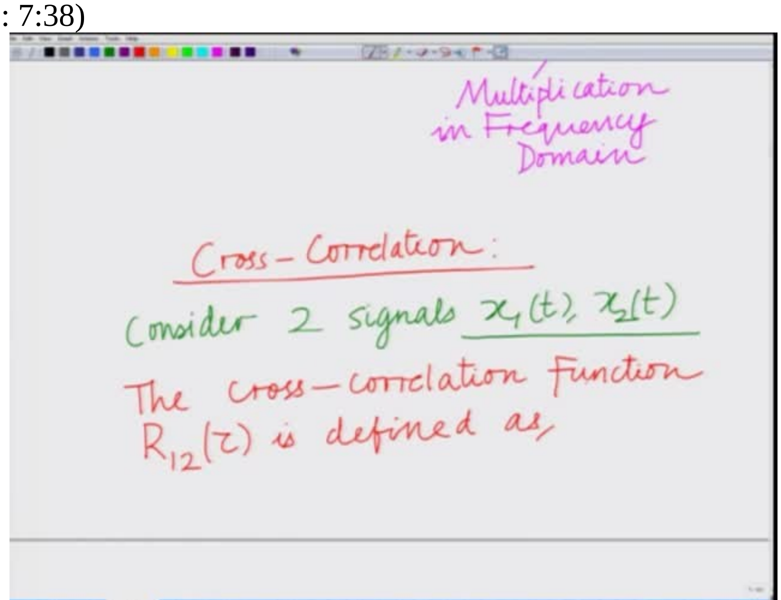


$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)\, x_2^*(t-\tau)\, dt$$

Auto correlation For lag τ

Characterizes the extent or degree of SIMILARITY between $x_1(t), x_2(t)$ for a shift of τ.

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)\, x_2^*(-(\tau-$$

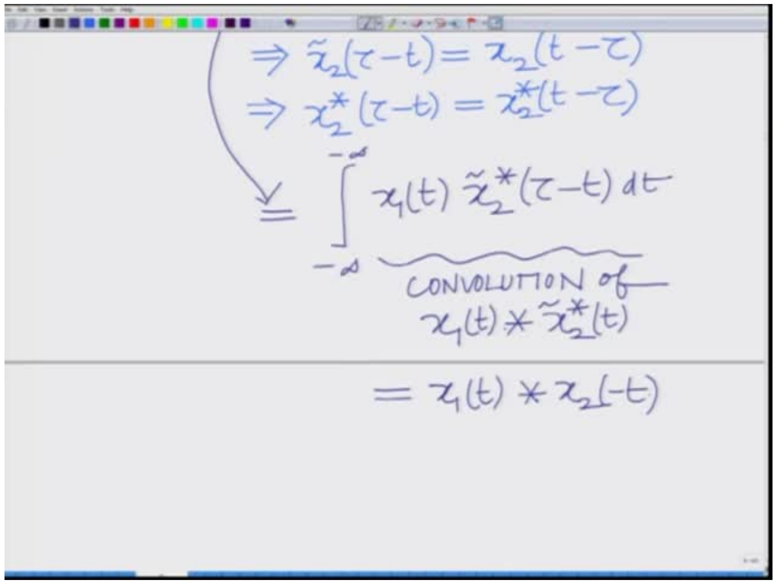
a shift of τ.

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)\, \underbrace{x_2^*(-(\tau-t))}_{\tilde{x}_2(\tau-t)}\, dt$$

$$\tilde{x}_2(t) = x_2(-t)$$

$$\Rightarrow \tilde{x}_2(\tau-t) = x_2(t-\tau)$$

$$\Rightarrow x_2^*(\tau-t) = x_2^*(t-\tau)$$



So we will start looking at another concept that is the cross correlation. The cross so consider for this purpose considered 2 signals x(t) and y(t) or 2 signals x1 t, x2 t, correct? The cross correlation function R12 t the cross correlation function are 1 2 tao is defined as, well R12 tao equals integral - infinity to infinity x1 t x2 conjugate t - tao dt.

X1 t x2 conjugate t - tao dt, this is R12 tao, this is the auto correlation for a lag tao that is what we are doing is getting x1 t. Multiplying it by the shifted version of x2, that is x2 conjugate t - tao, multiplying x1 t by x2 conjugate t - tao and integrating - from - infinity to infinity, this basically this is a measure of the extent of similarity between x1 t and x2 t for the lag tao.

So this characterizes, this is important to know the meaning of the auto correlation characterizes the extent or degree of similarity between x1 t, x2 t for a lag or a shift of tao. For a shift the extent of similarity between x1 t and x2 t for a shift of tao that is corresponding to the shift of tao. What is the extent of similarity between the signal x1 t and x2 t - tao?

Alright, now simplify this. That is we have R1 tao or R12 tao we have written this as x1 t - infinity to infinity x1 t x2 conjugate t - tao I am going to write this as x2 conjugate - tao - t dt. Now let us define x2 tilde t as x2 of - t this implies x2 conjugate x2 tilde of tao - t equals x2 of t - tao this implies x2 conjugate of tao - t equals x2 conjugate of t - tao therefore here instead of x2 conjugate t - tao I can substitute, here I can substitute x2 tilde of tao - t.

So therefore this becomes a cross correlation becomes this is integral - infinity to infinity x1 t x2 conjugate tao - t dt and notice this is nothing but convolution of x1 t and x2 conjugate of t. This basically you will see realize that from all definition this is convolution of, it follows from the definition of convolution that this is a convolution of x1 t with x2 conjugate of t but realize x2 conjugate that is x2 tilde, I am sorry this is x2 tilde conjugate tao - t.

This is a convolution of x1 t with x2 tilde conjugate t but x2 tilde conjugate t is equal to x2 x2 conjugate - t. So this is a convolution of x1 t with x2 of - t, okay.

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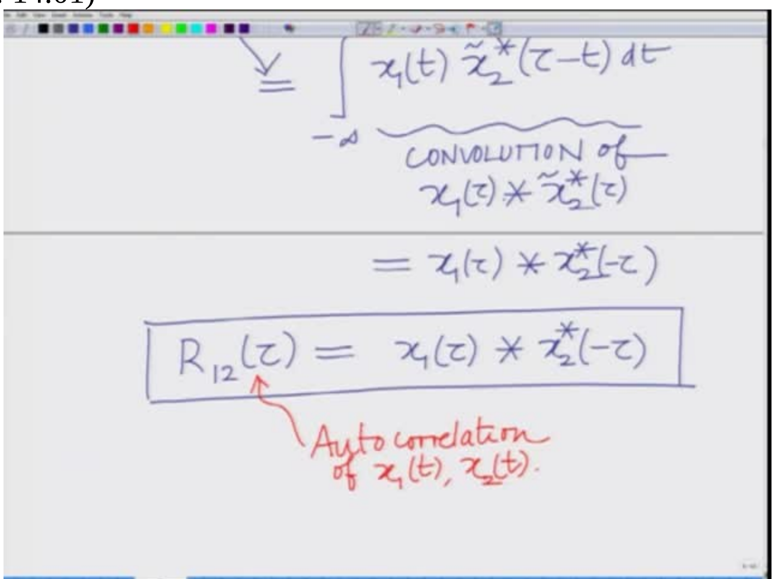

So R12 or you can also replace this by tao that is R12 or you can also replace this by tao. Since t here is the variable of integration, so this is tao this is - tao, so I have R12 of tao equals x1 tao convolved with x2 conjugate of this is x1 tao convolved with x2 conjugate of - tao, remember this is the auto correlation of x1 t and x2 t corresponding to a lag of tao. This is the auto correlation function of x1 t, 2 different signals x1 t and x2 t corresponding to a lag of tao.

We have shown that this is basically the convolution between x1 tao and x2 conjugate of - tao. This is convolution between x1 and the conjugate version of x2. There is conjugate x2 and flip it, x2 of - tao x2 conjugate of - tao is nothing but flip it about the origin and take the complex conjugate, alright. And this auto correlation is an important measure because it is it is a measure of the degree of similarity between x1 and x2 for a lag of tao, alright.

And this has a lot of uses as we are going to see later in our study of communication system. This measure of out of this measure that is the auto correlation between 2 different signals x1 t and x2 t, okay.

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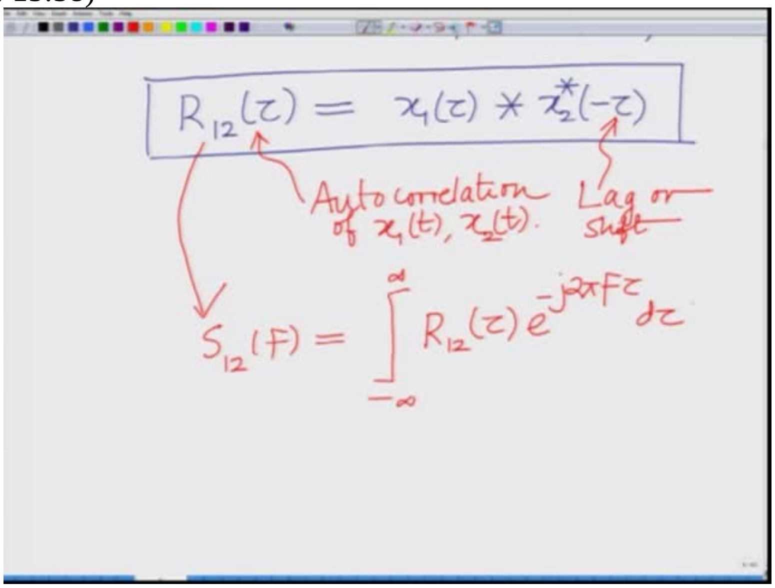

Now let us try to look at this in the now let us look at this in the frequency domain naturally convolution and the time is that is convolution in the tao domain, of course now this is not even time this is tao, remember this is the lag or shift now if we take to the Fourier transform of this that is denote this by S12 of F equals integral - infinity to infinity R12 of tao e to the power of - j 2 pi F tao d tao then we know that this convolution in the Tao domain is multiplication in the frequency domain.

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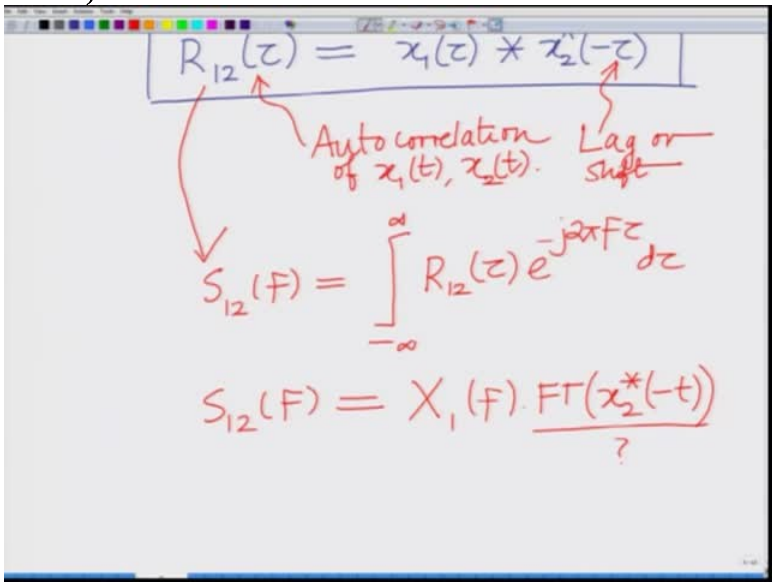

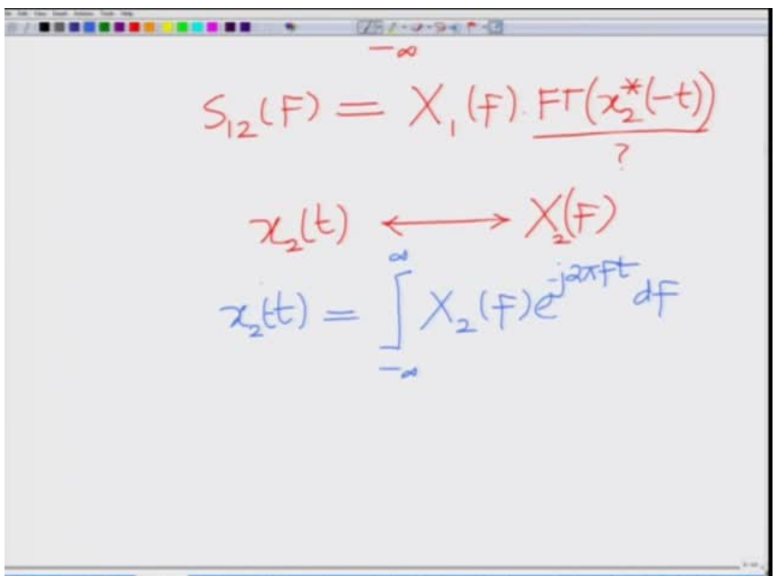

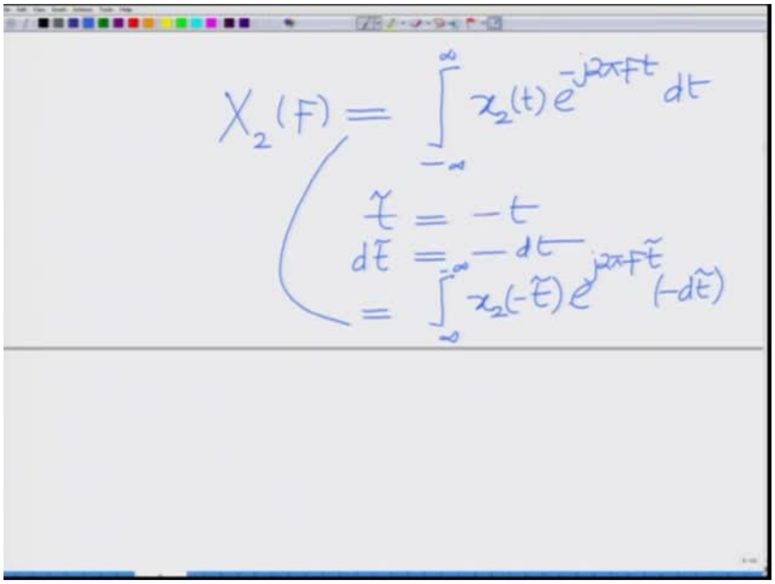

Therefore we can write from the property of linear system or from the property of convolution S12 F is x1 F times the Fourier transform of x2 tilde or x2 conjugate Fourier transform of x2 conjugate - t. Now what we have to do is we have to derive the Fourier transform of F 2 conjugate - t that is what is the Fourier transform of F 2 conjugate - t? Now we know that if x2 t forms a Fourier transform pair with x2 F then x2 t is given by the inverse Fourier transform that is x2 F e to the power of j 2 pi Ft d F consider the conjugate of this, well if I replace this by - t this becomes over here this becomes - t.

And now if I consider the conjugate, correct? Now if I for instance let us look at this rather than the inverse Fourier transform, let us look at the Fourier transform that is x2 of F equals - infinity to infinity x2 of t e to the power of - j 2 pi Ft, correct? Now if I replace dt, correct? Now if I replace t by - t so let us now replace or let us say t tilde equals - t then what we have is dt tilde equals - dt. So this integral becomes t tilde equals - t, so this integral becomes infinity to - infinity x2 of - t tilde e to the power of j 2 pi F t tilde - d tilde. I can remove this - sign by interchanging the order of integration.

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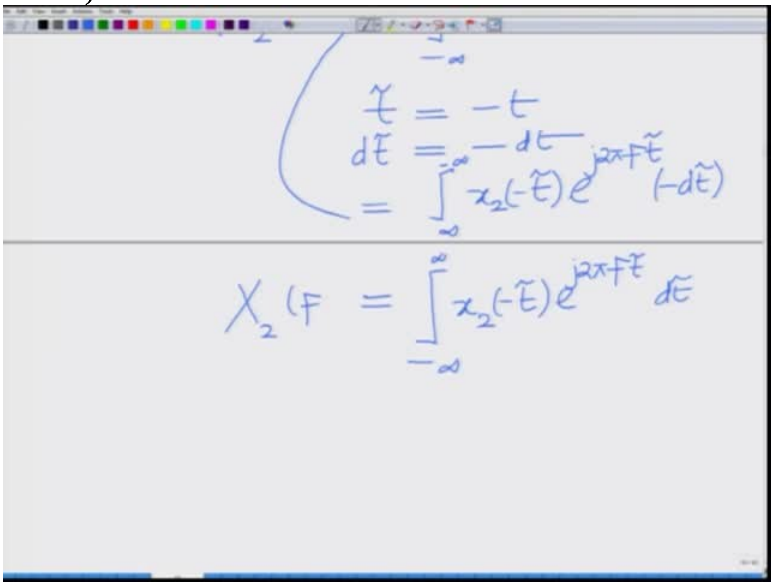

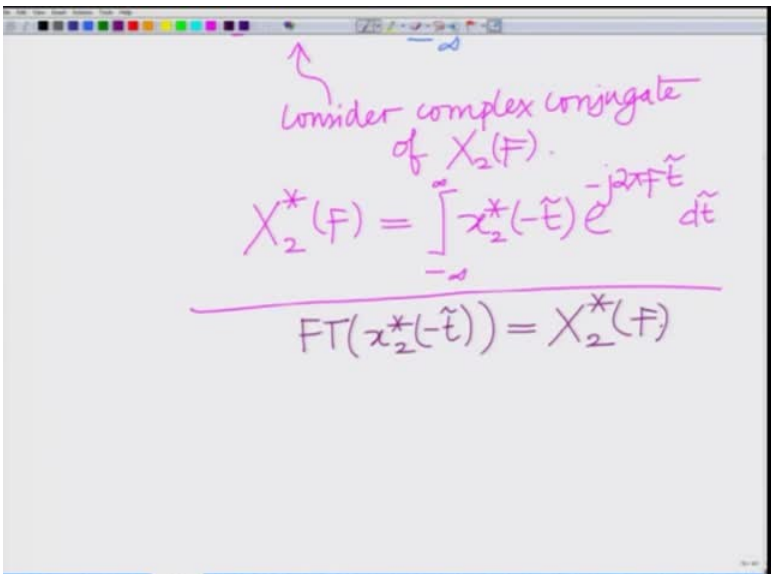

So this is if I flip is from infinity to - infinity equals - infinity to infinity therefore x2 of - t tilde e to the power of j 2 pi F d tilde, the - sign in d tilde goes because I change the other of integration. Now if we consider the complex conjugate. Now consider complex conjugate if we consider the complex conjugate of x2 F I have x2 conjugate of F equals - infinity to infinity x2 conjugate of - t tilde e to the power of - j 2 pi F t tilde d tilde.

And therefore now, what you can see from this? What you can see from this is that the Fourier Fourier transform of x2 conjugate - t tilde the Fourier transform x2 conjugate - t tilde is x2 conjugate F. That is FT of x2 or x2 conjugate - t tilde equals x2 conjugate of F.

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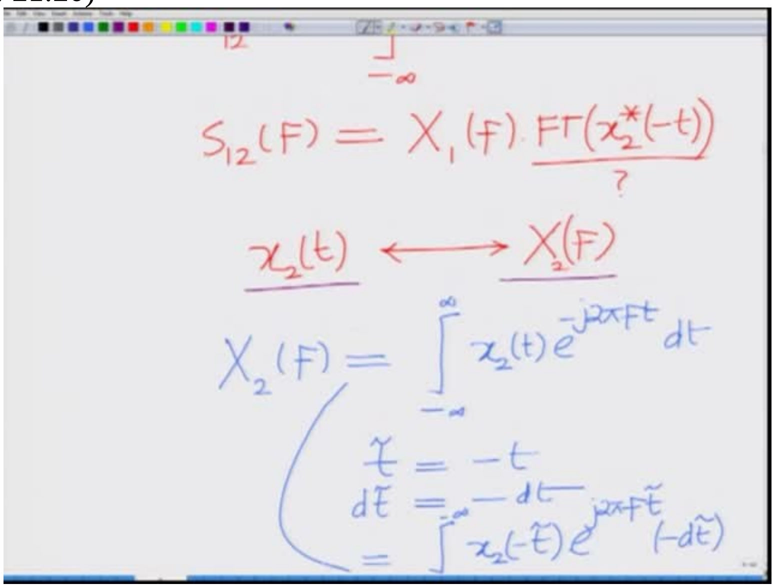

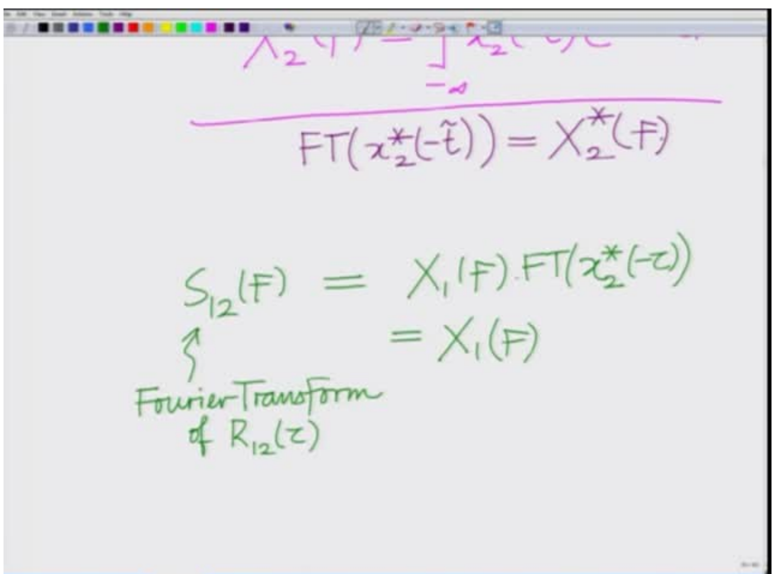

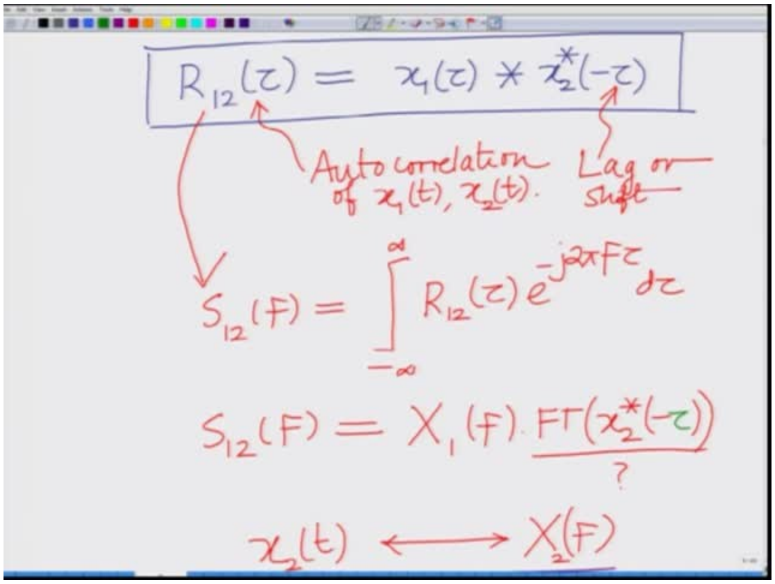

And therefore what we have over here that is if x2, a has Fourier transform if to x2 F than x2 conjugate or x2 conjugate - t has Fourier transform x2 conjugate F. Therefore now we can write the Fourier transform S12 F, remember S12 F is Fourier transform of your auto correlation function R12 tao, that is x1 of F times Fourier transform x2 conjugate of - t or x2 conjugate of - tao which is basically x1 of F, this is the Fourier transform of x2 conjugate of - tao. This is the Fourier transform x1 of F times x2. Product of the Fourier that is x1 of F times x2 conjugate of F.

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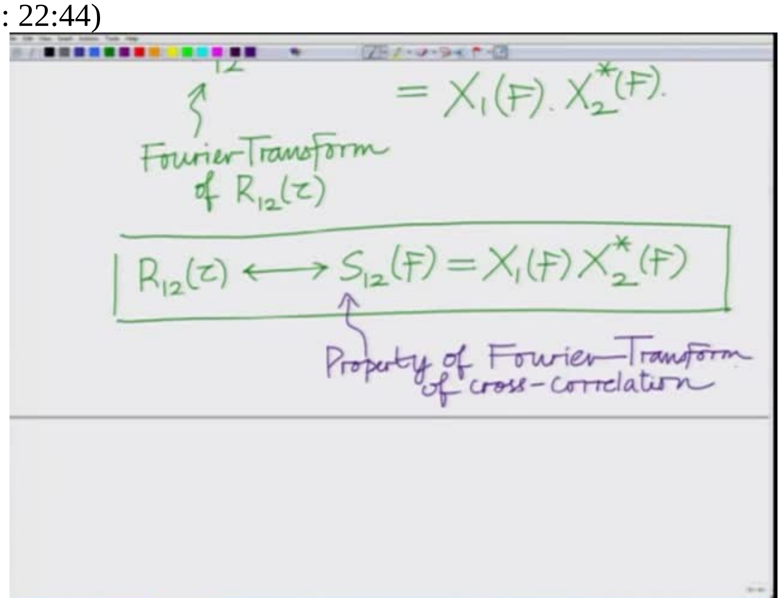

Therefore what we have is R12 of tao which has Fourier transform S12 of F which is equal to x1 of F times x2 conjugate of F. This is the property of the Fourier transform of the cross correlation property of the, sorry this has to be the cross correlation therefore this is the Fourier transform of the cross correlation between 2 functions x1, this is the Fourier transform of the cross correlation that is we are considering 2 different signals x1 t and x2 t.

So this is the property of the Fourier transform of the cross correlation that is R12 t denotes the cross correlation between x1 t and x2 t for a lag of tao, okay. Now what we are going to do is we are going to consider a special case of the cross correlation that is the auto correlation, okay.

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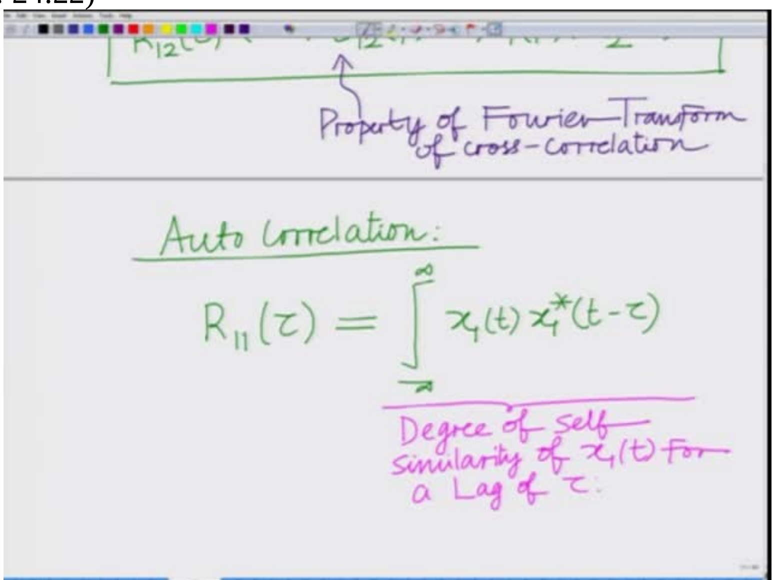

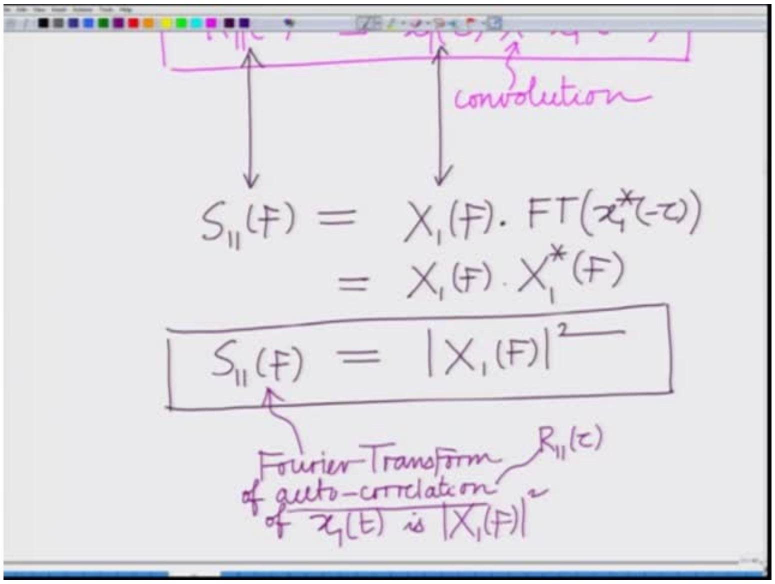

So now we are going to consider the auto correlation which is basically now the that is if I replace x2 t by x1 t itself that is the degree of self similarity of the signal that is characterize the degree of self similarity of the signal x1 t for a lag of tao it becomes the auto correlation. So the auto correlation is R11 of tao is the degree of self similarity of x1 t x1 conjugate t - tao. What is this? This measures the degree of self similarity.

That is this is the degree of self similarity of x1 t for a for a lag of tao s So therefore R11 t from the property that we have deal shown before simply, we have simply replaced x1 t by x2 t by x1 t. So this is the convolution of x1 tao with x1 conjugate of - tao that is your R11 of tao.

That is the auto correlation, this is the convolution operator and further if you look at the Fourier transforms that is S 11 of F that is Fourier transform the auto correlation is Fourier transform of X1 tao which is X1 F times the product in the time domain Fourier transform of x1 conjugate - tao which is nothing but as we have shown previously Fourier transform of X1 conjugate - tao is X1 conjugate of F.

So we have X1 F into X1 conjugate of F, so this is magnitude X1 F square. So S 11 F that is the Fourier transform of the auto correlation of x1 t. This is the Fourier transform and this is a very interesting property this is the Fourier transform of auto correlation of x1 of t is magnitude X1 of F square. That is this auto correlation which is R11 of tao is magnitude X1 F square where X1 F is the Fourier transform of x1 t, right? So x1 t has Fourier transform X1 F, alright.

Okay, so this is important to keep in mind. So what we have shown is a very interesting property that is we have looked at the auto correlation function, alright. That is R11 tao which characterises the self similarity of the (si) signal x1 t for a lag of tao its Fourier transform is nothing but the Fourier transform of magnitude square magnitude X1 F square where X1 F is a Fourier transform of x1 tao, alright.

So this is a very interesting property. Again we are going to rely on these properties several times that is the cross correlation between 2 signals x1 and x2 t, the Fourier transform the cross correlation between x1 and x2, alright. X1 t andx2 t and also the auto correlation of the signal x1 t and the Fourier transform of the auto correlation of x1 t, alright. So these so we have looked at these concepts today studied several properties and we are going to rely on these properties in analysis of different communications (sys) systems, alright. So let us stop this module here and we will continue with other aspects, thank you.